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A. M. Plichko, M. M. Popov

# SOME OPEN PROBLEMS ON BANACH SPACES

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Notes of the Problem Session which has been held on the section of Banach Spaces during the International conference dedicated to the 120-th anniversary of Stefan Banach in Lviv (Ukraine), September 17–21, 2012.

А. Н. Пличко, М. М. Попов. *Некоторые открытые проблемы о банаховых пространствах* // Мат. Студії. – 2012. – Т.38, №2. – С.203–211.

Записки проблемной сессии секции банаховых пространств на международной конференции, посвященной 120-летию со дня рождения Стефана Банаха в г. Львов (Украина) 17–21 сентября 2012 г.

## 1. A question about renormings of the James space (V. M. Kadets).

*Kharkiv V.N. Karazin National University (Kharkiv, Ukraine)*, vova1kadets@yahoo.com

Let us say that a Banach space $X$ possesses the 2-*Daugavet property* if every finite rank operator $T\colon X \to X$ satisfies

$$\|\mathrm{Id} + T\|^2 \geq 1 + \|T\|^2.$$

**Problem 1.1.** *Is it true that the James space $J$ possesses the 2-Daugavet property in some equivalent norm?*

It is known ([1]) that the 2-Daugavet property of $X$ implies the absence of an unconditional basis in $X$. So, the 2-Daugavet property could be a reasonable object of study. Unfortunately, there is a little obstacle: all the known examples of spaces with the 2-Daugavet property in fact possess (in the original norm or after a renorming) the much stronger *Daugavet property*: every finite rank operator $T\colon X \to X$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

Every space with the Daugavet property contains a isomorphic copy of $\ell_1$ (see [2], so the James space $J$ does not have this property in any equivalent norm. So a positive answer to our problem would justify the very definition of the 2-Daugavet property. On the other hand, the only way to give a negative answer is to find a new isomorphic consequence of the

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2-Daugavet property, different from the absence of an unconditional basis, and this is also an interesting task.

1. V.M. Kadets, *Some remarks concerning the Daugavet equation*, Quaestiones Math., **19** (1996), 225–235.
2. V.M. Kadets, R.V. Shvidkoy, G.G. Sirotkin, D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc., **352** (2000), 855–873.

## 2. Bidual spaces with the Daugavet property (V. M. Kadets, D. Werner).

*Kharkiv V.N. Karazin National University (Kharkiv, Ukraine)*, vova1kadets@yahoo.com
*Free University Berlin (Berlin, Germany)*, werner@math.fu-berlin.de

A Banach space $X$ has the *Daugavet property* if every finite rank operator $T\colon X \to X$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|. \tag{1}$$

Equivalently, $X$ has the Daugavet property iff (1) holds for every rank-1 operator iff (1) holds for every operator not fixing a copy of $\ell_1$. Among the examples of spaces with the Daugavet property let us mention $C(K)$ if the compact space $K$ does not contain an isolated point, $L_1(\mu)$ and $L_\infty(\mu)$ if the measure $\mu$ does not have any atoms, the algebra $H^\infty$ of bounded analytic functions on the unit disc, and the space of Lipschitz functions on a convex subset of a Banach space. Among the examples of Banach spaces with the Daugavet property there are several dual Banach spaces (e.g., $L_\infty(\mu)$, $H^\infty$ or $\mathrm{Lip}(Q)$ in the above list), but we are not aware of any bidual example. So we ask:

**Problem 2.1.** *Do there exist bidual Banach spaces with the Daugavet property?*

Clearly, if $X^{**}$ has the Daugavet property, then so do $X^*$ and $X$. It is known ([1]) that a space with the Daugavet property contains a copy of $\ell_1$ and fails the RNP; hence if $X$ has the Daugavet property, then $X^*$ and a fortiori $X^{**}$ is not separable. It is also known that there is no bidual space among the $C(K)$- or $L_1(\mu)$-spaces with the Daugavet property. The results of [2] imply that the same is true for their noncommutative counterparts, i.e., $C^*$-algebras or preduals of von Neumann algebras.

1. V.M. Kadets, R.V. Shvidkoy, G.G. Sirotkin, D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc., **352** (2000), 855–873.
2. T. Oikhberg, *The Daugavet property of $C^*$-algebras and non-commutative $L_p$-spaces*, Positivity, **6** (2002), 59–73.

**3. On Maurey and narrow operators on the spaces $L_p$ for $1 \le p < \infty$ (I. Krasikova).**

*Zaporizhzhya National University (Zaporizhzhya, Ukraine)*, yudp@mail.ru

We consider two classes of "small" (continuous linear) operators on the spaces $L_p = L_p[0,1]$ for $1 \le p < \infty$ — narrow operators and Maurey operators. These classes generalize compact operators. So, it is natural to ask, what is the connection between these operators.

Let $\mathcal{B}$ denote the Borel $\sigma$-algebra on $[0,1]$. Recall that an operator $T \in \mathcal{L}(L_p)$ is called *narrow* if for each $A \in \mathcal{B}$, and $\varepsilon > 0$ there is $x \in L_p$ such that $x^2 = \mathbf{1}_A$, $\int_{[0,1]} x\, d\mu = 0$ and $\|Tx\| < \varepsilon$. For an information on narrow operators we refer the reader to the recent monograph [4].

To define Maurey operators, we denote by $Z$ the closed unit ball of $L_\infty$ endowed with the weak* topology $\sigma(L_\infty, L_1)$. For every $A \in \mathcal{B}$ we set

$$Z(A) = \left\{ h \in Z \colon h^2 = \mathbf{1}_A, \ \int_{[0,1]} h\, d\mu = 0 \right\}.$$

In other words, $h \in Z(A)$ if and only if $h = \mathbf{1}_B - \mathbf{1}_C$ for some $B, C \in \mathcal{B}$ with $A = B \sqcup C$ and $\lambda(B) = \lambda(C)$. We consider $Z(A)$ with the topology induced by $Z$. Following [2], for every $T \in \mathcal{L}(L_p)$ and every $A \in \mathcal{B}$ we consider the maps

$$\widetilde{M}_T(A) = \limsup_{Z(A) \ni h \to 0} \int_{[0,1]} h\, Th\, d\mu, \quad \widetilde{m}_T(A) = \liminf_{Z(A) \ni h \to 0} \int_{[0,1]} h\, Th\, d\mu,$$

and set

$$M_T(A) = \inf \left\{ \sum_{k=1}^{n} \widetilde{M}_T(A_k) \colon n \in \mathbb{N}, \ \ A = \bigsqcup_{k=1}^{n} A_k, \ \ A_k \in \mathcal{B} \right\},$$
$$m_T(A) = \sup \left\{ \sum_{k=1}^{n} \widetilde{m}_T(A_k) \colon n \in \mathbb{N}, \ \ A = \bigsqcup_{k=1}^{n} A_k, \ \ A_k \in \mathcal{B} \right\}.$$

The previous two maps are countably additive measures on $\mathcal{B}$, which we call the *upper* and *lower Maurey measures*, respectively. Each of these measures has the Radon-Nikodým derivative, that is, for every $A \in \mathcal{B}$ one has the representations

$$M_T(A) = \int_A F_T d\mu, \ m_T(A) = \int_A f_T d\mu,$$

where $F_T, f_T \in L_\infty$ are some functions called the *upper* and *lower Maurey derivatives of* $T$. Remark that for every operator $T \in \mathcal{L}(L_p)$ one has $f_{-T} = -F_T$, and the number

$$\|T\|_M = \max\bigl\{ \|f_T\|_\infty, \|F_T\|_\infty \bigr\}$$

is a semi-norm on $\mathcal{L}(L_p)$ which we call the *Maurey semi-norm* of $T$. We say that an operator $T \in \mathcal{L}(L_p)$ is a *Maurey operator* if $\|T\|_M = 0$.

Every compact operator $T \in \mathcal{L}(L_p)$ with $1 \le p < \infty$ is Maurey, and the set $\mathcal{M}(L_p)$ of all Maurey operators is a closed linear subspace of $\mathcal{L}(L_p)$. Moreover, every operator $T \in \mathcal{L}(L_p)$ which is not Maurey is an isomorphic embedding when being restricted to a suitable subspace $E \subseteq L_p$ isomorphic to $L_p$, see [1].

In spite of similarity of some properties, in general the classes of narrow and Maurey operators are incomparable. Indeed, on the one hand, there exists a Maurey operator on $L_2$ which is an onto isometry, and hence, is not narrow. On the other hand, for each $p \in (1, +\infty)$ there is a narrow operator $T \in \mathcal{L}(L_p)$, which is not Maurey. The latter fact follows from the existence of a decomposition of the identity $Id$ of $L_p$ as a sum of two narrow operators ([3, p. 59]), and the linearity of $\mathcal{M}(L_p)$.

**Problem 3.1.** *Does there exist a non-narrow Maurey operator on $L_p$ for $p \neq 2$?*

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## 4. Injective Banach lattices (A. G. Kusraev).

*Southern Mathematical Institute of Vladikavkaz;*
*Science Center of the Russian Academy of Sciences*
*(Vladikavkaz, Russia)*, kusraev@smath.ru

**Definition 4.1.** Let $1 \leq \lambda \in \mathbb{R}$. A real Banach lattice $X$ is said to be $\lambda$-*injective*, if for every Banach lattice $Y$, closed sublattice $Y_0 \subset Y$, and positive operator $T_0 \colon Y_0 \to X$ there exists a positive extension $T \colon Y \to X$ with $\|T\| \leq \lambda \|T_0\|$.

It was proved in [1] that every finite-dimensional $\lambda$-injective Banach lattice is lattice isomorphic to $\left(\sum_{j\leq k}^{\oplus} l_1(n_j)\right)_{l_\infty}$, while it was shown in [2] that every order continuous $\lambda$-injective Banach lattice is lattice isomorphic to $L_1(\mu)$ space. But the general question is still open:

**Problem 4.2.** *Is every $\lambda$-injective Banach lattice order isomorphic to* 1*-injective Banach lattice?*

One of the intriguing problems is the classification of Banach space whose duals are isometric to $AL$-spaces, see [3]. I believe that the injective version of this problem deserves an independent study.

**Problem 4.3.** *Classify and characterize Banach spaces whose duals are injective Banach lattices.*

As is seen from [4] an injective Banach lattice $X$ has a mixed $LM$-structure. Thus, the dual $X'$ should have, in a sense, an $ML$-structure. Hence a natural question arises.

**Problem 4.4.** *What kind of duality theory is there for injective Banach lattices?*

**Definition 4.5.** An *injective envelope* of a Banach lattice $X$ is a pair $({}^{\epsilon}X, \iota)$ with ${}^{\epsilon}X$

an injective Banach lattice and $\iota\colon X \to {}^{\mathfrak{e}}X$ a lattice isometry such that the only sublattice of ${}^{\mathfrak{e}}X$ that is injective and contains $\iota(X)$ is ${}^{\mathfrak{e}}X$ itself, cf. [5].

**Problem 4.5.** *Does every Banach lattice have an injective envelope?*


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## 5. Approximation properties which are metric with respect to Banach operator ideals (A. Lissitsin, E. Oja).

*University of Tartu (Tartu, Estonia)*, aleksei.lissitsin@ut.ee,
*University of Tartu (Tartu, Estonia)*, eve.oja@ut.ee

Let $X$ and $Y$ be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from $X$ to $Y$ and by $\mathcal{F}(X,Y)$ its subspace of finite-rank operators. Let $I_X$ denote the identity operator on $X$. Recall that $X$ has the *approximation property* (AP) if there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha \to I_X$ uniformly on compact subsets of $X$. If $(S_\alpha)$ can be chosen with $\sup_\alpha \|S_\alpha\| \le 1$, then $X$ is said to have the *metric AP* (MAP). These are classical notions due to Grothendieck.

Let $\mathcal{A} = (\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach operator ideal. According to [1], we say that $X$ has the *MAP for* $\mathcal{A}$ if for every Banach space $Y$ and every operator $T \in \mathcal{A}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha \to I_X$ uniformly on compact subsets of $X$ and $\limsup_\alpha \|TS_\alpha\|_{\mathcal{A}} \le \|T\|_{\mathcal{A}}$.

Having a natural partial ordering on the class of all Banach operator ideals, let us look at the chain

$$\mathcal{N} \subset \mathcal{SI} \subset \mathcal{I} \subset \mathcal{W} \subset \mathcal{L},$$

where $\mathcal{N}$, $\mathcal{SI}$, $\mathcal{I}$, $\mathcal{W}$, and $\mathcal{L}$ denote, respectively, the Banach operator ideals of nuclear, strictly integral, integral, weakly compact, and of all bounded linear operators. Obviously, the MAP for $\mathcal{L}$ is just the MAP. By a definition in [2], the MAP for $\mathcal{W}$ is the *weak MAP*. Further on, by [1], both the MAP for $\mathcal{I}$ and the MAP for $\mathcal{SI}$ are again equal to the MAP, but, in turn, the MAP for $\mathcal{N}$ equals the weak MAP.

"Zooming in" between $\mathcal{I}$ and $\mathcal{W}$, one can see, e.g., the ideal $\mathcal{P}$ of absolutely summing operators and its dual ideal $\mathcal{P}^{\mathrm{dual}}$. By [1], the MAP for $\mathcal{P}^{\mathrm{dual}}$ equals the MAP.

**Problem 5.1** (cf. Problem 5.3 in [4])**.** *Describe the MAP for* $\mathcal{P}$.

More generally, recalling that $\mathcal{N} = \mathcal{N}_1$, $\mathcal{I} = \mathcal{I}_1$, $\mathcal{P} = \mathcal{P}_1$, one is interested in the following.

**Problem 5.2** (cf. Problem 5.4 in [4])**.** *Describe the MAP for* $\mathcal{N}_p$, $\mathcal{I}_p$, $\mathcal{P}_p$, $1 < p \le \infty$.

A Banach operator ideal is called *classical* if its operator ideal norm is the usual operator norm. Classical Banach operator ideals are, e.g., the ideal $\mathcal{K}$ of compact operators, $\mathcal{W}$, $\mathcal{L}$, the ideal $\mathcal{RN}$ of Radon–Nikodým operators, the ideal $\mathcal{U}$ of unconditionally summing operators, the ideal $\mathcal{V}$ of completely continuous operators. Here we have the chains

$$\mathcal{K} \subset \mathcal{W} \subset \mathcal{RN} \subset \mathcal{U} \subset \mathcal{L},$$
$$\mathcal{K} \subset \mathcal{W} \subset \mathcal{RN}^{\mathrm{dual}} \subset \mathcal{U}^{\mathrm{dual}} \subset \mathcal{L},$$
$$\mathcal{K} \subset \mathcal{V} \subset \mathcal{U}.$$

By [1] and [3], the MAP for any classical Banach operator ideal between $\mathcal{K}$ and $\mathcal{RN}^{\mathrm{dual}}$ equals the weak MAP.

**Problem 5.3.** *Describe the MAP for $\mathcal{RN}$, $\mathcal{V}$, $\mathcal{U}$, $\mathcal{V}^{\mathrm{dual}}$, $\mathcal{U}^{\mathrm{dual}}$.*

Finally, it is essential to stress that it is an open *problem* whether the weak MAP is strictly weaker than the MAP. Since the AP and the weak MAP are equivalent for dual Banach spaces (see [2]), the latter problem is intimately related to the long-standing famous open *AP-implies-MAP problem* whether the AP of a dual Banach space implies the MAP. For an overview around the AP-implies-MAP problem, see [4]; see also the recent survey [5] for results and references concerning Problems 5.1–5.3.

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## 6. Separable complementation property (A. M. Plichko).

*Cracow University of Technology, Poland*, aplichko@usk.pk.edu.pl

Let $X$ be a Banach space. We say that $X$ has the *separable complementation property* (SCP) if for every separable subspace $E$ of $X$ there is a separable complemented subspace $E \subset F \subset X$. The *density character* of $X$ is the minimal cardinality $\operatorname{dens} X$ of its dense subsets. Let $\mathfrak{m}$ be an arbitrary cardinal. We say that $X$ has the $\mathfrak{m}$-*complementation property* ($\mathfrak{m}$-CP) if for every subspace $E$ of $X$ with $\operatorname{dens} E = \mathfrak{m}$ there is a complemented subspace $E \subset F \subset X$ with $\operatorname{dens} F = \mathfrak{m}$.

**Problem 6.1.** *Does the SCP imply the $\mathfrak{m}$-CP?*

For SCP and $\mathfrak{m}$-CP see [1].

1. A.M. Plichko, D. Yost, *Complemented and uncomplemented subspaces of Banach spaces*, Extracta Math., **15** (2000), 335–371.

## 7. A covariant version for Stinespring's type construction (M. Pliev).

*Southern Mathematical Institute of Vladikavkaz*
*Science Center of the Russian Academy of Sciences*
*(Vladikavkaz, Russia)*, plimarat@yandex.ru

Stinespring's representation theorem is a fundamental theorem in the theory of completely positive maps. The study of completely positive maps is motivated by applications of the theory of completely positive maps to quantum information theory, where operator valued completely positive maps on $C^\star$-algebras are used as a mathematical model for quantum operations, and quantum probability. A completely positive map $\varphi\colon A \to B$ of $C^\star$-algebras is a linear map with the property that $[\varphi(a_{ij})]_{i,j=1}^n$ is a positive element in the $C^\star$-algebra $M_n(B)$ of all $n \times n$ matrices with entries in $B$ for all positive matrices $[(a_{ij})]_{i,j=1}^n$ in $M_n(A)$, $n \in \mathbb{N}$. Stinespring has shown that a completely positive map $\varphi\colon A \to L(H)$ is of the form $\varphi(\cdot) = S^\star\pi(\cdot)S$, where $\pi$ is a $\star$-representation of $A$ on a Hilbert space $K$ and $S$ is a bounded linear operator from $H$ to $K$. Hilbert $C^\star$-modules are generalizations of Hilbert spaces and $C^\star$-algebras. In [1] Asadi had considered a version of the Stinespring theorem for completely positive maps on Hilbert $C^\star$-modules. Later Joita in [3, 4] had proved a covariant version of the Stinespring theorem and Radon-Nikodym's theorem. In [2] we had proved a version of the Stinespring theorem for completely $n$-positive map on Hilbert $C^\star$-modules.

**Definition 7.1.** A map $\Phi = (\Phi_1, \dots, \Phi_n)\colon V_n \to L(H_1^n, H_2^n)$ is called *completely $n$-positive* if there exists a completely $n$-positive map $[\varphi]\colon A \to L(H_1)$ such that

$$[\langle \Phi_i(x_i), \Phi_j(y_j)\rangle]_{i,j=1}^n = [\varphi_{ij}\langle x_i, y_j\rangle]_{i,j=1}^n$$

for every $(x_1, \dots, x_n), (y, \dots, y_n) \in V^n$.

**Theorem 7.2.** *Let $A$ be a unital $C^\star$-algebra, $V$ a Hilbert $A$-module, $\Phi = (\Phi_1, \dots, \Phi_n)\colon V^n \to L(H_1^n, H_2^n)$ an $n$-completely positive map on $V$ and $[\varphi_{ij}]_{i,j=1}^n\colon A \to L(H_1)$ an $n$-completely positive map associated with $\Phi$. Then there exists a data $(\pi, S_1, \dots, S_n, K_1)$, $(\Psi, R_1, \dots, R_n, K_2)$, where*

*(1) $K_1$ and $K_2$ are Hilbert spaces;*

*(2) $\pi\colon A \to L(K_1)$ is a unital $\star$-homomorphism, $\Psi\colon V \to L(K_1, K_2)$ is $\pi$-morphism, $S_i\colon H_1 \to K_1$, $W_i\colon H_2 \to K_2$ are bounded linear operators for every $i \in \{1, \dots, n\}$, such that*

$$\varphi_{ij}(a) = S_i^\star \pi_A(a) S_j \text{ for all } a \in A;\ i, j \in \{1, \dots, n\} \text{ and}$$

$$(\Phi_1(x_1), \dots, \Phi_n(x_n)) = \sum_{i=1}^n W_i^\star \Psi(x_1, \dots, x_n) S_i$$

*for all $(x_1, \dots, x_n) \in V^n$.*

**Problem 7.3.** *Prove a covariant version of Theorem 1 in the sense of [3].*

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## 8. Narrow and strictly singular operators (M. Popov, B. Randrianantoanina).

*Chernivtsi National University (Chernivtsi, Ukraine)*, misham.popov@gmail.com,
*Miami University (Oxford, OH, USA)*, randrib@muohio.edu

We use standard notation. Let $\mathcal{L}(X,Y)$ be the Banach space of all linear bounded maps from $X$ to $Y$, $([0,1],\Sigma,\mu)$ the Lebesgue measure space, $\mathbf{1}_A$ the characteristic function of a set $A\in\Sigma$. By a Köthe Banach space on $[0,1]$ we mean a Banach space $E\subseteq L_1$, $\mathbf{1}_{[0,1]}\in E$ and for each $x\in L_1$ and $y\in E$ the condition $|x|\le|y|$ implies $x\in E$ and $\|x\|\le\|y\|$. Let $E$ be a Köthe Banach space on $[0,1]$ and $X$ a Banach space. An operator $T\in\mathcal{L}(E,X)$ is called

- *narrow* if for each $A\in\Sigma$ and $\varepsilon>0$ there is $x\in E$ such that $x^2=\mathbf{1}_A$, $\int_{[0,1]}x\,d\mu=0$ and $\|Tx\|<\varepsilon$;
- *strictly singular* if the restriction $T|_Y$ of $T$ to any infinite dimensional subspace $Y$ of $E$ is not an into-isomorphism;
- *$Z$-strictly singular* if $Z$ is an infinite dimensional Banach space, and the restriction $T|_Y$ of $T$ to any subspace $Y$ of $E$ isomorphic to $Z$ is not an into-isomorphism.

Evidently, every compact operator is strictly singular, and every strictly singular operator is $Z$-strictly singular for every $Z$. It is well known that the converse to the first of the above statements is not true (having the identity $J\colon\ell_p\to\ell_r$ for $1\le p<r<\infty$ strictly singular and noncompact, one can easily construct a similar example of an operator from $L_p$ to $\ell_r$). It is also well known that, if $E$ has an absolutely continuous norm on the unit (that is, $\lim_{\mu(A)\to0}\|\mathbf{1}_A\|=0$) then every compact operator $T\in\mathcal{L}(E,X)$ is narrow ([3, Prop. 2.1]). If, in addition, $E$ is an r.i. space then there exists a narrow projection of $E$ onto a subspace $E_0$ isometrically isomorphic to $E$, which is obviously neither compact, nor strictly singular (moreover, not $Z$-strictly singular for any $Z$ isomorphically embedded to $E$, [3, Cor. 4.16]). There is an interesting problem posed in 1990 by Plichko and Popov ([2]).

**Problem 8.1.** *Is every strictly singular operator narrow?*

The answer is affirmative for operators from $L_1$ to any Banach space $X$. On the other hand there exists a nonnarrow functional $f\in L_\infty^*$ ([3, Ex. 11.46]) which is obviously strictly singular. However, no counterexample is known if the norm of $E$ is absolutely continuous on the unit. Another version of Problem 8.1, which was also posed in [2], expects that an affirmative answer to it could be much stronger.

**Problem 8.2.** *Is every $\ell_2$-strictly singular operator narrow?*

The strongest partial answer to Problem 2 was recently obtained in [1]: for every $1\le p<\infty$ every $\ell_2$-strictly singular operator from $L_p$ to a Banach space with an unconditional basis is narrow. We also would like to point out the following versions of the above problems, in which $E$ is assumed to be a Köthe Banach space on $[0, 1]$ with an absolutely continuous norm on the unit, $X$ a Banach space and $T \in \mathcal{L}(E, X)$.

**Problem 8.3.** *Does $T$ have to be narrow, provided that $T$ is $Z$-strictly singular for an appropriately chosen infinite dimensional subspace $Z$ of $E$?*

**Problem 8.4.** *Does $T$ have to be narrow whenever $T$ is $E$-strictly singular?*

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Editors affiliation:

Politechnika Krakowska im. Tadeusza Kosciuszko
aplichko@pk.edu.pl

Chernivtsi National University
misham.popov@gmail.com

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