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THE SYSTEM M^θ/G/1/m WITH TWO OPERATION MODES AND HYSTERETIC CONTROL OF QUEUE

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We consider the $\mathrm{M}^\theta/\mathrm{G}/1/\mathrm{m}$ queue with two service modes (basic and postthreshold) with the distribution functions of service time $F(x)$ and $\widetilde{F}(x)$ respectively. The postthreshold mode is used in conjunction with blocking of input flow if at the beginning of service of the next customer the number of customers in the system $\xi(t)$ satisfies the condition $\xi(t) > h_2$. Return to the basic mode and stop blocking carried out at the beginning of service of the next customer, if $\xi(t) \le h_1$, where $h_1 \le h_2$. Laplace transforms for distributions of the number of customers in the system during the busy period and for the busy time distribution function are found. The mean duration of the busy time is found, and formulas for the stationary distribution of number of customers in the system and stationary characteristics of queue are obtained. The case of $m = \infty$ is considered.

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Для системы $\mathrm{M}^\theta/\mathrm{G}/1/\mathrm{m}$ применяются два режима обслуживания (основной и послепороговый) с функциями распределения времени обслуживания $F(x)$ и $\widetilde{F}(x)$ соответственно. Послепороговый режим сопровождается блокировкой входного потока и начинает функционировать, если в момент t начала обслуживания очередной заявки число заявок в системе $\xi(t)$ удовлетворяет условию $\xi(t) > h_2$. Возвращение к основному режиму и прекращение блокировки осуществляется в момент начала обслуживания той заявки, для которой $\xi(t) \le h_1$, где $h_1 \le h_2$. Найдены значения преобразования Лапласа для распределения числа заявок в системе в течение периода занятости и для функции распределения периода занятости, определена средняя продолжительность периода занятости, получены формулы для стационарного распределения числа заявок в системе и стационарных характеристик. Рассмотрен случай $m = \infty$.

1. Introduction. In application of queueing systems, the following two problems are of great importance: a) we want the queue to be not too long; b) it is desirable that the total idle time of the server is as short as possible. These requirements are in opposition to each other. This situation is readily clarified in the following example. Let us consider a standard M/G/1-type queueing system and let ρ be its traffic intensity. It is well known that if $0 < \rho < 1$ then the mean value of the queue length is finite, but it increases towards infinity as $\rho \uparrow 1$. On the other hand, the mean value of the total idle time has a tendency to increase as $\rho \downarrow 0$. So, problem a) requires ρ to be close to 0, while problem b) requires ρ to be close to 1. We

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propose to satisfy both these requirements using a special type of queuing system with an oscillating stochastic process.

Many schemes of control the processes of transmission information in telecommunication networks are built on the idea of oscillating stochastic process, first studied by J. H. B. Kemperman ([1]). In these schemes the idea of threshold value of queue length is used to control of the service speed and intensity of the input flow (see [2–6]). For example, D. Choi, C. Knessl and C. Tier ([2]) consider the queue with one server and with Poisson input flow. If, at the moment of customer service start, the queue length is less than the threshold value of h (respectively, greater than or equal to h), then the service time distribution function of this customer is F (respectively $\widetilde{F}$). Generalization of this approach is the strategy of switching hysteresis modes with two threshold values of $h_1 \le h_2$, which is used, in particular, in the papers of A. Dudin ([7]) and M. Bratiichuk ([8]) to $\mathrm{M}^\theta/\mathrm{G}/1$ and G/M/1 queues, respectively. In the first paper, there are two modes of operation with different intensities of input flow and speed of service, and in the second paper there are two modes of operation with different intensities of speed service. The stationary distribution of the number of customers is found in [7] by using the method of an embedded Markov chain. This stationary distribution corresponds to the completion of service of customers and for $\mathrm{M}^\theta/\mathrm{G}/1/\mathrm{m}$ system with limited queue, in general, does not coincide with a stationary distribution for arbitrary moments of time ($t \to \infty$).

In connection with rapid development of INTERNET, interest to study of queues with finite buffer (limited queue) increases recent years ([9]). Unlike [7], we study the $\mathrm{M}^\theta/\mathrm{G}/1/\mathrm{m}$ system with limited queue and with blocking of input flow during use postthreshold mode of service. Systems with blocking of input flow were studied, in particular, by A. Bratiichuk ([10]) and K. Zhernovyi ([11–14]). Comparing to [7], here we are going to use another approach which was proposed by V. Korolyuk ([15]) in studying boundary problems for compound Poisson processes and which he called the potential method. This approach allows one to obtain an efficient algorithm for finding stationary distribution of number of customers and study not only stationary, but the transition mode of the $\mathrm{M}^\theta/\mathrm{G}/1/\mathrm{m}$ queue.

2. Description of the model. Let us consider a finite-capacity *queueing system* (QS) $\mathrm{M}^\theta/\mathrm{G}/1/\mathrm{m}$, in which arrivals can occur in a group. Such a system can be formally defined by sequences of the random variables $\{\alpha_n\}$, $\{\theta_n\}$, $\{\beta_n\}$ or $\{\widetilde{\beta}_n\}$ ($n \ge 1$), which represent the interarrival times of the groups of customers, the size of the n-th group and the service time of the n-th customer respectively. All these random variables are supposed to be totally independent and

$$\mathbf{P}\{\alpha_n < x\} = 1 - e^{-\lambda x}\ (\lambda > 0),\ \mathbf{P}\{\theta_n = i\} = a_i\ (i \ge 1),\ \sum\nolimits_{i=1}^{\infty} a_i = 1,$$
$$\mathbf{P}\{\beta_n < x\} = F(x)\ (x \ge 0),\ F(0) = 0;\ \mathbf{P}\{\widetilde{\beta}_n < x\} = \widetilde{F}(x)\ (x \ge 0),\ \widetilde{F}(0) = 0.$$

The service discipline is “first come — first served” and the system has only m waiting places, so if the n-th group of customers arrives in the system having already k customers in the line, then only $\min\{\theta_n, m + 1 - k\}$ customers of this group are added to the line, but the others are lost.

Let $\xi(t)$ denote the number of customers in the system at a moment t, and we introduce two thresholds $h_1 \le h_2$. If t is the moment of customer service start and $\xi(t) > h_2$ ($1 \le h_2 \le m - 1$), then the input flow is blocked during service of this customer (customers are not allowed to enter the QS). Admission process of customers is restored at the moment t of customer service start, for which $\xi(t) \le h_1$ ($1 \le h_1 \le m - 1$). Since the moment of the first customer service start during blocking until the completion of the blocking of input, flow service time is distributed by law $\widetilde{F}(x)$.

We denote the described QS as $\mathrm{M}^{\theta}_{\mathrm{h_1,h_2}}/\mathrm{G},\widetilde{\mathrm{G}}/1/\mathrm{m}$. If $h_1 = h_2 = h$, then we obtain the system $\mathrm{M}^{\theta}_{\mathrm{h}}/\mathrm{G},\widetilde{\mathrm{G}}/1/\mathrm{m}$ with a single threshold, which is studied in [13, 14]. The system $\mathrm{M}^{\theta}_{\mathrm{h,m-1}}/\mathrm{G},\widetilde{\mathrm{G}}/1/\mathrm{m}$, for which $h_1 < h_2 = m-1$, is considered in [11]. If $h_1 = h_2 = m$, then we get the system $\mathrm{M}^{\theta}/\mathrm{G}/1/\mathrm{m}$ with limited queue.

3. Distribution of the number of customers in the system on the busy period. Denote by $\mathbf{P}_{F,n}$ ($\mathbf{P}_{\widetilde{F},n}$) the conditional probability, provided that at the initial time the number of customer of QS is $n \geq 0$ ($n \geq h_1 + 1$) and service begins with the service time distributed according to the law $F(x)$ $\big(\widetilde{F}(x)\big)$, and by $\mathbf{E}$ ($\mathbf{P}$) the conditional expectation (conditional probability) if the system starts to work at the time of arrival of the first group of customers.

We introduce the following notation: $\eta(x)$ is the number of customers arriving in the system during the time interval $[0;x)$; a_i^{k*} is the k-convolution of the distribution a_i; $a(s,z) = s + \lambda\big(1-\alpha(z)\big)$; $\rho = \lambda M b$. We put

$$f(s) = \int_0^\infty e^{-sx} dF(x),\ M = \int_0^\infty x dF(x) < \infty,\quad \overline{F}(x) = 1 - F(x),$$
$$\widetilde{f}(s) = \int_0^\infty e^{-sx} d\widetilde{F}(x),\ \widetilde{M} = \int_0^\infty x d\widetilde{F}(x) < \infty,\quad \overline{\widetilde{F}}(x) = 1 - \widetilde{F}(x),$$
$$b = \sum\nolimits_{k=1}^\infty k a_k < \infty,\ \alpha(z) = \sum\nolimits_{k=1}^\infty z^k a_k,\ \overline{a}_n = \sum\nolimits_{k=n}^\infty a_k,$$
$$\overline{p}_n(s) = \sum\nolimits_{k=n}^\infty p_k(s),\ \overline{q}_n(s) = \sum\nolimits_{k=n}^\infty q_k(s).$$

For $\operatorname{Re} s \geq 0$ we define the sequences $p_i(s)$ ($i \in \{-1,0,1,\ldots\}$) and $q_i(s)$ ($i \in \{0,1,\ldots\}$) using the relations

$$p_i(s) = \frac{1}{f(s)} \int_0^\infty e^{-sx} \mathbf{P}\{\eta(x) = i+1\} dF(x) = \frac{1}{f(s)} \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!} dF(x), \quad (1)$$
$$q_i(s) = \int_0^\infty e^{-sx} \mathbf{P}\{\eta(x) = i\} \overline{F}(x) dx = \sum_{k=0}^{i} a_i^{k*} \int_0^\infty e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!} \overline{F}(x) dx. \quad (2)$$

Functions $R_k(s)$ ($k \in \{1,2,3,\ldots\}$) are defined using the equality

$$\sum_{k=1}^\infty z^k R_k(s) = \frac{z}{f\big(a(s,z)\big) - z}, \quad |z| < \nu_-(s), \quad (3)$$

where $\nu_-(s)$ is the unique solution of the equation $f\big(a(s,z)\big) = z$ on the interval $[0,1]$.

The sequence $p_i(s)$ with $s > 0$ can be treated as a distribution of jumps of some down skip free random walk corresponding to the distribution function $F(x)$ of the main mode of service.

Let

$$p_i = \lim_{s\to+0} p_i(s), \quad R_i = \lim_{s\to+0} R_i(s), \quad q_i = \lim_{s\to+0} q_i(s), \quad (4)$$

then from equalities (1)–(4) we obtain

$$p_i = \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^k}{k!} dF(x) \ (i \geq -1), \ q_0 = \frac{1 - f(\lambda)}{\lambda}, q_k = \sum_{i=1}^{k} a_i q_{k-i} - \frac{p_{k-1}}{\lambda} \ (k \geq 1),$$

$$R_1 = \frac{1}{p_{-1}}, \quad R_{k+1} = \frac{R_k - \sum_{i=0}^{k-1} p_i R_{k-i}}{p_{-1}} \quad (k \geq 1). \tag{5}$$

Let $\tau(m) = \inf\{t \geq 0 \colon \xi(t) = 0\}$ denote the first busy period for the system $\mathrm{M}_{\mathrm{h_1,h_2}}^{\theta}/\mathrm{G}$, $\widetilde{\mathrm{G}}/1/\mathrm{m}$, and

$$\varphi_{F,n}(t,k) = \mathbf{P}_{F,n}\{\xi(t) = k, \ \tau(m) > t\} \ (1 \leq n, k \leq m+1),$$
$$\varphi_{\widetilde{F},n}(t,k) = \mathbf{P}_{\widetilde{F},n}\{\xi(t) = k, \tau(m) > t\} \ (h_1 + 1 \leq n \leq m+1, 1 \leq k \leq m+1),$$
$$\varphi_n(t,k) = \begin{cases} \varphi_{F,n}(t,k), & 1 \leq n \leq h_2, \\ \varphi_{\widetilde{F},n}(t,k), & h_2 + 1 \leq n \leq m+1, \end{cases} \quad \Phi_n(s,k) = \int_0^\infty e^{-st} \varphi_n(t,k) dt, \ \mathrm{Re}\, s > 0.$$

It is obvious that $\varphi_0(t,k) = 0$. We introduce the notations

$$L_n(s) = \widetilde{f}^{m-h_1}(s)\overline{p}_{m-n}(s) + \sum\nolimits_{j=h_2+1}^{m-1} p_{j-n}(s)\widetilde{f}^{j-h_1}(s),$$
$$M_n(s,k) = q_{k-n}(s) + I\{k = m+1\}\overline{q}_{m+2-n}(s) + f(s)\Big(I\{h_1 + 1 \leq k \leq m\}\overline{p}_{m-n}(s)\widetilde{f}^{m-k}(s) +$$
$$+ \sum\nolimits_{j=h_2+1}^{m-1} p_{j-n}(s)\widetilde{f}^{j-k}(s) I\{h_1 + 1 \leq k \leq j\}\Big) \frac{1 - \widetilde{f}(s)}{s},$$
$$r_n(s) = R_n(s) - f(s)\sum\nolimits_{i=1}^{n} R_i(s)p_{n-i}(s), \ \Delta_1(s) = 1 + f(s)\sum\nolimits_{i=1}^{h_2-h_1} R_i(s)L_{h_1+i}(s),$$
$$\Delta(s) = \Delta_1(s)r_{h_2}(s) - f(s)r_{h_2-h_1}(s)\sum\nolimits_{i=1}^{h_2} R_i(s)L_i(s).$$

Here, $I\{A\}$ is 1 or 0, depending on whether the event A occurs or not.

Theorem 1. *For the system* $\mathrm{M}_{\mathrm{h_1,h_2}}^{\theta}/\mathrm{G}, \widetilde{\mathrm{G}}/1/\mathrm{m}$ *we have*

$$\Phi_n(s,k) = r_{h_2-n}(s)\Phi_{h_2}(s,k) -$$
$$- f(s)\sum\nolimits_{i=1}^{h_2-n} R_i(s)L_{n+i}(s)\Phi_{h_1}(s,k) - \sum\nolimits_{i=1}^{h_2-n} R_i(s)M_{n+i}(s,k), \ 1 \leq n \leq h_2 - 1, \tag{6}$$
$$\Phi_n(s,k) = \widetilde{f}^{n-h_1}(s)\Phi_{h_1}(s,k) +$$
$$+ I\{h_1 + 1 \leq k \leq n\}\widetilde{f}^{n-k}(s)\frac{1 - \widetilde{f}(s)}{s}, \ h_2 + 1 \leq n \leq m+1, \tag{7}$$

where $1 \leq k \leq m+1$, $\mathrm{Re}\, s > 0$, *and*

$$\Phi_{h_1}(s,k) = \frac{1}{\Delta(s)}\Big(r_{h_2-h_1}(s)\sum\nolimits_{i=1}^{h_2} R_i(s)M_i(s,k) - r_{h_2}(s)\sum\nolimits_{i=1}^{h_2-h_1} R_i(s)M_{h_1+i}(s,k)\Big),$$
$$\Phi_{h_2}(s,k) = \frac{1}{\Delta(s)}\sum\nolimits_{i=1}^{h_2} R_i(s)\Big(\Delta_1(s)M_i(s,k) - f(s)L_i(s)\sum\nolimits_{j=1}^{h_2-h_1} R_j(s)M_{h_1+j}(s,k)\Big). \tag{8}$$

Proof. Using the formula of total probability, we obtain the equalities

$$\varphi_n(t,k)=\sum_{j=0}^{m-n}\int_0^t \mathbf{P}\{\eta(x)=j\}\varphi_{n+j-1}(t-x,k)dF(x)+$$
$$+\int_0^t \mathbf{P}\{\eta(x)\ge m+1-n\}\varphi_m(t-x,k)dF(x)+\big(P\{\eta(t)=k-n\}+$$
$$+I\{k=m+1\}\mathbf{P}\{\eta(t)\ge m+2-n\}\big)\overline{F}(t),\quad 1\le n\le h_2,$$
$$\varphi_n(t,k)=\int_0^t \mathbf{P}\Big\{\sum_{i=1}^{n-h_1}\widetilde{\beta}_i\in dx\Big\}\varphi_{h_1}(t-x,k)+I\{h_1+1\le k\le n-1\}\times$$
$$\times\int_0^t \mathbf{P}\Big\{\sum_{i=1}^{n-k}\widetilde{\beta}_i\in dx\Big\}\overline{\widetilde{F}}(t-x)+I\{k=n\}\overline{\widetilde{F}}(t),\ h_2+1\le n\le m+1.$$

Let us pass to Laplace transforms in these equations. To determine the functions $\Phi_n(s,\,k)$ $(1\le n\le m+1)$, using (1), (2), we obtain the system of equations, which consists of the equations

$$\Phi_n(s,k)=f(s)\sum\nolimits_{j=0}^{m-n}p_{j-1}(s)\Phi_{n+j-1}(s,k)+f(s)\overline{p}_{m-n}(s)\Phi_m(s,k)+$$
$$+q_{k-n}(s)+I\{k=m+1\}\overline{q}_{m+2-n}(s),\ 1\le n\le h_2, \tag{9}$$

the equations (7) and the boundary condition

$$\Phi_0(s,k)=0. \tag{10}$$

Taking from (7) $\Phi_n(s,k)$ for $h_2+1\le n\le m$ and substituting them into relations (9), we obtain the equations

$$\Phi_n(s,k)-f(s)\sum\nolimits_{j=-1}^{h_2-n-1}p_j(s)\Phi_{n+j}(s,k)=$$
$$=f(s)L_n(s)\Phi_{h_1}(s,k)+f(s)p_{h_2-n}(s)\Phi_{h_2}(s,k)+M_n(s,k),\quad 1\le n\le h_2. \tag{11}$$

Writing solutions of equations (11) in the same form as similar in structure the system in [11], we obtain the equalities (6). Putting in (6) first $n=h_1$, and then $n=0$, taking into account boundary condition (10), we obtain a system of two linear equations for $\Phi_{h_1}(s,k)$ and $\Phi_{h_2}(s,k)$. Solutions of this system are determined by formulas (8). □

4. The busy period and stationary distribution for the systems $\mathbf{M^{\theta}_{h_1,h_2}/G,\widetilde{G}/1/m}$ and $\mathbf{M^{\theta}_{h_1,h_2}/G,\widetilde{G}/1}$. If the system $\mathrm{M}^{\theta}_{h_1,h_2}/\mathrm{G},\widetilde{\mathrm{G}}/1/\mathrm{m}$ starts functioning when the first group of customers arrives, then

$$\int_0^\infty e^{-st}\mathbf{P}\{\xi(t)=k,\tau(m)>t\}dt=\sum_{n=1}^m a_n\Phi_n(s,k)+\overline{a}_{m+1}\Phi_{m+1}(s,k). \tag{12}$$

To obtain a representation for $\int_0^\infty e^{-st}\mathbf{P}\{\tau(m)>t\}dt$, we should pass in equality (12) to summation on k from 1 to $m+1$. Denoting $\sum_{k=1}^{m+1} M_n(s,k)$ by $M_n(s)$, we find

$$M_n(s) = \frac{1-f(s)}{s} + f(s)\Big(\frac{1-\widetilde{f}^{m-h_1}(s)}{s}\overline{p}_{m-n}(s) + \sum\nolimits_{j=h_2+1}^{m-1} p_{j-n}(s)\frac{1-\widetilde{f}^{j-h_1}(s)}{s}\Big),$$

$$\sum\nolimits_{k=1}^{m+1} \Phi_{h_1}(s,k) = \frac{D_{h_1}(s)}{\Delta(s)},\ \sum\nolimits_{k=1}^{m+1} \Phi_{h_2}(s,k) = \frac{D_{h_2}(s)}{\Delta(s)},$$

$$D_{h_1}(s) = r_{h_2-h_1}(s)\sum\nolimits_{i=1}^{h_2} R_i(s)M_i(s) - r_{h_2}(s)\sum\nolimits_{i=1}^{h_2-h_1} R_i(s)M_{h_1+i}(s),$$

$$D_{h_2}(s) = \sum\nolimits_{i=1}^{h_2} R_i(s)\Big(\Delta_1(s)M_i(s) - f(s)L_i(s)\sum\nolimits_{j=1}^{h_2-h_1} R_j(s)M_{h_1+j}(s)\Big),$$

and from (12) we obtain

$$\int_0^\infty e^{-st}\mathbf{P}\{\tau(m)>t\}dt = \frac{1}{\Delta(s)}\Big(\sum\nolimits_{n=1}^{h_2} a_n r_{h_2-n}(s)D_{h_2}(s) - D_{h_1}(s)\times$$
$$\times\Big(f(s)\sum\nolimits_{n=1}^{h_2-1} a_n\sum\nolimits_{i=1}^{h_2-n} R_i(s)L_{n+i}(s) - \sum\nolimits_{n=h_2+1}^{m} a_n\widetilde{f}^{n-h_1}(s) - \overline{a}_{m+1}\widetilde{f}^{m+1-h_1}(s)\Big)\Big)-$$
$$-\sum\nolimits_{n=1}^{h_2-1} a_n\sum\nolimits_{i=1}^{h_2-n} R_i(s)M_{n+i}(s) + \sum\nolimits_{n=h_2+1}^{m} a_n\frac{1-\widetilde{f}^{n-h_1}(s)}{s} + \overline{a}_{m+1}\frac{1-\widetilde{f}^{m+1-h_1}(s)}{s}. \quad (13)$$

We have

$$L_n(0) = \overline{p}_{h_2+1-n},\ M_n(0) = M + \widetilde{M}\Big(\sum\nolimits_{j=h_2+1}^{m-1}(j-h_1)p_{j-n} + (m-h_1)\overline{p}_{m-n}\Big).$$

Using equalities (5) and the following equalities from [11]

$$\sum\nolimits_{i=1}^{n} R_i\overline{p}_{n-i} = R_n - 1\ (n \geq 1), \quad (14)$$

we have $r_n(0) = p_{-1}R_{n+1},\ \Delta_1(0) = \Delta(0) = p_{-1}R_{h_2-h_1+1}$. Passing to the limit as $s \to +0$ in equality (13), we obtain an expression for the mean duration of the busy time. So, we have proved the following statement.

Theorem 2. *The mean duration of the busy time for the system* $\mathrm{M}^{\theta}_{\mathrm{h_1,h_2}}/\mathrm{G},\widetilde{\mathrm{G}}/1/\mathrm{m}$ *has the form*

$$\mathbf{E}\tau(m) = MT_0(h_1,h_2) + \widetilde{M}T_1(m,h_1,h_2), \quad (15)$$

where

$$T_0(h_1,h_2) = \sum_{i=1}^{h_2} R_i\overline{a}_{h_2+1-i} - R(h_1,h_2)\sum_{i=1}^{h_2-h_1} R_i,$$

$$T_1(m,h_1,h_2) = \sum_{i=1}^{h_2} R_i\sum_{j=h_2+1}^{m-1}\Big((j-h_1)p_{j-i} + (m-h_1)\overline{p}_{m-i}\Big)-$$
$$-\sum_{n=1}^{h_2-1} a_n\sum_{i=1}^{h_2-n} R_i\sum_{j=h_2+1}^{m-1}\Big((j-h_1)p_{j-n-i} + (m-h_1)\overline{p}_{m-n-i}\Big) + \sum_{n=h_2+1}^{m}(n-h_1)a_n+$$
$$+(m+1-h_1)\overline{a}_{m+1} - R(h_1,h_2)\sum_{i=1}^{h_2-h_1} R_i\sum_{j=h_2+1}^{m-1}\Big((j-h_1)p_{j-h_1-i} + (m-h_1)\overline{p}_{m-h_1-i}\Big), \quad (16)$$

$$R(h_1,h_2) = \frac{1}{R_{h_2-h_1+1}}\Big(R_{h_2+1} - \sum_{n=1}^{h_2} a_n R_{h_2+1-n}\Big).$$

Putting in (12) $s=0$, we obtain

$$\int_0^\infty \mathbf{P}\{\xi(t)=k,\tau(m)>t\}dt=\sum\nolimits_{i=1}^{h_2} R_iM_i(k)-\sum\nolimits_{n=1}^{h_2-1} a_n\sum\nolimits_{i=1}^{h_2-n} R_iM_{n+i}(k)+$$
$$+\widetilde{M}\Big(\sum\nolimits_{n=h_2+1}^{m} a_nI\{h_1+1\le k\le n\}+\overline{a}_{m+1}I\{h_1+1\le k\le m+1\}\Big)-$$
$$-R(h_1,h_2)\sum\nolimits_{i=1}^{h_2-h_1} R_iM_{h_1+i}(k), \qquad (17)$$

where

$$M_n(k)=M_n(0,k)=q_{k-n}+I\{k=m+1\}\overline{q}_{m+2-n}+$$
$$+\widetilde{M}\Big(\sum\nolimits_{j=h_2+1}^{m-1} I\{h_1+1\le k\le j\}p_{j-n}+I\{h_1+1\le k\le m\}\overline{p}_{m-n}\Big).$$

Let $\rho_k(m)=\lim_{t\to\infty}\mathbf{P}\{\xi(t)=k\}$ be the stationary probability of the presence of k customers in the system $\mathrm{M}^{\theta}_{\mathrm{h_1,h_2}}/\mathrm{G},\widetilde{\mathrm{G}}/1/\mathrm{m}$ and $\pi_k(m)=\rho_k(m)/\big(\lambda\rho_0(m)\big)$. In the notation of similar characteristics of the system with unlimited queue $\mathrm{M}^{\theta}_{\mathrm{h_1,h_2}}/\mathrm{G},\widetilde{\mathrm{G}}/1$, we replace m with ∞. From (17), reasoning as in the proof of Theorem 2 ([12]), we obtain the following statement.

Theorem 3. *For the stationary distribution of the number of customers in the system* $\mathrm{M}^{\theta}_{\mathrm{h_1,h_2}}/\mathrm{G},\widetilde{\mathrm{G}}/1/\mathrm{m}$, *the following representations are valid*

$$\rho_0(m)=\frac{1}{1+\lambda\mathbf{E}\tau(m)},\quad \pi_k(m)=\sum_{i=1}^{k}R_iq_{k-i}-\sum_{n=1}^{k-1}a_n\sum_{i=1}^{k-n}R_iq_{k-n-i}\ (k\in\{1,\dots,h_1\}),$$

$$\pi_k(m)=\sum_{i=1}^{k}R_iq_{k-i}-\sum_{n=1}^{k-1}a_n\sum_{i=1}^{k-n}R_iq_{k-n-i}-$$
$$-R(h_1,h_2)\Big(\sum_{i=1}^{k-h_1}R_iq_{k-h_1-i}-\widetilde{M}\Big)\ (k\in\{h_1+1,\dots,h_2\}),$$

$$\pi_k(m)=\sum_{i=1}^{h_2}R_i\big(q_{k-i}+\widetilde{M}\overline{p}_{k-i}\big)-\sum_{n=1}^{h_2-1}a_n\sum_{i=1}^{h_2-n}R_i\big(q_{k-n-i}+\widetilde{M}\overline{p}_{k-n-i}\big)+$$
$$+\widetilde{M}\overline{a}_k-R(h_1,h_2)\sum_{i=1}^{h_2-h_1}R_i\big(q_{k-h_1-i}+\widetilde{M}\overline{p}_{k-h_1-i}\big)\ (k=\in\{h_2+1,\dots,m\}), \qquad (18)$$

$$\pi_{m+1}(m)=\sum_{i=1}^{h_2}R_i\overline{q}_{m+1-i}-\sum_{n=1}^{h_2-1}a_n\sum_{i=1}^{h_2-n}R_i\overline{q}_{m+1-n-i}+\widetilde{M}\overline{a}_{m+1}-R(h_1,h_2)\sum_{i=1}^{h_2-h_1}R_i\overline{q}_{m+1-h_1-i}.$$

According to Lemma 1 ([13]), for sequences $\{p_i\}$, $\{R_i\}$ the following equalities are performed

$$\sum\nolimits_{j=0}^{\infty}\overline{p}_j=\rho,\quad \sum\nolimits_{i=1}^{k}R_i\sum\nolimits_{j=0}^{k-i}\overline{p}_j=\sum\nolimits_{i=1}^{k}R_i-k. \qquad (19)$$

Theorem 4. *For the mean duration of the busy time and the stationary distribution of number of customers in the system* $\mathrm{M}^{\theta}_{\mathrm{h_1,h_2}}/\mathrm{G},\widetilde{\mathrm{G}}/1$, *the following representations are valid*

$$\mathbf{E}\tau(\infty)=MT_0(h_1,h_2)+\widetilde{M}\Big(b+(\rho-1)T_0(h_1,h_2)\Big), \qquad (20)$$

$$\rho_0(\infty)=\frac{1}{1+\lambda\mathbf{E}\tau(\infty)},\ \pi_k(\infty)=\pi_k(m)\ (k\ge 1), \qquad (21)$$

where $\pi_k(\infty) = \rho_k(\infty)/\big(\lambda\rho_0(\infty)\big)$, *and* $T_0(h_1, h_2)$, $\pi_k(m)$ *are defined by formulas (16), (18).*

Proof. If $m \uparrow \infty$ then such parameters of the QS as a state of the process $\xi(t)$ at a fixed time t, busy time $\tau(m)$ and stationary probabilities $\rho_k(m)$ have the property of stochastic monotonicity ([16, p.116]). Putting $m \to \infty$ in (15) and using the monotone convergence theorem ([17, p.405]), we obtain

$$\lim_{m\to\infty} \mathbf{E}\tau(m) = \mathbf{E}\tau(\infty) = MT_0(h_1, h_2) + \widetilde{M}T_1(\infty, h_1, h_2), \tag{22}$$

where

$$T_1(\infty, h_1, h_2) = \sum_{i=1}^{h_2} R_i \sum_{j=h_2+1}^{\infty} (j-h_1)p_{j-i} - \sum_{n=1}^{h_2-1} a_n \sum_{i=1}^{h_2-n} R_i \sum_{j=h_2+1}^{\infty} (j-h_1)p_{j-n-i} +$$
$$+ \sum_{n=h_2+1}^{\infty} (n-h_1)a_n - R(h_1,h_2) \sum_{i=1}^{h_2-h_1} R_i \sum_{j=h_2+1}^{\infty} (j-h_1)p_{j-h_1-i}.$$

Taking into account relations (19), using simple transformations we obtain

$$\sum_{j=h_2+1}^{\infty} (j-h_1)p_{j-i} = (h_2-h_1+1)\overline{p}_{h_2+1-i} + \sum_{j=h_2+2-i}^{\infty} \overline{p}_j = (h_2-h_1)\overline{p}_{h_2+1-i} + \rho - \sum_{j=0}^{h_2-i} \overline{p}_j,$$

$$\sum_{i=1}^{h_2} R_i \sum_{j=h_2+1}^{\infty} (j-h_1)p_{j-i} = \sum_{i=1}^{h_2} R_i\Big(\rho - \sum_{j=0}^{h_2-i} \overline{p}_j + (h_2-h_1)(\overline{p}_{h_2-i} - p_{h_2-i})\Big) =$$
$$= h_2 + (h_2-h_1)(R_{h_2}-1) - \sum_{i=1}^{h_2} R_i + \sum_{i=1}^{h_2} R_i\Big(\rho - (h_2-h_1)p_{h_2-i}\Big) =$$
$$= (\rho-1)\sum_{i=1}^{h_2} R_i + h_1 + (h_2-h_1)R_{h_2+1}p_{-1}. \tag{23}$$

By transforming the expressions

$$\sum\nolimits_{i=1}^{h_2-n} R_i \sum\nolimits_{j=h_2+1}^{\infty} (j-h_1)p_{j-n-i}, \quad \sum\nolimits_{i=1}^{h_2-h_1} R_i \sum\nolimits_{j=h_2+1}^{\infty} (j-h_1)p_{j-h_1-i},$$

as it was done in (23), taking into account the equality

$$\sum\nolimits_{n=h_2+1}^{\infty} (n-h_1)a_n = b - \sum\nolimits_{n=1}^{h_2} na_n - h_1\overline{a}_{h_2+1},$$

from (22) we obtain equality (20). Putting $m \to \infty$ in equalities (18) and using again the monotone convergence theorem, we get (21). □

5. Determination of stationary characteristics. Relying on ergodicity property of the process $\xi(t)$, we obtain the formula of the probability of service $\mathbf{P}_{\mathrm{sv}}(m)$ for the system $\mathrm{M}^{\theta}_{\mathrm{h_1,h_2}}/\mathrm{G},\widetilde{\mathrm{G}}/1/\mathrm{m}$ as the ratio of the mean number of served customers to the mean number of arrived per unit time. The mean number of arrived customers is λb, and the mean number of served customers during the same time is $\big(1-\rho_0(m)\big)/\overline{M}$, where $\overline{M}$ is a mean service time of one customer, which we find by the formula

$$\frac{1}{\overline{M}} = \frac{\mathbf{E}\tau_0(m)}{M\mathbf{E}\tau(m)} + \frac{\mathbf{E}\widetilde{\tau}(m)}{\widetilde{M}\mathbf{E}\tau(m)}.$$

Here $\mathbf{E}\tau_0(m)$ and $\mathbf{E}\widetilde{\tau}(m)$ are mean durations of parts of the busy time corresponding of basic and postthreshold mode of service, respectively. As a result, we get the formula for the probability of service

$$\mathbf{P}_{\text{sv}}(m) = \frac{T_0(h_1,h_2) + T_1(m,h_1,h_2)}{b\big(1+\lambda\mathbf{E}\tau(m)\big)}. \tag{24}$$

For the system $\mathrm{M}^{\theta}_{\mathrm{h_1,h_2}}/\mathrm{G},\widetilde{\mathrm{G}}/1$ equality (24) takes the form

$$\mathbf{P}_{\text{sv}}(\infty) = \frac{b + \rho T_0(h_1,h_2)}{b\big(1+\lambda\mathbf{E}\tau(\infty)\big)}.$$

Stationary characteristics of queue of the system $\mathrm{M}^{\theta}_{\mathrm{h_1,h_2}}/\mathrm{G},\widetilde{\mathrm{G}}/1/\mathrm{m}$ such as the mean queue length $\mathbf{E}Q(m)$ and the mean waiting time $\mathbf{E}w(m)$, we find by the formulas

$$\mathbf{E}Q(m) = \sum\nolimits_{k=1}^{m} k\rho_{k+1}(m), \quad \mathbf{E}w(m) = \frac{\mathbf{E}Q(m)}{\lambda b\mathbf{P}_{\text{sv}}(m)}.$$

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