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SUBHARMONIC FUNCTIONS OF FINITE (γ, ε)-TYPE IN A HALF-PLANE

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We obtain criterions for δ-subharmonic function to belong to the class of functions of finite (γ, ε)-type in a half-plane. These criterions are formulated in terms of Fourier coefficients of a function.

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Получены критерии принадлежности δ-субгармонической функции классу функций конечного (γ, ε)-типа в полуплоскости. Эти критерии формулируются в терминах коэффициентов Фурье функции.

1. Introduction. In this paper we use the Fourier series method for the study of properties subharmonic functions. This method was introduced by L. A. Rubel and B. A. Taylor ([1]). Further the Fourier series method was used by J. B. Miles ([2]), D. F. Shea, A. A. Kondratyuk ([3]–[5]) and others.

We call a strictly positive continuous unbounded increasing function $\gamma(r)$ on $[0, \infty)$ a *growth function*. Let f be a meromorphic function in the complex plane, let $Z(f)$ $(W(f))$ be the set of its zeros (poles), $T(r, f)$ its Nevanlinna characteristic. A function f is called a *function of finite* γ*-type* if there exist positive constants A and B such that $T(r, f) \le A\gamma(Br)$ for all $r > 0$. We denote the class of such functions by Γ, and we denote by Γ_E the class of entire functions of finite γ-type. Below we use letters A, B, … to denote positive constants, not necessarily the same throughout the paper. Let

$$c_k(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| e^{-ik\theta}\, d\theta,\ k \in \mathbb{Z}\,,$$

be the Fourier coefficients of f. L. A. Rubel and B. A. Taylor ([1]) proved the equivalence of the following three properties:

(1) $f \in \Gamma$;

(2) the sequence $Z(f)$ (or the sequence $W(f)$) has finite γ-density and $|c_k(r, f)| \le A\gamma(Br)$, $k \in \mathbb{Z}$, for some positive A, B, and all $r > 0$;

(3) the sequences $Z(f)$ and $W(f)$ have finite γ-densities and

$$|c_k(r, f)| \le \frac{A}{|k| + 1}\gamma(Br),\ k \in \mathbb{Z},$$

for some (not necessarily the same, but independent of k) positive constants A, B, and all $r > 0$.

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Using the Fourier series method, J. B. Miles ([2]) solved the problem (which has been remained unsolved for several years) of the representation of a meromorphic function $f \in \Gamma$ as a ratio of two entire functions from the class Γ_E: $\Gamma = \Gamma_E/\Gamma_E$. Important results in these directions were obtained in 80-th by A. A. Kondratyuk ([3]–[5]), who generalized the Levin-Pflüger theory of entire functions of completely regular growth to meromorphic functions of arbitrary γ-type.

The method of Fourier coefficients of δ-subharmonic functions in the plane was developed further by P. Noverraz ([6]), Ya. V. Vasyl'kiv, K. G. Malyutin ([7]). These authors extended of the results mentioned above to δ-subharmonic functions in the complex plane. At the beginning of the 21th century, the first-named author of this paper extended the results of L. A. Rubel, B. A. Taylor, J. B. Miles to δ-subharmonic functions in a half-plane ([8]).

In 2003, B. N. Khabibullin ([9]) introduced the classes of meromorphic in $\mathbb{C}$ functions of finite (γ,ε)-type which are a generalizations of the class Γ. Yu. S. Protsyk ([10]) extended the results of L. A. Rubel, B. A. Taylor, J. B. Miles to the classes of subharmonic functions of finite (γ,ε)-type in $\mathbb{R}^n$ $(n \geq 2)$.

In the present paper we extend some of the mentioned results to functions of finite (γ,ε)-type in a half-plane. First, we prove our results for δ-subharmonic functions. The passage to a half-plane leads to complications that are due to the complicated behavior of a δ-subharmonic function near the boundary. The difference from the plane case already seen in obtaining tests for a δ-subharmonic function to belong to a fixed class. For instance, no generalization or analogue of property (3) in Rubel–Taylor criterion is possible in a half-plane.

2. Main result. Let $J\delta$ be the class of proper δ-subharmonic functions, and $J\delta((\gamma,\varepsilon))$ be the class of proper δ-subharmonic functions of finite (γ,ε)-type in the upper half-plane (we present definitions of these classes below). For $v \in J\delta$ we set

$$c_k(\theta,r,v) = \frac{2\sin k\theta}{\pi}\int_0^{\pi} v(re^{i\varphi})\sin k\varphi d\varphi, \quad \theta \in [0,\pi],\ k \in \mathbb{N}.$$

The functions $c_k(\theta,r,v)$ are called the *spherical harmonics associated with a subharmonic function* v.

The main result of this paper is the following theorem.

Theorem 1. *Let γ be a growth function, ε be a function of the class $\mathcal{E}$, and let $v \in J\delta$. Then the following two properties are equivalent*:

(1) $v \in J\delta((\gamma,\varepsilon))$;

(2) *the measure $\lambda_+(v)$ (or $\lambda_-(v)$) has finite (γ,ε)-density, and*

$$|c_k(\theta,r,v)| \leq \frac{A\gamma(r+B\varepsilon(r)r)}{(\varepsilon(r))^{\alpha}}, \quad k \in \mathbb{N}, \tag{1}$$

for some positive α, A, B and all $r > 0$.

Here $\lambda(v)=\lambda_+(v)-\lambda_-(v)$ is the complete measure corresponding to the function v.

3. Classes of functions in $\mathbb{C}_+$. In this paper we use terminology from [8], [11]. Let $\mathbb{C}_+ = \{z\colon\ \operatorname{Im} z > 0\}$ be the upper half-plane. We denote by $C(a,r)$ the open disc of radius r with center at a, and by Ω_+ the intersection of a set Ω with the half-plane $\mathbb{C}_+\colon \Omega_+ = \Omega \cap \mathbb{C}_+$. A subharmonic function v in $\mathbb{C}_+$ is said to be *proper subharmonic* if $\limsup_{z\to t} v(z) \leq 0$ for each $t \in \mathbb{R}$. The class of proper subharmonic functions in $\mathbb{C}_+$ will be denoted by JS. Let SK be the class of subharmonic functions in $\mathbb{C}_+$ possessing a positive harmonic majorant in each bounded subdomain of $\mathbb{C}_+$. Functions from SK have the following properties ([11]).

(a) $v(z)$ has non-tangential limits $v(t)$ almost everywhere on the real axis and $v(t) \in L^1_{loc}(-\infty, \infty)$;

(b) there exists a signed measure ν on the real axis such that

$$\lim_{y\to+0}\int_a^b v(t+iy)dt = \nu([a,b]) - \frac{1}{2}\nu(\{a\}) - \frac{1}{2}\nu(\{b\}).$$

The measure ν is called the *boundary measure* of v;

(c) $d\nu(t) = v(t)dt + d\sigma(t)$, where σ is a singular measure with respect to the Lebesgue measure.

For a function $v \in SK$, following [11] we define the corresponding complete measure λ by the formula

$$\lambda(K) = 2\pi \int_{\mathbf{C}_+\cap K} \operatorname{Im}\zeta d\mu(\zeta) - \nu(K),$$

where μ is the Riesz measure of v. The measure λ has the following properties:

(1) λ is a finite measure on each compact subset K of $\mathbb{C}$;

(2) λ is a positive measure outside $\mathbb{R}$;

(3) λ vanishes in the half-plane $\mathbb{C}_- = \{z\colon \operatorname{Im} z < 0\}$.

Conversely, if λ is a measure with properties (1)–(3), then there exists a function $v \in SK$ with complete measure λ. The collection of properties (1)–(3) will be denoted by $\{\mathbf{G}\}$ in what follows; if, in addition, λ is a non-negative measure on $\mathbb{R}$ then we denote the corresponding collection by $\{\mathbf{G}^+\}$.

The complete measure of a function $v \in JS$ is a positive measure, which explains the term "proper subharmonic function".

Let us now introduce the class of proper δ-subharmonic functions $J\delta = JS - JS$. Note that $J\delta$ is the broadest class of δ-subharmonic functions in the half-plane for which one can define the Nevanlinna characteristic. In fact, all Nevanlinna characteristics of a δ-subharmonic functions are defined in terms of the corresponding measure. For instance, if a function is δ-subharmonic in the entire plane, then one should consider its Riesz measure, while if it is in the class $J\delta$, then one should consider its complete measure (whose definition takes into account the boundary measure ν). The boundary behavior of an arbitrary function outside the class $J\delta$ is determined by some generalized function on the real axis that is not a measure.

Note that $J\delta = SK - SK$ ([11]).

For a fixed measure λ let

$$d\lambda_m(\zeta) = \frac{\sin m\varphi}{\sin\varphi}\tau^{m-1}d\lambda(\zeta)\ (\zeta = \tau e^{i\varphi}), \quad \lambda_m(r) = \lambda_m\left(\overline{C(0,r)}\right),$$

where $\frac{\sin m\varphi}{\sin\varphi}$ is defined for $\varphi = 0, \pi$ by continuity.

The next relation is Carleman's formula in Grishin's notation:

$$\frac{1}{r^k}\int_0^\pi v\left(re^{i\varphi}\right)\sin k\varphi d\varphi = \int_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}}dt + \frac{1}{r_0^k}\int_0^\pi v(r_0e^{i\varphi})\sin k\varphi d\varphi; \tag{2}$$

in particular, for $k=1$ we have

$$\frac{1}{r}\int_0^\pi v(re^{i\varphi})\sin\varphi d\varphi=\int_{r_0}^r\frac{\lambda(t)}{t^3}dt+\frac{1}{r_0}\int_0^\pi v(r_0e^{i\varphi})\sin\varphi d\varphi \tag{3}$$

for all $r>r_0$.

We provide also another inequality, which is useful in what follows:

$$\left|\lambda_m(r)\right|=\left|\iint_{\overline{C(0,r)}}d\lambda_m(\zeta)\right|=\left|\iint_{\overline{C(0,r)}}\frac{\sin m\varphi}{\sin\varphi}\tau^{m-1}d\lambda(\zeta)\right|\le$$

$$\le m\iint_{\overline{C(0,r)}}\tau^{m-1}d|\lambda|(\zeta)\le mr^{m-1}|\lambda|(r). \tag{4}$$

Functions $v\in J\delta$ have representation in the half-disc $C_+(0,R)$

$$v(z)=-\frac{1}{2\pi}\iint_{\overline{C_+(0,R)}}K(z,\zeta)d\lambda(\zeta)+\frac{R}{2\pi}\int_0^\pi\frac{\partial G(z,Re^{i\varphi})}{\partial n}v(Re^{i\varphi})d\varphi, \tag{5}$$

where $G(z,\zeta)$ is the Green function of the the half-disc, $\frac{\partial G}{\partial n}$ is its derivative in the inward normal direction, and the function $K(z,\zeta)=\frac{1}{\operatorname{Im}\zeta}G(z,\zeta)$ is extended by continuity to the points on the real axis.

Using the theory of elliptic functions (see, for instance, [12], Chapter VIII) one can obtain expansions of the kernel in formulae (5), $z=re^{i\theta}$ and $\zeta=\tau e^{i\varphi}$:

$$G(z,Re^{i\varphi})=\begin{cases}2\sum\limits_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{\tau}\right)^m\left(1-\frac{\tau^{2m}}{R^{2m}}\right)\sin m\theta\sin m\varphi, & 0\le r<\tau\le R,\\ 2\sum\limits_{m=1}^{\infty}\frac{1}{m}\left(\frac{\tau}{r}\right)^m\left(1-\frac{r^{2m}}{R^{2m}}\right)\sin m\theta\sin m\varphi, & 0\le \tau<r\le R.\end{cases} \tag{6}$$

$$\frac{\partial G(z,Re^{i\varphi})}{\partial n}=4\sum_{m=1}^{\infty}\frac{r^m}{R^{m+1}}\sin m\theta\sin m\varphi. \tag{7}$$

4. Fourier coefficients of functions of class $J\delta$. The *Fourier coefficients of a function* $v\in J\delta$ are defined as usual ([7])

$$c_k(r,v)=\frac{2}{\pi}\int_0^\pi v(re^{i\theta})\sin k\theta d\theta,\quad k\in\mathbb{N}.$$

Then $c_k(\theta,r,v)=\sin\theta c_k(r,v)$. From (2) we obtain the following representations for the spherical harmonics for $r>r_0$

$$c_k(\theta,r,v)=\sin\theta\alpha_kr^k+\frac{2r^k\sin\theta}{\pi}\int_{r_0}^r\frac{\lambda_k(t)}{t^{2k+1}}dt,\quad k\in\mathbb{N}, \tag{8}$$

where $\alpha_k=r_0^{-k}c_k(r_0,v)$ (here r_0 is a fixed number, for example $r_0=1$).

Applying the formula of integration by parts to the integral in (8) we obtain

$$c_k(\theta,r,v)=\sin\theta\alpha_kr^k+\frac{r^k\sin\theta}{\pi kr_0^{2k}}\iint_{\overline{C_+(0,r_0)}}\frac{\sin k\varphi}{\operatorname{Im}\zeta}\tau^kd\lambda(\zeta)+$$

$$+\frac{r^k \sin\theta}{\pi k}\iint_{D_+(r_0,r)}\frac{\sin k\varphi}{\tau^k \operatorname{Im}\zeta}d\lambda(\zeta)-\frac{\sin\theta}{r^k\pi k}\iint_{\overline{C_+(0,r)}}\frac{\sin k\varphi}{\operatorname{Im}\zeta}\tau^k d\lambda(\zeta), \tag{9}$$

where $\zeta=\tau e^{i\varphi}$.

5. Subharmonic and δ-subharmonic functions of finite (γ,ε)-type. For $v\in J\delta$ let $v=v_+-v_-$, let λ be the complete measure of v and let $\lambda=\lambda_+-\lambda_-$ be the Jordan decomposition of λ.

We set

$$m(r,v):=\frac{1}{r}\int_0^\pi v_+(re^{i\varphi})\sin\varphi d\varphi,\ N(r,v):=\int_{r_0}^r\frac{\lambda_-(t)}{t^3}dt,\ T(r,v):=m(r,v)+N(r,v)+m(r_0,-v),$$

where r_0 is an arbitrary positive number and $r>r_0$; one can take $r_0=1$.

In this notation Carleman's formula (3) can be written as follows

$$T(r,v)=T(r,-v). \tag{10}$$

We assume that the growth function γ satisfies the condition

$$\liminf_{r\to\infty}\frac{\gamma(r)}{r}>0. \tag{11}$$

Let γ be the growth function. Let $\varepsilon(r)$ be a non-increasing function on $[0;+\infty]$ such that $\varepsilon(0)=1$ and the inequality

$$\varepsilon(r+r\varepsilon(r))\geq(\varepsilon(r))^\eta \tag{12}$$

is valid for all large enough r.

The class of such functions is defined by $\mathcal{E}$.

Following N. B. Khabibullin, we will provide a definition.

Definition 1. Let γ be a growth function and $\varepsilon\in\mathcal{E}$. A function $v\in J\delta$, $0\notin\operatorname{supp}\lambda_v$, $v(0)=0$, is called a *function of finite* (γ,ε)*-type* if there exist constants α, A and $B>0$ such that

$$T(r,v)\leq\frac{A}{r(\varepsilon(r))^\alpha}\gamma(r+B\varepsilon(r)r).$$

We denote the class of such functions by $J\delta((\gamma,\varepsilon))$. Let $JS((\gamma,\varepsilon))$ be the class of proper subharmonic functions of finite (γ,ε)-type.

Lemma 1. *The class $J\delta((\gamma,\varepsilon))$ is a real vector space and $JS((\gamma,\varepsilon))$ is a cone.*

Lemma 1 is a consequence of (10) and the inequality $T\left(r,\sum v_j\right)\leq\sum T\left(r,v_j\right)$.

A positive measure λ on the complex plane is called a *measure of finite* (γ,ε)*-type* if there exist positive constants α, A and B such that for all $r>0$,

$$\lambda(r)\leq\frac{Ar}{(\varepsilon(r))^\alpha}\gamma(r+B\varepsilon(r)r). \tag{13}$$

A positive measure λ has a *finite* (γ,ε)*-density* if there exist positive constants α, A and B such that

$$N(r,\lambda):=\int_{r_0}^r\frac{\lambda(t)}{t^3}dt\leq\frac{A}{r(\varepsilon(r))^\alpha}\gamma(r+B\varepsilon(r)r). \tag{14}$$

Lemma 2. *If λ is a measure of finite (γ,ε)-density, then it is a measure of finite (γ,ε)-type.*

Proof. We have

$$N(r(1+\varepsilon(r)),\lambda)=\int_{r_0}^{r(1+\varepsilon(r))}\frac{\lambda(t)}{t^3}dt\geq\int_{r}^{r(1+\varepsilon(r))}\frac{\lambda(t)}{t^3}dt\geq\frac{\lambda(r)}{r^2(1+\varepsilon(r))^2}\ln(1+\varepsilon(r)).$$

The above inequality and the elementary inequality $\ln(1+x)\geq x/(1+x)$ $(x\geq 0)$ yield

$$N(r(1+\varepsilon(r)),\lambda)\geq\frac{\lambda(r)\varepsilon(r)}{r^2(1+\varepsilon(r))^3}. \tag{15}$$

Further, by inequality (12) we obtain

$$N(r(1+\varepsilon(r)),\lambda)\leq\frac{A\gamma(r(1+\varepsilon(r))+B\varepsilon(r(1+\varepsilon(r)))r(1+\varepsilon(r)))}{r(1+\varepsilon(r))(\varepsilon(r(1+\varepsilon(r))))^\alpha}\leq\frac{A\gamma(r+r\varepsilon(r)+2Br\varepsilon(r))}{r(1+\varepsilon(r))(\varepsilon(r))^{\alpha\eta}}.$$

Inequality (13) for some constants α, A and B follows from the last inequality and (15). □

6. Proof of the main theorem. Let us prove the implication 1) $\Longrightarrow$ 2). We shall need the following lemma.

Lemma 3. *Let $v\in J\delta((\gamma,\varepsilon))$. Then each of the measures $\lambda_+(v)$ and $\lambda_-(v)$ has finite (γ,ε)-density, and the following inequality is valid*

$$\int_0^\pi|v(re^{i\varphi})|\sin\varphi d\varphi\leq\frac{A}{(\varepsilon(r))^\alpha}\gamma(r+B\varepsilon(r)r). \tag{16}$$

Proof. The measure $\lambda_-(v)$ has finite (γ,ε)-density by the definition of the class $J\delta((\gamma,\varepsilon))$. The fact that $\lambda_+(v)$ has finite (γ,ε)-density is a consequence of (10). The same formula yields

$$\int_0^\pi v_\pm(re^{i\varphi})\sin\varphi d\varphi\leq\frac{A}{(\varepsilon(r))^\alpha}\gamma(r+B\varepsilon(r)r).$$

This implies (16). The proof of the lemma is complete. □

From (16) it follows that for a function $v\in JS((\gamma,\varepsilon))$

$$|c_k(\theta,r,v)|\leq\int_0^\pi|v(re^{i\varphi})||\sin k\varphi|d\varphi\leq\frac{Ak}{(\varepsilon(r))^\alpha}\gamma(r+B\varepsilon(r)r). \tag{17}$$

Formula (2) yields

$$c_k(\theta,r,v)=\frac{c_k(\theta,r(1+\varepsilon(r)),v)}{(1+\varepsilon(r))^k}-\frac{2r^k\sin k\theta}{\pi}\int_r^{r(1+\varepsilon(r))}\frac{\lambda_k(t)}{t^{2k+1}}dt. \tag{18}$$

By

$$\frac{k}{(1+a)^k}\leq\frac{2}{a},\quad 0<a\leq 1,$$

and (17), we can deduce an estimate of the first addend on the right-hand side of (18).

$$\left|\frac{c_k(\theta, r(1+\varepsilon(r)), v)}{(1+\varepsilon(r))^k}\right| \leq \frac{Ak\gamma(r(1+\varepsilon(r))+B\varepsilon(r(1+\varepsilon(r)))r(1+\varepsilon(r)))}{(1+\varepsilon(r))^k(\varepsilon(r(1+\varepsilon(r))))^\alpha} \leq \leq \frac{2A\gamma(r+B_1\varepsilon(r)r)}{(\varepsilon(r))^{\alpha+1}}. \tag{19}$$

By (4) and (13), find now an estimate of the second addend in the right-hand side of (18)

$$\left|\frac{2r^k\sin k\theta}{\pi}\int_r^{r(1+\varepsilon(r))}\frac{\lambda_k(t)}{t^{2k+1}}dt\right| \leq \frac{2kr^k}{\pi}\int_r^{r(1+\varepsilon(r))}\frac{|\lambda(t)|}{t^{k+2}}dt \leq \leq \frac{Akr^k\gamma(r+B\varepsilon(r)r)}{(\varepsilon(r))^\alpha}\int_r^{r(1+\varepsilon(r))}\frac{dt}{t^{k+1}} \leq \frac{A\gamma(r+B\varepsilon(r)r)}{(\varepsilon(r))^\alpha}. \tag{20}$$

Relation (1) now follows from (19) and (20).

The proof of the implication 1) $\Longrightarrow$ 2) is complete. Let us prove the implication 2) $\Longrightarrow$ 1). Assume now that condition 2) from the theorem holds. Then it follows, by the inequality

$$|c_1(r,v)| \leq A\frac{\gamma(r+B\varepsilon(r)r)}{(\varepsilon(r))^\alpha}$$

and formula (3), that if one of the measures $\lambda_+(v)$ and $\lambda_-(v)$ has finite (γ,ε)-density, then the other measure also has finite (γ,ε)-density, and therefore $|\lambda|$ has finite (γ,ε)-density. We can now find an estimate of $v_+(z)$ using formula (5). By considering expansion (7) in the Fourier series, we obtain

$$\left|\frac{R}{2\pi}\int_0^\pi\frac{\partial G(z,Re^{i\varphi})}{\partial n}v(Re^{i\varphi})d\varphi\right| \leq \left|\frac{2}{\pi}\int_0^\pi\sum_{m=1}^\infty\frac{r^m}{R^m}\sin m\theta\sin m\varphi v(Re^{i\varphi})d\varphi\right| = = \left|\sum_{m=1}^\infty\left(\frac{r}{R}\right)^m c_m(\theta,R,v)\right| \leq \frac{A\gamma(R+B\varepsilon(R)R)}{(\varepsilon(R))^\alpha}\sum_{m=1}^\infty\left(\frac{r}{R}\right)^m, \quad z=re^{i\theta}.$$

Set $R = r(1+\varepsilon(r))$. Then

$$\left|\frac{R}{2\pi}\int_0^\pi\frac{\partial G(z,Re^{i\varphi})}{\partial n}v(Re^{i\varphi})d\varphi\right| \leq \frac{A\gamma\big(r+r\varepsilon(r)+Br(1+\varepsilon(r))\varepsilon(r+r\varepsilon(r))\big)}{\big(\varepsilon(r+r\varepsilon(r))\big)^\alpha}\times \times\sum_{k=1}^\infty\frac{1}{(1+\varepsilon(r))^k} \leq \frac{A_1\gamma\big(r+B_1r\varepsilon(r)\big)}{\big(\varepsilon(r)\big)^{\alpha_1}}\sum_{k=1}^\infty\frac{1}{(1+\varepsilon(r))^k} = \frac{A_1\gamma\big(r+B_1r\varepsilon(r)\big)}{\big(\varepsilon(r)\big)^{\alpha_1+1}}.$$

Since the function $K(z,\zeta)$ in (5) is positive, one has

$$v_+(z) \leq \frac{1}{2\pi}\iint_{\overline{C_+(0,R)}}K(z,\zeta)d\lambda_-(\zeta) + \frac{A_1\gamma\big(r+B_1r\varepsilon(r)\big)}{\big(\varepsilon(r)\big)^{\alpha_1+1}}.$$

Now using the orthogonality of the system of polynomials $\{\sin k\theta\}$, $k=1,2,...$, on the interval $[0,\pi]$ and formula (6), we obtain

$$\int_0^\pi v_+(z)\sin\theta d\theta \leq \frac{1}{2\pi}\int_0^\pi\left\{\left[\iint_{\overline{C_+(0,r)}}+\iint_{D_+(r,R)}\right]K(z,\zeta)d\lambda_-(\zeta)\right\}\sin\theta d\theta+$$

$$+\frac{2A_1\gamma\big(r+B_1r\varepsilon(r)\big)}{\big(\varepsilon(r)\big)^{\alpha_1+1}}\leq\frac{1}{2}\iint_{\overline{C_+(0,r)}}\frac{\sin\varphi}{\operatorname{Im}\zeta}\frac{\tau}{r}d\lambda_-(\zeta)+$$

$$+\frac{1}{2}\iint_{D_+(r,R)}\frac{\sin\varphi}{\operatorname{Im}\zeta}\frac{r}{\tau}d\lambda_-(\zeta)+\frac{2A_1\gamma\big(r+B_1r\varepsilon(r)\big)}{\big(\varepsilon(r)\big)^{\alpha_1+1}}\leq\frac{1}{2r}\iint_{\overline{C_+(0,R)}}d\lambda_-(\zeta)+$$

$$+\frac{2A_1\gamma\big(r+B_1r\varepsilon(r)\big)}{\big(\varepsilon(r)\big)^{\alpha_1+1}}\leq\frac{\lambda_-(R)}{2r}+\frac{2A_1\gamma\big(r+B_1r\varepsilon(r)\big)}{\big(\varepsilon(r)\big)^{\alpha_1+1}}.$$

Since the measure λ_- has finite (γ,ε)-density, we have $v\in J\delta((\gamma,\varepsilon))$.

Theorem 2. *Let γ be a growth function, $\varepsilon\in\mathcal{E}$ and let $v\in JS$. Then the following properties are equivalent*:

1) $v\in JS((\gamma,\varepsilon))$;
2) $|c_k(\theta,r,v)|\leq\dfrac{A\gamma(r+B\varepsilon(r)r)}{(\varepsilon(r))^{\alpha}}$, $k\in\mathbb{N}$, *for some positive α, A, B and $r>0$.*

This is an immediate consequence of Theorem 1, because the measure λ_- vanishes for functions in the class JS.

REFERENCES

1. L.A. Rubel, B.A. Taylor, *Fourier series method for meromorphic and entire functions*, Bull. Soc. Math. France., **96** (1968), 53–96.
2. J.B. Miles, *Quotient representations of meromorphic functions*, J. d′ Analyse Math., **25** (1972), 371–388.
3. A.A. Kondratyuk, *The Fourier series method for entire and meromorphic functions of completely regular growth*, Mat. Sb., **106(148)** (1978), №3(7), 386–408; English transl. in Math. USSR-Sb., **35** (1979), №1, 63–84.
4. A.A. Kondratyuk, *The Fourier series method for entire and meromorphic functions of completely regular growth. II*, Mat. Sb., **113(155)** (1980), №1(9), 118–132; English transl. in Math. USSR-Sb., **41** (1982), №1, 101–113.
5. A.A. Kondratyuk, *The Fourier series method for entire and meromorphic functions of completely regular growth. III*, Mat. Sb., **120(162)** (1983), №3, 331–343; English transl. in Math. USSR-Sb., **48**(1984), №3, 327–338.
6. P. Noverraz, *Fonctions plurisousharmoniques et analtiques dans les espaces vectoriels topologiques complexes*, Ann. Inst. Fourier, **19** (1969), №2, 419–493.
7. K.G. Malyutin, *Fourier series and δ-subharmonic functions*, Trudy Inst. Problem Mat. Mekh. Akad. Nauk Ukr., **3** (1998), 146–157. (in Russian)
8. K.G. Malyutin, *Fourier series and δ-subharmonic functions of finite γ-type in a half-plane*, Mat. Sb., **192** (2001), №6, 51–70; English transl. in Sb. Math., **192** (2001), №6, 843–861.
9. B.N. Khabibullin, *Growth of entire functions with given zeros and representation of meromorphic functions*, Mathematical Notes, **73** (2003), №1–2, 110–124. (in Russian)
10. Yu.S. Protsyk, *Subharmonic functions of finite (γ,ε)-type*, Mat. Stud., **24** (2005), №1, 39–56.
11. A.F. Grishin, *Continuity and asymptotical continuity of subharmonic functions*, Mathematical Physics, Analysis and Geometry, **1** (1994), №2, 193–215. (in Russian)
12. N.I. Ahiezer, Elements of the theory of elleptic functions, Nauka, Moscow, 1970; English transl., Amer. Math. Soc., Providence, RI, 1990.

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