FINITELY GENERATED SUBGROUPS AS VON NEUMANN RADICALS OF AN ABELIAN GROUP

Let $G$ be an infinite Abelian group. We give a complete characterization of those finitely generated subgroups of $G$ which are the von Neumann radicals for some Hausdorff group topologies on $G$. It is proved that every infinite finitely generated Abelian group admits a complete Hausdorff minimally almost periodic group topology. The latter result resolves a particular case of Comfort’s problem.

The following proposition (proved in Section 3) is a simple corollary of the main result of [6].

**Theorem 1.** Every infinite Abelian group admits a complete non-discrete Hausdorff group topology with trivial von Neumann radical.
A much deeper question is whether every infinite Abelian group admits a Hausdorff group topology with a non-zero von Neumann radical. A positive answer to this question was given by M. Ajtai, I. Havas and J. Komlós ([1]). E. G. Zelenyuk and I. V. Protasov ([15]) proved that every infinite Abelian group $G$ admits a complete Hausdorff group topology for which characters do not separate points. I. V. Protasov ([14]) posed the question whether every infinite Abelian group admits a minimally almost periodic group topology. A simple example of a bounded group $G$ which does not admit any Hausdorff group topology $\tau$ such that $(G, \tau)$ is minimally almost periodic is given by D. Remus ([5]). This justifies the following problem.

**Question (Comfort’s Problem 521 [5]).** Does every Abelian group which is not of bounded order admit a minimally almost periodic topological group topology? What about the countable case?

Moreover, it was not known even whether every infinite finitely generated Abelian group $G$ admits a Hausdorff minimally almost periodic group topology. We answer this question in the affirmative theorem.

**Theorem 2.** Every infinite finitely generated Abelian group $G$ admits a complete Hausdorff minimally almost periodic group topology.

Let $G$ be an infinite Abelian group and $H$ its infinite finitely generated subgroup. By Theorem 2, there is a Hausdorff MinAP group topology $\tau'$ on $H$. Let $\tau$ be a group topology on $G$ such that $H \in \tau$ and $\tau|_H = \tau'$. Then the von Neumann radical of $(G, \tau)$ is $H$ (see Lemma 9 below). So, every infinite finitely generated subgroup of an Abelian group $G$ can be considered as the von Neumann radical for some Hausdorff group topology on $G$. Noting that every finite group is finitely generated, it is natural to ask, for which finite subgroup $H$ of an infinite Abelian group $G$ there is a Hausdorff group topology $\tau$ on $G$ such that $H$ is the von Neumann radical of $(G, \tau)$.

Let $G$ be an infinite Abelian group. We denote by $\mathcal{N\mathcal{R}}(G)$ (by $\mathcal{N\mathcal{RC}}(G)$) the set of all subgroups $H$ of $G$ for which there exists a (complete) non-discrete Hausdorff group topology $\tau$ on $G$ such that $n(G, \tau) = H$. It is clear that $\mathcal{N\mathcal{RC}}(G) \subseteq \mathcal{N\mathcal{R}}(G)$. Therefore, by Theorem 1, $\{0\} \in \mathcal{N\mathcal{RC}}(G)$, and, by [15], $\mathcal{N\mathcal{RC}}(G) \neq \{\{0\}\}$. The general question of describing the sets $\mathcal{N\mathcal{R}}(G)$ and $\mathcal{N\mathcal{RC}}(G)$ was raised in [9].

The main goal of the paper is to describe all finitely generated subgroups of an infinite Abelian group $G$ which are contained in $\mathcal{N\mathcal{R}}(G)$.

For an Abelian group $G$, the symbols $\mathcal{FGS}(G)$ and $\mathcal{FS}(G)$ denote the set of all finitely generated subgroups and finite subgroups $G$, respectively.

Theorem 2 is an immediate consequence of the following theorem.

**Theorem 3.** Let $G$ be an Abelian group that is not bounded. Then for every finitely generated subgroup $H$ of $G$ there exists a complete Hausdorff group topology $\tau$ on $G$ such that $H = n(G, \tau)$, i.e., $\mathcal{FGS}(G) \subseteq \mathcal{N\mathcal{RC}}(G)$.

The case of bounded groups is more complicated. The direct sum of $\omega$ copies of an Abelian group $H$ we denote by $H^{(\omega)}$.

**Theorem 4.** Let $G$ be an infinite Abelian bounded group. Let $H \in \mathcal{FS}(G) = \mathcal{FGS}(G)$. Then the following statements are equivalent: 1) $G$ contains a subgroup of the form $H^{(\omega)}$; 2) $H \in \mathcal{N\mathcal{RG}}(G)$; 3) $H \in \mathcal{N\mathcal{R}}(G)$.
As an evident corollary of Theorems 3 and 4 we obtain the following result resolving [9, Problem 3].

**Corollary 1.** Let \( G \) be an infinite Abelian group and \( H \in \mathcal{FGS}(G) \). Then \( H \in \mathcal{NR}(G) \) if and only if \( H \in \mathcal{NRC}(G) \), i.e.,
\[
\mathcal{FS}(G) \cap \mathcal{NR}(G) = \mathcal{FS}(G) \cap \mathcal{NRC}(G), \quad \text{and} \quad \mathcal{FGS}(G) \cap \mathcal{NR}(G) = \mathcal{FGS}(G) \cap \mathcal{NRC}(G).
\]

By Corollary 1, in all subsequent theorems and corollaries of this section only the (simpler) option \( \mathcal{NR}(G) \) is considered.

Also as a trivial corollary of Theorems 3 and 4 we obtain the main result of [9].

**Corollary 2 ([9]).** An Abelian group \( G \) admits a Hausdorff group topology with non-trivial finite von Neumann radical if and only if it is not torsion free.

**Proof.** Clearly, if \( G \) admits a Hausdorff group topology with non-trivial von Neumann radical it must contain a nonzero element of finite order. Conversely, since every finite subgroup is finitely generated and since any infinite Abelian bounded group contains a subgroup of the form \( \mathbb{Z}(p^\omega) \) for some prime \( p \), the corollary immediately follows from Theorems 3 and 4. \( \square \)

The following problem was posed in [9, Problem 6]: describe all infinite Abelian groups \( G \) such that \( \mathcal{FS}(G) \subset \mathcal{NR}(G) \). (We note that this inclusion is strict since \( \mathcal{NR}(G) \) contains a countably infinite subgroup.) A solution to this problem is provided by the following theorem.

**Theorem 5.** Let \( G \) be an infinite Abelian group. Then the following statements are equivalent: 1. \( \mathcal{FGS}(G) \subseteq \mathcal{NR}(G) \); 2. \( \mathcal{FS}(G) \subset \mathcal{NR}(G) \); 3. \( G \) satisfies one of the following conditions: 1) \( \exp G = \infty \); 2) \( \exp G = m \) is finite and \( G \) contains a subgroup of the form \( \mathbb{Z}(p^\omega) \).

We can reformulate Theorem 5 for bounded groups as follows. It is well known that a bounded group \( G \) has the form \( G = \bigoplus_{p \in M} \bigoplus_{k=1}^{n_{p,k}} \mathbb{Z}(p^k) \), where \( M \) is a finite set of prime numbers. Leading Ulm-Kaplansky invariants of \( G \) are the cardinal numbers \( k_{n_{p,k}} \).

**Corollary 3.** All finite subgroups \( H \) of an infinite bounded Abelian group \( G \) belong to \( \mathcal{NR}(G) \) if and only if all leading Ulm-Kaplansky invariants of \( G \) are infinite.

The article is organized as follows. In Section 2 we prove some auxiliary lemmas that will be used to prove the main results. In Section 3 special \( T \)-sequences are constructed for some Abelian groups. These \( T \)-sequences are used to define the topologies with the desired property in Theorem 3. In the last Section 4 we prove Theorems 3, 4 and 5.

**2. Auxiliary lemmas.** Let us recall that a subset \( X \) of an Abelian group \( G \) is called independent provided that for every finite sequence \( x_1, \ldots, x_n \) of pairwise distinct elements of \( X \) and each sequence \( m_1, \ldots, m_n \) of integer numbers, if \( m_1x_1 + \cdots + m_nx_n = 0 \) then \( m_ix_i = 0 \) for all \( i \in \{1, \ldots, n\} \).

**Lemma 1.** Let \( \{b_n\}_{n \in \omega} \) be an independent sequence of an Abelian group \( G \). Then for every nonzero element \( g \) of \( G \) there is \( n_0 \) such that the set \( \{g, b_{n_0}, b_{n_0+1}, \ldots\} \) is independent.
Proof. Set \( H = \bigoplus_{n \in \omega} \langle b_n \rangle \). If the intersection \( H \cap \langle g \rangle \) is trivial then one can take \( n_0 = 0 \). Otherwise, \( H \cap \langle g \rangle \) is a subgroup of \( \langle g \rangle \), hence \( H \cap \langle g \rangle = \langle mg \rangle \neq 0 \) for some \( m \in \mathbb{N} \). The support of \( mg \in \bigoplus_{n=0}^\infty \langle b_n \rangle \) is finite, so there exists \( k \) such that \( mg \in \bigoplus_{n=0}^k \langle b_n \rangle \). Thus, \( H \cap \langle g \rangle = \langle mg \rangle \subseteq \bigoplus_{n=0}^k \langle b_n \rangle \). Therefore, \( \langle g \rangle \cap \bigoplus_{n=k+1}^\infty \langle b_n \rangle = 0 \). Putting \( n_0 = k + 1 \) we obtain that the set \( \{ g, b_{n_0}, b_{n_0+1}, \ldots \} \) is independent. \( \square \)

As usual, for an element \( g \) of an Abelian group \( G \), we denote by \( \langle g \rangle \) the subgroup of \( G \) generated by \( g \).

For the proof of Theorem 4 we need the following lemma.

Lemma 2. Let \( G \) be an infinite Abelian group and \( e_1, \ldots, e_q \in G \). Then the following assertions are equivalent:

1. \( G \) contains a subgroup of the form \( \langle e_1 \rangle^{(\omega)} \oplus \langle e_2 \rangle^{(\omega)} \oplus \cdots \oplus \langle e_q \rangle^{(\omega)} \);
2. \( G \) contains a subgroup of the form \( \langle e_i \rangle^{(\omega)} \) for every \( 1 \leq i \leq q \).

Proof. We need to prove only the implication \((2) \Rightarrow (1)\). It is easy to see that we can restrict ourselves to the case when \( e_i \) has a finite order \( n_i \) for every \( 1 \leq i \leq q \).

Let \( p_1^{b_1} \ldots p_l^{b_l} \) be the prime decomposition of the least common multiple of \( n_1, \ldots, n_q \). Since any \( p_j^{b_j} \) is a divisor of some \( n_{k(j)} \), by hypothesis, \( G \) contains a subgroup of the form \( \bigoplus_{n=1}^\infty H_n^j \), where \( H_n^j \cong \mathbb{Z}(p_j^{b_j}) \).

Thus, \( G \) contains the following subgroup

\[
\bigoplus_{i=1}^q \left( \bigoplus_{n=1}^\infty H_{nq+i}^1 \oplus \bigoplus_{n=1}^\infty H_{nq+i}^2 \oplus \cdots \oplus \bigoplus_{n=1}^\infty H_{nq+i}^l \right) .
\]

Evidently, the group \( \bigoplus_{n=1}^\infty H_{nq+i}^1 \oplus \bigoplus_{n=1}^\infty H_{nq+i}^2 \oplus \cdots \oplus \bigoplus_{n=1}^\infty H_{nq+i}^l \) contains a subgroup of the form \( \langle e_i \rangle^{(\omega)} \) for every \( 1 \leq i \leq q \). \( \square \)

Let us consider the group \( \mathbb{Z}(p^\infty) \) with discrete topology. Then \( \mathbb{Z}(p^\infty)^\wedge = \Delta_p \) is the compact group of \( p \)-adic integers which elements are denoted by \( x = (a_i), 0 \leq a_i < p \), and the identity is \( \mathbf{1} = (1, 0, 0, \ldots) \). By [10, Remark 10.6], \( \langle \mathbf{1} \rangle \) is dense in \( \Delta_p \) and, by [10, 25.2], \( (\lambda, 1) = \exp(2\pi i \cdot \lambda) \) for every \( \lambda \in \mathbb{Z}(p^\infty) \). Following [10, 10.4], we denote by \( \Lambda_k, k \geq 1 \), the set of all \( \mathbf{x} = (x_0, \ldots, x_{k-1}, x_k, \ldots) \in \Delta_p \) such that \( x_0 = \cdots = x_{k-1} = 0 \) and put \( \Lambda_0 = \Delta_p \). Note that \( \Lambda_k \) is just \( p^k \Delta_p \).

A group \( G \) with the discrete topology is denoted by \( G_d \). If \( H \) is a subgroup of \( (G_d)^\wedge \) then \( H^\perp := \{ g \in G \mid (g, h) = 1 \ \forall h \in H \} \). We use the following lemma to prove Theorem 3.

Lemma 3. Let \( G = \mathbb{Z}(p^\infty) + H \), where \( H \) is a finite group, endowed with the discrete topology. Let \( H_1 \) be a finite group such that \( G = \mathbb{Z}(p^\infty) \oplus H_1 \). Then there exist \( k \geq 0 \) and a finite set \( S_0 \subseteq \langle \mathbf{1} \rangle \oplus H_1^\wedge \) such that \( H^\perp = \langle S_0 \rangle + \Lambda_k \). In particular, the finitely generated subgroup \( \langle S_0 \rangle \cup \{ p^k \mathbf{1} \} \) is dense in \( H^\perp \).

Proof. Let \( H = \langle e_1 \rangle \oplus \cdots \oplus \langle e_l \rangle \oplus \langle e_{l+1} \rangle \oplus \cdots \oplus \langle e_q \rangle \), where \( o(e_i) = p^{w_i} \) for \( 1 \leq i \leq l \) and \( o(e_i) = p^{w_i}, p_i \neq p \), for \( l < i \leq q \). Then for some integer \( t \) we have

\[
H_1 = \langle g_1 \rangle \oplus \cdots \oplus \langle g_l \rangle \oplus \langle e_{l+1} \rangle \oplus \cdots \oplus \langle e_q \rangle ,
\]

where \( o(g_i) = p^{r_i} \) for some natural number \( r_i \) and \( e_i = a_{i0}g_0 + a_{i1}g_1 + \cdots + a_{it}g_t, 1 \leq i \leq l \), where \( g_0 = \frac{1}{p^m} \in \mathbb{Z}(p^\infty) \), and \( 0 \leq a_{ij} < p^{r_i} \) for every \( 0 \leq j < t \). Since \( H_1 \) is finite, we will
identify $H_1$ with $H_1^\wedge$. Let $\omega = x + \lambda_1 g_1 + \cdots + \lambda_t g_t + \lambda_{t+1} e_{t+1} + \cdots + \lambda_{t+q-l} e_q \in G^\wedge$, where $x = (x_0, x_1, \ldots) \in A_p$, $0 \leq \lambda_i < p^i$ for $1 \leq i \leq t$ and $0 \leq \lambda_{t+i} < p^i_{t+i}$ for $1 \leq j \leq q - l$. By definition, $\omega \in H^\perp$ if $(\omega, e_i) = 1$ for every $1 \leq i \leq q$. In particular, for every $1 \leq j \leq q - l$,

$$(\omega, e_{t+j}) = \frac{\lambda_{t+j}}{p_{t+j}^{w_{t+j}}} = 0 \text{ (mod 1)}.$$ 

Since $\lambda_{t+j} < p_{t+j}^{w_{t+j}}$, we have $\lambda_{t+j} = 0$ for every $1 \leq j \leq q - l$. So, $\omega \in H^\perp$ if and only if it has the form $\omega = x + \lambda_1 g_1 + \cdots + \lambda_t g_t$ and $(\omega, e_i) = 1, \forall 1 \leq i \leq l$. Thus, by [10, 25.2], $\omega \in H^\perp$ if and only if $\lambda_{t+j} = 0$, for every $1 \leq j \leq q - l$, and for any $1 \leq i \leq l$, (mod 1)

$$\frac{a_i^0}{p^i} (x_0 + x_1 p + \cdots + x_{r_0-1} p^{r_0-1}) + \frac{\lambda_1 a_i^1}{p^1} + \cdots + \frac{\lambda_t a_i^t}{p^t} = 0. \quad (1)$$

Denote by $S_0$ the set of all $\omega \in H^\perp$ which have the form

$$\omega = x + \lambda_1 g_1 + \cdots + \lambda_t g_t, \quad x = (x_0, x_1, \ldots, x_{r_0-1}, 0, \ldots),$$

where $x$ and $\lambda_1, \ldots, \lambda_t$ satisfy (1). By definition, $S_0 \subset H^\perp \cap (\{1\} \oplus H_1^\wedge)$ and for every $\omega \in H^\perp$ there is $\omega_0 \in S_0$ such that $\omega - \omega_0 = (0, \ldots, 0_{r_0-1}, x_{r_0}, x_{r_0+1}, \ldots) \in \Lambda_{r_0}$. Set $k = r_0$. Then $H^\perp \subseteq \langle S_0 \rangle + \Lambda_k$. The converse inclusion follows from (1). By [10, Remark 10.6], $\langle p^k 1 \rangle$ is dense in $\Lambda_k$. So, $\langle S_0 \cup \{p^k 1\} \rangle$ is dense in $H^\perp$. \hfill \square

3. Construction of $T$-sequences. In this section we construct $T$-sequences which will be needed for the proofs of the main results.

Following E. G. Zelenyuk and I. V. Protasov ([15], [16]), we say that a sequence $d = \{d_n\}$ in a group $G$ is a $T$-sequence if there is a Hausdorff group topology on $G$ with respect to which $d_n$ converges to zero. The group $G$ equipped with the finest group topology with this property is denoted by $(G, d)$. We note also that, by [16, Theorem 2.3.11], the group $(G, d)$ is complete.

For a sequence $\{d_n\}$ and $k, m \in \mathbb{N}$, one defines ([15])

$$A(k, m) = \{n_1 d_{r_1} + \cdots + n_s d_{r_s} : m \leq r_1 < \cdots < r_s,$$ 

$$n_1, n_2, \ldots, n_s \in \mathbb{Z} \setminus \{0\}, \quad \sum_{i=1}^s |n_i| \leq k + 1 \} \cup \{0\}.$$ 

In this section we make extensive use of the following Protasov-Zelenyuk’s criterion.

**Theorem 6** ([15]). A sequence $\{d_n\}$ of elements of an Abelian group $G$ is a $T$-sequence if and only if, for every integer $k \geq 0$ and for each element $g \in G$ with $g \neq 0$, there is an integer $m$ such that $g \notin A(k, m)$.

For a prime $p$ and $n \in \mathbb{N}$ we set $f_n = p^{n^3-n^2} + \cdots + p^{n^3-2n} + p^{n^3-n} + p^n \in \mathbb{Z}$. Then $f_n < 2p^{n^3} \leq p^{n^3+1}$. For $0 < r_1 < r_2 < \cdots < r_v$ and integers $l_1, l_2, \ldots, l_v$ such that $\sum_{i=1}^v |l_i| \leq k + 1$, we have

$$|l_1 f_{r_1} + l_2 f_{r_2} + \cdots + l_v f_{r_v}| < (k + 1) f_{r_v} \leq (k + 1)p^{r_v^3+1}. \quad (2)$$

Lemmas 4 and 5 are slight modifications of items (1) and (2) in the proof of [9, Theorem 1].
Lemma 4. Let \( G = \langle e_0 \rangle \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_{q-1} \rangle \), where \( \langle e_0 \rangle \cong \mathbb{Z} \). Given a prime \( p \) and \( \varepsilon_n \in \{-1,0,1\} \) for \( n \geq 1 \), the formulas

\[
d_{2n} = p^n e_0 \quad \text{and} \quad d_{2n-1} = f_n e_0 + \varepsilon_n e_{n_{\text{mod} q}}
\]
define a \( T \)-sequence \( \{d_n\} \) in \( G \).

Proof. Fix an integer \( k \geq 0 \) and an element \( g \in G \) with \( g \neq 0 \). By Theorem 6, it suffices to prove that \( g \notin A(k,m) \) for some \( m \in \mathbb{N} \). Let \( g = b e_0 + a_1 e_1 + \cdots + a_{q-1} e_{q-1} \), where \( b \in \mathbb{Z} \), \( 0 \leq a_i < o(e_i) \) if \( o(e_i) < \infty \) and \( a_i \in \mathbb{Z} \) if \( o(e_i) = \infty \). Let \( t = \left( |b| + |a_1| + \cdots + |a_{q-1}| \right) (k+1) \) and \( m = 20t \). We are going to check that \( g \notin A(k,m) \). To accomplish this, we pick an arbitrarily \( \sigma \in A(k,m) \) with \( \sigma \neq 0 \) and prove that \( g \neq \sigma \). To this end, we prove that \( |\phi_0| > b \), where \( \phi_0 \) is the coefficient of \( e_0 \) in \( \sigma \).

Since the sequence \( d_n \) is defined by two different subsequences, we have to consider some particular cases to estimate \( \phi_0 \).

a) Assume that

\[
\sigma = l_1 d_{2r_1} + l_2 d_{2r_2} + \cdots + l_s d_{2r_s} = (l_1 p^{r_1} + \cdots + l_s p^{r_s}) e_0 = p^{r_1} \cdot \sigma' \cdot e_0,
\]

where \( m \leq 2r_1 < 2r_2 < \cdots < 2r_s \) and \( \sigma' \in \mathbb{Z} \). Since \( \sigma' \neq 0 \), we have \( p^{r_1} > p^{5|b|} > |b| \) and \( \sigma \neq g \).

b) Assume that \( \sigma = l_1 d_{2r_1-1} + l_2 d_{2r_2-1} + \cdots + l_s d_{2r_s-1} \), where \( m \leq 2r_1 - 1 < 2r_2 - 1 < \cdots < 2r_s - 1 \) and the integers \( l_1, l_2, \ldots, l_s \) are such that \( l_s \neq 0 \) and \( \sum_{i=1}^s |l_i| \leq k + 1 \). Then

\[
\sigma = (l_1 f_{r_1} + \cdots + l_{s-1} f_{r_{s-1}} + l_s f_{r_s}) e_0 + l_1 \varepsilon_{r_1} e_{r_1_{\text{mod} q}} + \cdots + l_s \varepsilon_{r_s} e_{r_s_{\text{mod} q}}.
\]

Since \( n^3 < (n+1)^3 - (n+1)^2 \) and \( r_s > |b| + (k+1) \), by (2), we can estimate the coefficient \( \phi_0 \) of \( e_0 \) in \( \sigma \) as follows

\[
|\phi_0| \geq |l_1 f_{r_1} + \cdots + l_{s-1} f_{r_{s-1}} + l_s f_{r_s}| - (k+1) > f_{r_s} - k \cdot p^{r_s-1} - 1 > \frac{p^3}{p^2 - 1} > |b|.
\]

Hence \( \phi_0 \neq b \) and \( \sigma \neq g \).

c) Assume that \( \sigma = l_1 d_{2r_1-1} + l_2 d_{2r_2-1} + \cdots + l_s d_{2r_s-1} + l_{s+1} d_{2r_{s+1}} + \cdots + l_h d_{2r_h} \), where \( 0 < s < h \) and

\[
m \leq 2r_1 - 1 < 2r_2 - 1 < \cdots < 2r_s - 1,
\]

\[
m \leq 2r_{s+1} < 2r_{s+2} < \cdots < 2r_h, \quad l_i \in \mathbb{Z}, \quad \sum_{i=1}^h |l_i| \leq k + 1.
\]

Since the number of summands with different powers of \( p \) in \( f_{r_s} \) is \( r_s + 1 > 10(k+1) \) and \( h - s < k + 1 \), by a simple pigeon-hole principle, there exists \( r_s - 2 \geq i_0 > 2 \) such that for every \( 1 \leq w \leq h - s \) we have

either \( r_{s+w} < r_s^3 - (i_0 + 2) r_s \) or \( r_{s+w} > r_s^3 - (i_0 - 1) r_s \).

The set of all \( w \) such that \( r_{s+w} < r_s^3 - (i_0 + 2) r_s \) we denote by \( B \) (it can be empty or have the form \( \{1, \ldots, \delta\} \) for some \( 1 \leq \delta \leq h - s \)). Set \( D = \{1, \ldots, h - s\} \setminus B \). Thus,

\[
\sigma = l_1 e_{r_1_{\text{mod} q}} + \cdots + l_s e_{r_s_{\text{mod} q}} + \left( l_1 f_{r_1} + \cdots + l_{s-1} f_{r_{s-1}} \right) e_0 + \sum_{w \in B} l_{s+w} d_{2r_{s+w}} + \left( l_s p^{r_s - r_s^2} + \cdots + l_s p^{r_s - (i_0 + 2) r_s} \right) e_0 + \left( l_s p^{r_s - (i_0 - 1) r_s} + \cdots + l_s p^{r_s^2} \right) e_0 + \sum_{w \in D} l_{s+w} d_{2r_{s+w}}.
\]
Denote the coefficients of $e_0$ in lines 1, \ldots, 5 by $A_1, \ldots, A_5$ respectively. Then $\phi_0 = A_1 + \cdots + A_5$. We estimate $A_1, \ldots, A_5$ as follows. For $A_1$ we have

$$|A_1| \leq |l_1| + \cdots + |l_s| \leq k + 1 < p^{k+1} < p^{r^2-(i_0+1)r_s}.$$  \hspace{1cm} (3)

Since $l_s \neq 0$ and $kp < p^r p < p^{r_s}$, by (2), we have

$$|A_2| = |l_1 f_r + \cdots + l_s f_{r_{s-1}}| \leq k \cdot p^{r^2-r_s+1} < p^{r^2-r_s+r_s} \leq p^{(r_s-1)^3+r_s} < p^{r^2-(i_0+1)r_s}.$$ \hspace{1cm} (4)

Since $3(k+1) < r_s < p^{r_s}$, for $A_3$ we have

$$|A_3| = \sum_{w \in B} l_{s+w} p^{r^2+r_s} + \left(l_4 p^{r^2-r_s} + \cdots + l_s p^{r^2-(i_0+2)r_s}\right) < \sum_{w \in B} |l_{s+w}| p^{r^2-(i_0+2)r_s} + |l_s| 2p^{r^2-(i_0+2)r_s} < 3(k+1)p^{r^2-(i_0+2)r_s} < p^{r^2-(i_0+1)r_s}.$$ \hspace{1cm} (5)

For $A_4$ we have

$$p^{r^2-i_0r_s} < |A_4| = |l_s| p^{r^2-(i_0+1)r_s} + |l_s| p^{r^2-i_0r_s} < 2k \cdot p^{r^2-i_0r_s}. \hspace{1cm} (6)$$

For $A_5$ we have

$$A_5 = l_4 p^{r^2-(i_0-1)r_s} + \cdots + l_s p^{r^2} + \sum_{w \in D} l_{s+w} p^{r^2+r_s} = p^{r^2-(i_0-1)r_s} \cdot \sigma^\prime \prime,$$ \hspace{1cm} (7)

where $\sigma^\prime \prime \in \mathbb{Z}$. We distinguish between two cases.

Case 1. $\sigma^\prime \prime \neq 0$. By (3)–(7), we can estimate $\phi_0$ from below as follows

$$|\phi_0| \geq |A_5| - (|A_1| + |A_2| + |A_3| + |A_4|) > p^{r^2-(i_0-1)r_s} - 3p^{r^2-(i_0+1)r_s} - 2kp^{r^2-i_0r_s} > p^{r^2-(i_0-1)r_s} - (2k + 3)p^{r^2-i_0r_s} > p^{r^2-i_0r_s} > p^{r^2} > p^{|b|} > |b|.$$  

Hence $\phi_0 \neq b$ and $\sigma \neq g$.

Case 2. $\sigma^\prime \prime = 0$. Then, by (3)–(5),

$$|\phi_0| \geq |A_4| - (|A_1| + |A_2| + |A_3|) > p^{r^2-i_0r_s} - 3p^{r^2-(i_0+1)r_s} > p^{r^2-(i_0+1)r_s} > p^{r^2} > p^{|b|} > |b|.$$  

Hence $\phi_0 \neq b$ and $\sigma \neq g$ too. \hfill \Box

In the following lemma we consider $\mathbb{Z}(p^\infty)$ as a subgroup of $(-\frac{1}{2}, \frac{1}{2}]$ by modulo 1. For the sake of clarity, $|x|(\text{mod} \ 1)$ denotes the distance from a real number $x$ to the nearest integer. Putting

$$\tilde{f}_n = \frac{1}{p^{n^3-n^2}} + \cdots + \frac{1}{p^{n^3-2n}} + \frac{1}{p^{n^3-n}} + \frac{1}{p^{n^3}} \in \mathbb{Z}(p^\infty),$$

we obtain ([11])

$$0 < \tilde{f}_n = \frac{1}{p^{n^3-n^2}} + \cdots + \frac{1}{p^{n^3-2n}} + \frac{1}{p^{n^3-n}} + \frac{1}{p^{n^3}} < \frac{n+1}{p^{n^3-n^2}} \to 0. \hspace{1cm} (8)$$
Lemma 5. Let $G = \mathbb{Z}(p^\infty) + H$, where $H = \langle e_0 \rangle \oplus \cdots \oplus \langle e_{q-1} \rangle$ is finite. Define
\[
d_2n = \frac{1}{p^\alpha} \in \mathbb{Z}(p^\infty) \quad \text{and} \quad d_{2n-1} = \tilde{f}_n + e_{n(mod\ q)} \quad \text{for} \quad n \geq 1.
\]
Then $d = \{d_n\}$ is a $T$-sequence in $G$.

**Proof.** Let $k \geq 0$ be an integer and $g \in G$ with $g \neq 0$. Then $g = \frac{b}{p^\beta} + a_0e_0 + \cdots + a_{q-1}e_{q-1}$, where $0 \leq a_i < o(e_i)$ and $\frac{b}{p^\beta} \in \mathbb{Z}(p^\infty)$. Let $\pi: G \rightarrow \mathbb{Z}(p^\infty)$ be the projection. Then $\pi((g) + H) = (\frac{1}{p^\beta})$.

Set $t = p(k + 1) + \beta$ and $m = 20t$. By Theorem 6, it is enough to prove that $g \not\in A(k, m)$. To achieve this, we take $\sigma \in A(k, m) \setminus \{0\}$ arbitrarily and show that $g \neq \sigma$. To this end, we prove two inequalities (mod 1):

1) $0 < |\pi(\sigma)|$ and 2) if $\pi(g) \neq 0$, then $|\pi(\sigma)| < |\pi(g)|$. This gives $\sigma \neq g$.

Since the sequence $d_n$ is defined by the two different subsequences, we have to consider some particular cases to estimate $\pi(\sigma)$.

a) Assume that $\sigma = l_1d_{2r_1} + l_2d_{2r_2} + \cdots + l_sd_{2r_s}$, where $m \leq 2r_1 < 2r_2 < \cdots < 2r_s$. If $\pi(g) = 0$, then $\pi(\sigma) = \pi(g) = 0$. If $\pi(g) \neq 0$, then
\[
0 < |\sigma| = |\pi(\sigma)| = |l_1d_{2r_1} + l_2d_{2r_2} + \cdots + l_sd_{2r_s}| \leq \sum_{i=1}^{s} |l_i| \leq k + 1 + \frac{k + 1}{p^{k+1+\beta}} < \frac{1}{p^{\beta}} \leq |\pi(g)|.
\]
So $\pi(\sigma) = \pi(g)$ and $\sigma \neq g$.

b) Assume that $\sigma = l_1d_{2r_1} - l_2d_{2r_2} + \cdots + l_sd_{2r_s}$, where $m < 2r_1 < 2r_2 < \cdots < 2r_s - 1$ and the integers $l_1, l_2, \ldots, l_s$ are such that $l_s \neq 0$ and $\sum_{i=1}^{s} |l_i| \leq k + 1$. Since $n^3 < (n + 1)^3 - (n + 1)^2$ and $r_s > 5p(k + 1) + 5\beta$, we have
\[
\pi(\sigma) = \frac{z'}{p^{r_s}} + \frac{l_s}{p^{r_s}}, \quad \text{where} \quad z' \in \mathbb{Z}.
\]
Since $|l_s| \leq k + 1 < \frac{r_s}{p} < p^{-1}$, we have the following: if $\pi(\sigma) = \frac{z''}{p^s}, z'' \in \mathbb{Z}$, is an irreducible fraction then $\alpha > r_s^3 - r_s + 1 > 5\beta$. Hence $\pi(\sigma) \neq \pi(g)$ and $\sigma \neq g$.

c) Assume that $\sigma = l_1d_{2r_1} - l_2d_{2r_2} + \cdots + l_sd_{2r_s} + l_{s+1}d_{2r_{s+1}} + \cdots + l_hd_{2r_h}$, where $0 < s < h$ and
\[
m < 2r_1 - 1 < 2r_2 - 1 < \cdots < 2r_s - 1,
\]
\[
m < 2r_{s+1} < 2r_{s+2} < \cdots < 2r_h, \quad l_i \in \mathbb{Z} \setminus \{0\}, \quad \sum_{i=1}^{h} |l_i| \leq k + 1.
\]
Since the number of summands with different powers of $p$ in $\tilde{f}_r$, is $r_s + 1 > 10p(k + 1)$ and $h - s < k + 1$, by a simple pigeon-hole principle, there exists $r_s - 2 > i_0 > 2$ such that for every $1 \leq w \leq h - s - 1$ we have
\[
either \ r_{s+w} < r_s^3 - (i_0 + 2)r_s \quad or \quad r_{s+w} > r_s^3 - (i_0 - 1)r_s.
\]
The set of all $w$ such that $r_{s+w} < r_s^3 - (i_0 + 2)r_s$ we denote by $K$ (it can be empty or have the form $\{1, \ldots, a\}$ for some $1 \leq a \leq h - s$). Set $L = \{1, \ldots, h - s\} \setminus K$. Thus
\[
\sigma = (l_1e_{r_1(mod\ q)} + \cdots + l_se_{r_s(mod\ q)}) + l_1\tilde{f}_{r_1} + \cdots + l_{s-1}\tilde{f}_{r_{s-1}} + \sum_{w \in K} l_{s+w}d_{2r_{s+w}} + \frac{l_s}{p^{r_s-r_s}} + \cdots + \sum_{w \in L} l_{s+w}d_{2r_{s+w}}.
\]
The elements in the lines 1, 2 and 4 we denote by \( \sigma_1, \sigma_2 \) and \( \sigma_4 \) respectively. Since \( n^3 < (n + 1)^3 - (n + 1)^2 \) and \( r_s > \beta \), the projection on \( \mathbb{Z}(p^{\infty}) \) of every summand in lines 1 and 2 has the form \( \frac{c}{p^r} \), with \( \gamma \leq r_s^3 - (i_0 + 2)r_s \) and \( \delta \in \mathbb{Z} \). Thus,

\[
\pi(\sigma_1 + \sigma_2) = \frac{c}{p^{r_3^2 -(i_0+2)r_s}}, \text{ for some } c \in \mathbb{Z}.
\]

Hence

\[
\pi(\sigma) = \frac{c}{p^{r_3^2 -(i_0+2)r_s}} + \frac{l_s}{p^{r_3^2 -(i_0+1)r_s}} + \frac{l_s}{p^{r_3^2 -i_0r_s}} + \pi(\sigma_4). \quad (9)
\]

Since \( r_s > 10p(k+1) \), then \( \frac{1}{1-1/p^{r_s}} < \frac{1}{1-1/p^\beta} < \frac{32}{31} \) and \( 2k < p^{2k} < p^{r_s} \). Thus, we can estimate \( \pi(\sigma_4) \) as follows:

\[
|\pi(\sigma_4)| = \left| \frac{l_s}{p^{r_3^2 -(i_0-1)r_s}} + \cdots + \frac{l_s}{p^{r_s}} + \sum_{w \in \mathcal{L}} l_{s+w} \frac{1}{p^{r_{s+w}}} \right| < \\
\lesssim \frac{|l_s|}{p^{r_3^2 -(i_0-1)r_s}} \left( 1 + \frac{1}{p^{r_s}} + \frac{1}{p^{2r_s}} + \cdots \right) + \frac{1}{p^{r_3^2 -(i_0-1)r_s+1}} \sum_{w \in \mathcal{L}} |l_{s+w}| \lesssim \frac{|l_s|}{p^{r_3^2 -(i_0-1)r_s}} \times \\
\times \frac{1}{1- \frac{1}{p^{r_s}}} + \frac{k}{p^{r_3^2 -(i_0-1)r_s+1}} < \frac{1}{p^{r_3^2 -(i_0-1)r_s}} \left( \frac{32}{31} + k \frac{1}{p} \right) < \frac{2k}{p^{r_3^2 -(i_0-1)r_s}} < \frac{1}{p^{r_3^2 -i_0r_s}}. \quad (10)
\]

We distinguish between two cases.

Case 1. \( \pi(\sigma_4) \neq 0 \). By (10) we have the following. If \( \pi(\sigma_4) = \frac{c}{p^r} \) is an irreducible fraction, then \( \alpha > r_s^3 - i_0r_s > 5\beta \). Thus, by (9), we also have

\[
\pi(\sigma) = \frac{c''}{p^r} \neq 0, \text{ where } c'' \in \mathbb{Z} \text{ and } (c'', p) = 1.
\]

Since \( \pi(g) \in \langle \frac{1}{p^r} \rangle \) and \( \alpha > 5\beta \), we have \( \pi(\sigma) \neq \pi(g) \) and \( \sigma \neq g \).

Case 2. \( \pi(\sigma_4) = 0 \). Let \( l_s = p^\psi \cdot l'_s \), where \( (p, l'_s) = 1 \) and \( \psi < k < r_s \). Thus, by (9),

\[
\pi(\sigma) = \frac{c}{p^{r_3^2 -(i_0+2)r_s}} + \frac{l_s}{p^{r_3^2 -(i_0+1)r_s}} + \frac{l_s}{p^{r_3^2 -i_0r_s}} = \frac{c''}{p^{r_3^2 -i_0r_s-\psi}},
\]

where \( c'' \in \mathbb{Z} \) and \( (c'', p) = 1 \). Since \( r_s^3 - i_0r_s - \psi > r_s^3 - (i_0 + 1)r_s > 5\beta \), we have \( \pi(\sigma) \neq 0 \) and \( \pi(\sigma) \neq \pi(g) \). Thus \( \sigma \neq g \).

\[\square\]

Put \( S_0 = 0 \) and \( S_n = 1 + 2 + \cdots + n \) for \( n \in \mathbb{N} \).

**Lemma 6.** Let \( q \) be an integer with \( q \geq 2 \). Then \( (S_{n-1} + k)q + i \neq (S_{m-1} + l)q + j \) for every \( m, n \geq 1, 0 \leq i, j < q, 1 \leq k \leq n \) and \( 1 \leq l \leq m \) such that \( (n, i, k) \neq (m, j, l) \).

**Proof.** We have three cases:

1. The case \( n \neq m \). We may assume that \( n \leq m - 1 \). Then for every \( 0 \leq i, j < q \) and \( 1 \leq k \leq n \) we have

\[
(S_{n-1} + k)q + i \leq S_nq + (q - 1) = (S_n + 1)q - 1 < (S_{m-1} + 1)q + j.
\]

So \( (S_{n-1} + k)q + i \neq (S_{m-1} + l)q + j \) for every \( 0 \leq i, j < q, 1 \leq k \leq n \) and \( 1 \leq l \leq m \).

2. The case \( n = m \) and \( i \neq j \). It is clear that

\[
(S_{n-1} + k)q + i \neq (S_{n-1} + l)q + j \text{ for every } 1 \leq k, l \leq n.
\]

3. The case \( n = m, i = j \) and \( k \neq l \). It is clear that \( (S_{n-1} + k)q + i \neq (S_{n-1} + l)q + i \).

\[\square\]

As usual, \( o(g) \) denotes the order of an element \( g \) of an Abelian group \( G \).

In the following lemma we modify the construction of [15, Example 5] (or [16, Example 2.6.2]).
Lemma 7. Let $H = \langle e_0 \rangle \oplus \cdots \oplus \langle e_{q-1} \rangle$ and $G = H \oplus \bigoplus_{i=q}^{\infty} \langle e_i \rangle = \bigoplus_{i=0}^{\infty} \langle e_i \rangle$, where $u_i := o(e_i) < \infty$ for every $i \geq 0$. Define a sequence $d = \{d_n\}_{n \geq 2q-1}$ as follows. For even indices we set
\[ d_{2q} = e_q, \quad d_{2(q+1)} = 2e_q, \ldots, d_{2(q+u_q-2)} = (u_q - 1)e_q, \quad d_{2(q+u_q-1)} = e_{q+1}, d_{2(q+u_q)} = 2e_{q+1}, \ldots \]
For odd indices and for $0 \leq i < q$ and $n \geq 1$, we define
\[ d_{2q+i-1} = e_i + e_{(S_{n-1}+1)q+i} + e_{(S_{n-1}+2)q+i} + \cdots + e_{S_nq+i}. \]
Assume that one of the following two conditions holds:
\( a) \) there exists an integer $j_0 \geq 0$ such that $u_j = u_{j_0}$ for all integers $j \geq j_0$ and $u_{j_0}$ is divided by every $u_0, \ldots, u_{q-1}$, or \( b) \) $u_n \to \infty$.

Then $d = \{d_n\}$ is a $T$-sequence in $G$.

Proof. Let $k \geq 0$ be an integer and $g \in G$ with $g \neq 0$. By Theorem 6, we have to show that there is $m \in \mathbb{N}$ such that $g \not\in A(k, m)$.

Step 1. By construction, $d_{2m} = \lambda(n)e_{\mu(n)}$, where $1 \leq \lambda(n) < o(e_{\mu(n)})$ and $\mu(n) \to \infty$ at $n \to \infty$. Since also $(S_{n-1}+1)q+i \to \infty$ at $n \to \infty$, we have the following: for every $j \geq q$ there exists $m \in \mathbb{N}$ such that $A(k, m) \subset H \oplus \bigoplus_{i=j}^{\infty} \langle e_i \rangle$. Thus,
\[ \bigcap_{m=1}^{\infty} A(k, m) \subset \bigcap_{j \geq q} \left( H \oplus \bigoplus_{i=j}^{\infty} \langle e_i \rangle \right) = H. \]
So, the condition of the Protasov-Zelenyuk criterion holds for every $g \not\in H$. (Note that a similar inclusion was proved in [11, Proposition 3.3] for another special case of $T$-sequence.)

By Step 1, it remains to check the Protasov-Zelenyuk criterion only for non-zero elements of $H$. Thus, in what follows, we assume that $g \in H$ and $g \neq 0$.

Note also that the summands of all the elements $d_{2(nq+i)-1} - e_i$ are independent, where $0 \leq i < q$ and $n \geq 1$. Indeed, this follows from Lemma 6 and the independence of the sequence $\{e_i\}$.

Step 2. Let $g \in A(k, 2m)$ for some natural $m$. Then $g$ has the following representation
\[ g = l_1d_{2r_1-1} + l_2d_{2r_2-1} + \cdots + l_sd_{2r_s-1} + l_{s+1}d_{2r_{s+1}} + l_{s+2}d_{2r_{s+2}} + \cdots + l_{h}d_{2r_{h}}, \]
(11)
where all summands are nonzero, $\sum_{i=1}^{h} |l_i| \leq k + 1$, $0 < s \leq h$ (by the construction of $d$) and
\[ 2m < 2r_1 - 1 < 2r_2 - 1 < \cdots < 2r_s - 1, \quad 2m \leq 2r_{s+1} < 2r_{s+2} < \cdots < 2r_h. \]
Since all the summands of all the elements $d_{2(nq+i)-1} - e_i$ are independent and since $g \in H$, by the construction of the elements $d_{2n}$ and (11), there is a subset $\Omega$ of the set $\{s+1, \ldots, r_h\}$ such that
\[ l_s d_{2r_s-1} + \sum_{w \in \Omega} l_w d_{2r_w} \in H. \]
(12)

Step 3. By Step 2, to prove the lemma it is enough to find $m_0$ such that (12) does not hold. We consider two cases a) and b) separately.

Assume that a) holds. Set $m_0 = 4q(j_0+1)(k+1)$. Then $d_{2r_s-1} - e_{r_s(\text{mod } q)}$ contains exactly
\[ t = \frac{1}{q}(r_s - r_s(\text{mod } q)) > \frac{1}{q}(m_0 - q) \geq 4k + 3 \]
independent summands of the form $e_j$ with $j \geq \left(\frac{t(t-1)}{2} + 1\right) > m_0 > j_0$. Since $l_s d_{2r_s-1} \neq 0$ and $u_{j_0}$ is divided by every $u_0, \ldots, u_{q-1}$, we may assume that $l_s$ is not divided by $u_{j_0}$. So, $l_s d_{2r_s-1}$ contains at least $4k + 3$ non-zero independent summands of the form $l_s e_j$ with $j > j_0$. 
Since $|\Omega| < h - s \leq k$ and $l_n d_{2r_0}$ has the form $a_v e_v$, the conclusion of (12) does not hold. Thus, $g \not\in A(k, 2m_0)$.

Assume that b) holds. Choose $j_0 > q$ such that $u_j > 2(k + 1)$ for every $j > j_0$. Set $m_0 = 4j_0(q + 1)(k + 1)$. Then $d_{2r_s - 1} - e_{r_j(\mod q)}$ contains at least $\frac{1}{q}(r_s - r_j(\mod q)) > \frac{1}{q}(m_0 - q) > 4j_0(k + 1)$ summands that are multiples of $e_j$. So, since $|l_s| \leq k + 1, l_s d_{2r_s - 1}$ contains at least $3(k + 1)$ non-zero independent summands of the form $l_s e_j$ with $j > j_0$.

Since $|\Omega| < h - s \leq k$ and $l_n d_{2r_0}$ has the form $a_v e_v$, the conclusion of (12) does not hold. Thus, $g \not\in A(k, 2m_0)$. \hfill \square

4. Proofs of Theorems 3, 4 and 5. Following [3], we say that a sequence $u = \{u_n\}$ is a TB-sequence in a group $G$ if there is a precompact Hausdorff group topology on $G$ in which $u_n \to 0$.

Proof of Theorem 1. Let $G$ be an infinite Abelian group. It is known ([6]) that $G$ admits a non-trivial TB-sequence $u$. As it was noted in [8], a sequence $u$ is a TB-sequence if and only if it is a T-sequence and $(G, u)$ is maximally almost periodic. So $n(G, u) = 0$. Thus, $G$ admits a complete non-discrete Hausdorff group topology with trivial von Neumann radical. \hfill \square

Let $X$ be an Abelian topological group and $u = \{u_n\}$ a sequence of elements of $X^\wedge$. Following D. Dikranjan, C. Milan and A. Tonolo ([7]), we denote by $s_u(X)$ the set of all $x \in X$ such that $(u_n, x) \to 1$.

A proof of the following lemma can be found, for example, in [16, Example 2.6.3].

Lemma 8. Let $d_{2n} = \frac{1}{p^n} \in \mathbb{Z}(p^\infty)$ and $\hat{d} = \{d_{2n}\}$. Then $x \in s_{\hat{d}}(\Delta_p)$ if and only if there exists $m = m(x) \in \mathbb{Z}$ such that

$$\lambda, x = \exp(2\pi im\lambda) \text{ for all } \lambda \in \mathbb{Z}(p^\infty).$$

(13)

In other words, $x \in s_{\hat{d}}(\Delta_p)$ if and only if $x = m1$ for some $m \in \mathbb{Z}$. In particular, $\text{Cl} \left(s_{\hat{d}}(\mathbb{Z}(p^\infty))\right) = \Delta_p$.

The following theorem is the algebraic part of [8, Theorem 4]. It shall be used to compute von Neumann kernels.

Theorem 7. If $d = \{d_n\}$ is a T-sequence of an Abelian group $G$ then $n(G, d) = s_d((G_d)^\wedge)$ algebraically.

Another ingredient of the proof is the following reduction principle.

Lemma 9. Let $H$ be a subgroup of an Abelian group $G$. If there exists a subgroup $G'$ of $G$ containing $H$ such that $H \in \mathcal{N}RC(G')$ (or $H \in \mathcal{N}RC(G')$) then $H \in \mathcal{N}RC(G)$ (respectively, $H \in \mathcal{N}RC(G)$).

Proof. Since $H \in \mathcal{N}RC(G')$, there exists a Hausdorff group topology $\tau'$ on $G'$ such that $H = n(G', \tau')$. Furthermore, if $H \in \mathcal{N}RC(G')$ then $\tau'$ can be chosen to be complete. Let $\tau$ be the group topology on $G$ such that $G' \in \tau$ and $(G', \tau')$ is a subspace of $(G, \tau)$. We note that $\tau$ is complete whenever $\tau'$ is. Since $(G', \tau')$ is an open subgroup of $(G, \tau)$, one has $n(G', \tau') = n(G, \tau)$ (see also [8, Lemma 4] for a more general statement). This proves that $H = n(G, \tau)$. \hfill \square
Proof of Theorem 3. Our goal is to construct a $T$-sequence $d$ in $G$ satisfying
\[ s_d((G_d)^\wedge) = H. \] (14)

Combining this with Theorem 7, we obtain that $n(G, d) = H$. Since $(G, d)$ is complete, this shows that $H \in \mathcal{NRC}(G)$.

The rest of the proof is split into the following four cases.

(1) $H$ is infinite. (2) $H$ is finite and $G$ is not torsion. (3) $H$ is finite, $G$ is torsion but not reduced. (4) $H$ is finite and $G$ is both torsion and reduced.

Since $H$ is finitely generated, it is a direct finite sum of cyclic groups.

(1) $H$ is infinite. Then $H = \langle e_0 \rangle \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_q \rangle$, where $\langle e_0 \rangle \cong \mathbb{Z}$. Applying the reduction principle (Lemma 9), we may assume that $G = H$. Choose any prime $p$ and let $\varepsilon_n = 1$ for all $n \in \mathbb{N}$. Let $d = \{d_n\}$ be the $T$-sequence in $G$ as in Lemma 4. To establish (14), it suffices to prove that $s_d((G_d)^\wedge) = 0$. Let
\[ \omega = x_0 + x_1 + \cdots + x_{q-1} \in (G_d)^\wedge, \quad x_i \in \langle e_i \rangle^\wedge, \text{ and } (d_n, \omega) \to 1. \]

Then $(d_{2n}, \omega) = (p^n e_0, x_0) \to 1$. Hence $x_0 \in \mathbb{Z}(p^\infty)$ (see [2] or [4, Remark 3.8]). If $x_0 = \frac{x}{p^r}, \rho \in \mathbb{Z}, \tau > 0$, then for any $n = qs + i > \tau$ we have $(d_{2qs+i-1}, \omega) = (e_i, x_i)$ for every $0 \leq i < q$. So $(d_{2qs+i-1}, \omega) \to 1$ only if $x_i = 0$ for every $i$. Hence $\omega = 0$.

(2) $H$ is finite and $G$ is not torsion. Fix $e_0 \in G$ such that $\langle e_0 \rangle \cong \mathbb{Z}$. Since $H$ is finite, $H \cap \langle e_0 \rangle = 0$. Let $H = \langle e_1 \rangle \oplus \cdots \oplus \langle e_q \rangle$ be a direct decomposition of $H$. Then $G' = \langle e_0 \rangle \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_q \rangle \cong \mathbb{Z} \oplus H$ is a subgroup of $G$ containing $H$. By Lemma 9, we may assume that $G = G'$.

Choose any prime $p$ and set $\varepsilon_n = 1$ if $n \equiv 0 \mod q > 0$ and $\varepsilon_n = 0$ if $n \equiv 0 \mod q = 0$. Let $d = \{d_n\}$ be the $T$-sequence in $G$ as in Lemma 4. To establish (14), it suffices to show that $\text{Cl}(s_d(G_d^{\wedge})) = \mathbb{Z}^\wedge = \mathbb{T}$. Let
\[ \omega = x_0 + x_1 + \cdots + x_{q-1} \in (G_d)^\wedge, \quad x_i \in \langle e_i \rangle^\wedge, \text{ and } (d_n, \omega) \to 1. \]

Then $(d_{2n}, \omega) = (p^n e_0, x_0) \to 1$. Hence $x_0 \in \mathbb{Z}(p^\infty)$ (see [2] or [4, Remark 3.8]). Let $x_0 = \frac{x}{p^r}, \rho \in \mathbb{Z}, \tau > 0$. Then for any $n = qs + i > \tau$ we have $(d_{2qs+i-1}, \omega) = 1$ if $i = 0$, and $(d_{2qs+i-1}, \omega) = (e_i, x_i)$ if $0 < i < q$. So $(d_{2qs+i-1}, \omega) \to 1$ only if $x_i = 0$ for every $0 < i < q$. Thus $\omega = x_0$, where $x_0 \in \mathbb{Z}(p^\infty) \subset \mathbb{T}$. So $s_d(G_d^{\wedge}) \subset \mathbb{Z}(p^\infty)$. Let us prove the converse inclusion. Let $\omega = x_0 + \frac{x}{p^r} \in \mathbb{Z}(p^\infty), \rho \in \mathbb{Z}, \tau > 0$. By the definition of $d_m$ we have
\[ (d_{2n}, x_0) = \exp\left\{ 2\pi i \frac{p^r \rho}{p^r} \right\}, \quad (d_{2n-1}, x_0) = \exp\left\{ 2\pi i \frac{f \rho}{p^r} \right\}. \]

Thus, $(d_m, x_0) = 1$ for every $m > 2\tau$ and hence $s_d(G_d^{\wedge}) \supseteq \mathbb{Z}(p^\infty)$.

Hence $s_d(G_d^{\wedge}) = \mathbb{Z}(p^\infty)$ and $\text{Cl}(s_d(G_d^{\wedge})) = \mathbb{T}$.

(3) $H$ is finite, $G$ is torsion but not reduced. Then $G$ contains a subgroup isomorphic to $\mathbb{Z}(p^\infty)$ for some prime $p$. By Lemma 9, we may assume that $G \cong \mathbb{Z}(p^\infty) + H$. Let $H = \langle e_0 \rangle \oplus \cdots \oplus \langle e_q \rangle$ be a direct decomposition of $H$. Let $d = \{d_n\}$ be the $T$-sequence in $G$ as in Lemma 5. To establish (14), it suffices to prove that $H^\perp = \text{Cl}(s_d((G_d)^\wedge))$.

Let us prove first that $s_d((G_d)^\wedge) \subseteq H^\perp$. We use the notations from Lemma 3. Assume that
\[ \omega = x_0 + y \in s_d((G_d)^\wedge), \quad \text{where } x_0 \in \Delta_p, y \in H^\perp. \]

Then $(d_{2n}, \omega) = (d_{2n}, x_0) \to 1$. By (13), $x_0 = m \mathbf{1}$ for some $m \in \mathbb{Z}$ and $(\lambda, x_0) = \exp(2\pi im\lambda), \forall \lambda \in \mathbb{Z}(p^\infty)$. In particular, $(f_n, x_0) = \exp(2\pi imf_n)$ for every $n \geq 1$. By (8), we obtain that $(f_n, x_0) \to 1$. So, for every $0 \leq i < q$, we have $(s \to \infty)$.
\[(d_{2(sq+i)-1}, \omega) = (\tilde{f}_{sq} + e_i, \omega) = (\tilde{f}_{sq}, x_0) \cdot (e_i, \omega) \rightarrow (e_i, \omega) = 1.\]

So \(\omega \in \langle e_i \rangle\) for every 0 \(\leq i < q\). Hence \(\omega \in H^\perp\).

Let us show now the reverse inclusion \(H^\perp \subseteq \text{Cl}(s_d((G_d)^\wedge))\). By Lemma 3, it is enough to prove that \(\{S_0 \cup \langle p^k1 \rangle\} \subseteq s_d((G_d)^\wedge)\). Let \(\omega \in S_0\). Then, by the construction of \(S_0\), for \(\omega = x_0 + y\) and \(x_0_0 = (x_0, \ldots, x_{k-1}, 0, \ldots) = m \cdot 1\) we have

\[(d_{2n}, \omega) = \exp\left\{2\pi i \frac{1}{p^n} (x_0 + \cdots + x_{k-1}p^{k-1})\right\} \rightarrow 1, \text{ at } n \rightarrow \infty.\]

\[(d_{2(sq+i)-1}, \omega) = (\tilde{f}_{sq}, x_0) \cdot (e_i, \omega) = (\tilde{f}_{sq}, x_0) = \exp\left\{2\pi i \tilde{f}_{sq}m\right\} \rightarrow 1.\]

Hence \(S_0 \subseteq s_d((G_d)^\wedge)\). For \(p^k1\) we obtain

\[(d_{2n}, p^k1) = \exp\left\{2\pi i \frac{1}{p^n} \cdot p^k\right\} \rightarrow 1, \text{ at } n \rightarrow \infty.\]

\[(d_{2(sq+i)-1}, p^k1) = (\tilde{f}_{sq}, p^k1) = \exp\left\{2\pi i \tilde{f}_{sq}p^k\right\} \rightarrow 1.\]

Thus, \(H^\perp = \text{Cl}(s_d((G_d)^\wedge))\).

(4) \(H\) is finite and \(G\) is both torsion and reduced. Since \(G\) is not bounded, \(G\) contains an independent sequence \(\{b_n\}\) of elements such that \(o(b_n) \rightarrow \infty\). Let \(H = \langle e_0 \rangle \oplus \cdots \oplus \langle e_{q-1} \rangle\) be a direct decomposition of \(H\). Using \(q\) times Lemma 1, we can find \(m \in \mathbb{N}\) such that the sequence \(\{e_0, e_1, \ldots, e_{q-1}, b_m, b_{m+1}, \ldots\}\) is independent. Define \(e_{q+k} = b_{m+k}\) for all integers \(k \geq 0\). Clearly, \(u_i := o(e_i) < \infty\) for every \(i \geq 0\) and \(u_i \rightarrow \infty\). By Lemma 9, we may assume that

\[G = H \oplus \bigoplus_{i=q}^\infty \langle e_i \rangle = \bigoplus_{i=q}^\infty \langle e_i \rangle.\]

Then \((G_d)^\wedge = \prod_{i=q}^\infty \langle e_i \rangle\).

Let \(d = \{d_i\}\) be the \(T\)-sequence in \(G\) as in Lemma 7. To establish (14), it suffices to prove that

\[\text{Cl}(s_d((G_d)^\wedge)) = \prod_{i=q}^\infty \langle e_i \rangle.\] (15)

We modify the proof of [11, Proposition 3.3]. Let \(\omega = (a_0, a_1, \ldots) \in s_d((G_d)^\wedge)\). By definition, there exists \(N \in \mathbb{N}\) such that \(|1 - (d_{2n}, \omega)| < 0.1, \forall n > N\). Thus, there is \(N_0 > N\) such that \(|1 - (d_{2n}, \omega)| = |1 - (d_{2n}, a_j)| < 0.1, \forall j \geq 1, u_i - 1, \text{ for every } l > N_0\). This means that \(a_l = 0\) for every \(l > N_0\). So \(\omega \in \bigoplus_{i=q}^\infty \langle e_i \rangle \subseteq (G_d)^\wedge\). Since \((d_{2n(q+i)}, \omega) \rightarrow 1\) too and \((d_{2n(q+i)-1}, \omega) = (e_i, a_i)\) for all sufficiently large \(n\), we obtain that \(a_i = 0\) for any \(i = 0, \ldots, q - 1\). Thus \(s_d((G_d)^\wedge) \subseteq \bigoplus_{i=q}^\infty \langle e_i \rangle\). The converse inclusion is trivial. Hence \(s_d((G_d)^\wedge) = \bigoplus_{i=q}^\infty \langle e_i \rangle\) and it is dense in \(\prod_{i=q}^\infty \langle e_i \rangle\). So, \(\text{Cl}(s_d((G_d)^\wedge)) = \prod_{i=q}^\infty \langle e_i \rangle.\) \(\square\)

**Proof of Theorem 4.** Let us prove the implication (1) \(\Rightarrow\) (2). Assume that \(G\) contains a subgroup of the form \(H^\omega\). Let \(H = \langle e_0^0 \rangle \oplus \cdots \oplus \langle e_0^q \rangle\) with \(e_0^j \in G\). By our assumption, \(G\) contains a subgroup of the form \(Y_0 \oplus Y_1 \oplus \cdots \oplus Y_q\), where

\[Y_j = \bigoplus_{i=0}^\infty \langle e_i^j \rangle, \text{ } 0 \leq j \leq q, \text{ } e_i^j \in G;\]

and the order of \(c_i^j\) is equal to \(u_j\) for every \(i \geq 0\). By the reduction principle (Lemma 9), we may assume that \(G = Y_0 \oplus Y_1 \oplus \cdots \oplus Y_q\). Further, since the von Neumann radical of a product of topological groups is the product of their von Neumann radicals, it is enough to construct a Hausdorff group topology \(\tau_j\) on \(Y_j\) such that \(n(Y_j, \tau_j) = \langle c_0^j \rangle\). So, we can restrict
ourselves to the case $H = \langle e_0 \rangle$ and $G = H \oplus \bigoplus_{i=1}^{\infty} \langle e_i \rangle = \bigoplus_{i=0}^{\infty} \langle e_i \rangle$, where the order of $e_i$ is equal to $u$ for every $i \geq 0$.

Let $d = \{d_n\}$ be the $T$-sequence in $G$ as in Lemma 7. As in the proof of Theorem 3, we only need to show that equality (14) holds. To this end, it is enough to prove that $\text{Cl}(s_d((G_d)\rangle)) = \prod_{i=1}^{\infty} \langle e_i \rangle$. The proof of this equality is the same as the proof of equality (15) in item (4) of Theorem 3 (where one needs to take $q = 1$).

Implication (2) $\Rightarrow$ (3) is trivial.

Let us prove implication (3) $\Rightarrow$ (1). Let $H = \langle e_1 \rangle \oplus \cdots \oplus \langle e_q \rangle$. Assume that (1) fails. By Lemma 2, there exists $1 \leq i_0 \leq q$ such that $G$ does not contain a subgroup of the form $\langle e_{i_0} \rangle^{(\omega)}$. Set $n_{i_0} = o(e_{i_0})$. Let $n_{i_0} = p_1^{k_1} \cdots p_l^{k_l}$ and $\exp G = p_1^{a_1} \cdots p_l^{a_l} \cdot p_{l+1}^{a_{l+1}} \cdots p_t^{a_t}$, where $p_1, \ldots, p_t$ are distinct prime integers. For $1 \leq j \leq l$ we put $m_j = \exp G/p_j^{a_j-k_j+1}$. Set $\pi_j: G \to G$, $\pi_j(g) = m_j g$, and $G_j = \pi_j(G)$. Then $\pi_j(e_{i_0}) \neq 0$ for every $1 \leq j \leq l$.

(a) Let us prove that there exists $1 \leq j \leq l$ such that $G_j$ is finite.

Assume for a contradiction that $G_j$ is infinite for every $j$. Since $\exp G_j = p_j^{a_j-k_j+1}$, $G_j$ contains a subgroup of the form

$$\bigoplus_{i=1}^{\infty} \langle \tilde{b}_i \rangle, \quad \text{where } \tilde{b}_i \in G_j \text{ and } \langle \tilde{b}_i \rangle \cong \mathbb{Z}(p_j).$$

Thus, for every $i \geq 1$ there exists an element $b_i \in G$ such that $o(b_i) = p_j^{k_j}$ and $\pi_j(b_i) = \tilde{b}_i$. Indeed, if $y$ is any element such that $\pi_j(y) = \tilde{b}_i$, then we may put $b_i = c_j y$, where $c_j = \exp G/p_j^{k_j}$ (and $m_j = c_j \cdot p_j^{k_j-1}$).

Let us prove that the sequence $\{b_i\}$ is independent. Assuming the converse we obtain that

$$s_1 b_{i_1} + s_2 b_{i_2} + \cdots + s_w b_{i_w} = 0 \quad \text{and} \quad s_r b_{i_r} \neq 0, \ 1 \leq r \leq w. \quad (16)$$

Let $s_r = p_j^{v_r} \cdot A_r$, where $p$ and $A_r$ are coprime. Set $v = \min\{v_1, \ldots, v_w\}$. By our choice of $b_i$ we have $v < k_j$. Thus, if we multiply equality (16) by $c_j \cdot p_j^{k_j-v-1}$ then we obtain

$$A_1 p_j^{v_1-v} b_{i_1} + A_2 p_j^{v_2-v} b_{i_2} + \cdots + A_w p_j^{v_w-v} b_{i_w} = 0.$$

Since there exists $r$ such that $v_r = v$ and $A_r p_j^{v_r-v} b_{i_r} = A_r \tilde{b}_{i_r} \neq 0$, we obtain that the elements $\tilde{b}_i$ are dependent. Since the sequence $\{b_i\}$ is independent, we obtain a contradiction.

Since the sequence $\{b_i\}$ is independent, $G$ contains a subgroup of the form

$$\bigoplus_{i=1}^{\infty} \langle b_i \rangle, \quad \text{where } \langle b_i \rangle \cong \mathbb{Z}(p_j^{k_j}),$$

for every $1 \leq j \leq l$. Since $p_1, \ldots, p_t$ are coprime, $G$ contains a subgroup of the form $\langle e_{i_0} \rangle^{(\omega)}$. This is a contradiction. Thus there exists $1 \leq j \leq l$ such that $G_j$ is finite.

(b) Let us prove that there is no Hausdorff group topology $\tau$ such that $n(G, \tau) = H$. (We repeat the arguments of D. Remus (see [5])).

Let $\tau$ be any Hausdorff group topology on $G$ and let $j$ be such that $G_j$ is finite. Then $\text{Ker}(\pi_j)$ is open and closed. So $n(G, \tau) \subseteq \text{Ker}(\pi_j)$. Since $0 \neq \pi_j(e_{i_0}) \in H/\text{Ker}(\pi_j)$, we obtain that $H \neq n(G, \tau)$.

Proof of Theorem 5. (1) is equivalent to (2) by Corollary 1.

Let us prove that (2) yields (3). If $G$ does not satisfy condition (3) then $\exp G < \infty$. Let $\exp G = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$, where $p_1, \ldots, p_t$ are distinct prime integers. By Lemma 2, there
exists \( 1 \leq i_0 \leq t \) such that \( G \) does not contain a subgroup of the form \( \mathbb{Z}(p_{i_0}^{a(i)})^{(\omega)} \). Set \( H = \langle e_{i_0} \rangle \), where \( o(e_{i_0}) = p_{i_0}^{a(i)} \). Then \( H \) is finite and, by Theorem 4, \( H \not\in \mathcal{NR}(G) \). This is a contradiction. Thus, (2) yields (3).

Let us prove that (3) yields (1). If \( \exp G = \infty \), the assertion follows from Theorem 3. If \( \exp G < \infty \), the assertion follows from Theorem 4.

**Acknowledgement:** I am deeply indebted to Professor Shakhmatov for numerous suggestions which essentially improve the exposition of the article. It is a pleasure to thank A. Leiderman.

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Received 12.07.2012