УДК 512.544

S. S. Gabriyelyan

FINITELY GENERATED SUBGROUPS AS VON NEUMANN RADICALS OF AN ABELIAN GROUP

S. S. Gabriyelyan. *Finitely generated subgroups as von Neumann radicals of an Abelian group*, Mat. Stud. **38** (2012), 124–138.

Let G be an infinite Abelian group. We give a complete characterization of those finitely generated subgroups of G which are the von Neumann radicals for some Hausdorff group topologies on G. It is proved that every infinite finitely generated Abelian group admits a complete Hausdorff minimally almost periodic group topology. The latter result resolves a particular case of Comfort's problem.

С. С. Габриелян. *Конечно порожденные подгруппы абелевой группы G, являющиеся ее радикалами фон Неймана* // Мат. Студiї. – 2012. – Т.38, №2. – C.124–138.

В статье дается полная характеризация конечно порожденных подгрупп бесконечной абелевой группы G, являющихся радикалами фон Неймана для хаусдорфовых топологий на G. Доказано что каждая бесконечная конечно порожденная абелевая группа допускает полную хаусдорфовую минимально почти периодическую топологию. Последний результат частично решает проблему Комфорта.

1. Introduction. Let G be an Abelian group G. Recall that G is *bounded* if there exists a positive integer n such that $ng = 0$ for every $g \in G$, and the minimal integer n with this property is called the *exponent* of G denoted by $\exp(G)$. When G is not bounded, we write $\exp(G) = \infty$ and say that G has *infinite exponent*.

For an Abelian topological group X, $X^\wedge$ denotes the group of all continuous characters on X endowed with the compact-open topology and

$$\mathbf{n}(X) = \bigcap_{\chi \in X^\wedge} \ker\chi$$

denotes the *von Neumann radical* of X. The richness of the dual group $X^\wedge$ is one of the most important properties of X, and it is characterized by the von Neumann radical $\mathbf{n}(X)$.

Following J. von Neumann ([12]), a group X is called *minimally almost periodic* (MinAP) if $\mathbf{n}(X) = X$, and it is called *maximally almost periodic* if $\mathbf{n}(X) = 0$.

The following proposition (proved in Section 3) is a simple corollary of the main result of [6].

Theorem 1. *Every infinite Abelian group admits a complete non-discrete Hausdorff group topology with trivial von Neumann radical.*

2010 *Mathematics Subject Classification:* 22A10, 43A40, 54H11.
Keywords: characterized group, T-sequence, von Neumann radical, finitely generated subgroup.

A much deeper question is whether every infinite Abelian group admits a Hausdorff group topology with a *non-zero* von Neumann radical. A positive answer to this question was given by M. Ajtai, I. Havas and J. Komlós ([1]). E. G. Zelenyuk and I. V. Protasov ([15]) proved that every infinite Abelian group G admits a *complete* Hausdorff group topology for which characters do not separate points. I. V. Protasov ([14]) posed the question whether every infinite Abelian group admits a minimally almost periodic group topology. A simple example of a bounded group G which does not admit any Hausdorff group topology τ such that (G, τ) is minimally almost periodic is given by D. Remus ([5]). This justifies the following problem.

Question (Comfort's Problem 521 [5]). *Does every Abelian group which is not of bounded order admit a minimally almost periodic topological group topology? What about the countable case?*

Moreover, it was not known even whether every infinite finitely generated Abelian group G admits a Hausdorff minimally almost periodic group topology. We answer this question in the affirmative theorem.

Theorem 2. *Every infinite finitely generated Abelian group G admits a complete Hausdorff minimally almost periodic group topology.*

Let G be an infinite Abelian group and H its infinite finitely generated subgroup. By Theorem 2, there is a Hausdorff MinAP group topology τ' on H. Let τ be a group topology on G such that $H \in \tau$ and $\tau|_H = \tau'$. Then the von Neumann radical of (G, τ) is H (see Lemma 9 below). So, every *infinite* finitely generated subgroup of an Abelian group G can be considered as the von Neumann radical for some Hausdorff group topology on G. Noting that every finite group is finitely generated, it is natural to ask, for which *finite* subgroup H of an infinite Abelian group G there is a Hausdorff group topology τ on G such that H is the von Neumann radical of (G, τ).

Let G be an infinite Abelian group. We denote by $\mathcal{NR}(G)$ (by $\mathcal{NRC}(G)$) the set of all subgroups H of G for which there exists a (complete) non-discrete Hausdorff group topology τ on G such that $\mathbf{n}(G, \tau) = H$. It is clear that $\mathcal{NRC}(G) \subseteq \mathcal{NR}(G)$. Therefore, by Theorem 1, $\{0\} \in \mathcal{NRC}(G)$, and, by [15], $\mathcal{NRC}(G) \neq \{\{0\}\}$. The general question of describing the sets $\mathcal{NR}(G)$ and $\mathcal{NRC}(G)$ was raised in [9].

The main goal of the paper is to describe all finitely generated subgroups of an infinite Abelian group G which are contained in $\mathcal{NR}(G)$.

For an Abelian group G, the symbols $\mathcal{FGS}(G)$ and $\mathcal{FS}(G)$ denote the set of all finitely generated subgroups and finite subgroups G, respectively.

Theorem 2 is an immediate consequence of the following theorem.

Theorem 3. *Let G be an Abelian group that is not bounded. Then for every finitely generated subgroup H of G there exists a complete Hausdorff group topology τ on G such that $H = \mathbf{n}(G, \tau)$, i.e., $\mathcal{FGS}(G) \subseteq \mathcal{NRC}(G)$.*

The case of bounded groups is more complicated. The direct sum of ω copies of an Abelian group H we denote by $H^{(\omega)}$.

Theorem 4. *Let G be an infinite Abelian bounded group. Let $H \in \mathcal{FS}(G) = \mathcal{FGS}(G)$. Then the following statements are equivalent:* 1) *G contains a subgroup of the form $H^{(\omega)}$*; 2) $H \in \mathcal{NRC}(G)$; 3) $H \in \mathcal{NR}(G)$.

As an evident corollary of Theorems 3 and 4 we obtain the following result resolving [9, Problem 3].

Corollary 1. *Let G be an infinite Abelian group and $H \in \mathcal{FGS}(G)$. Then $H \in \mathcal{NR}(G)$ if and only if $H \in \mathcal{NRC}(G)$, i.e.,*

$$\mathcal{FS}(G) \cap \mathcal{NR}(G) = \mathcal{FS}(G) \cap \mathcal{NRC}(G), \text{ and } \mathcal{FGS}(G) \cap \mathcal{NR}(G) = \mathcal{FGS}(G) \cap \mathcal{NRC}(G).$$

By Corollary 1, in all subsequent theorems and corollaries of this section only the (simpler) option $\mathcal{NR}(G)$ is considered.

Also as a trivial corollary of Theorems 3 and 4 we obtain the main result of [9].

Corollary 2 ([9]). *An Abelian group G admits a Hausdorff group topology with non-trivial finite von Neumann radical if and only if it is not torsion free.*

Proof. Clearly, if G admits a Hausdorff group topology with non-trivial von Neumann radical it must contain a nonzero element of finite order. Conversely, since every finite subgroup is finitely generated and since any infinite Abelian bounded group contains a subgroup of the form $\mathbb{Z}(p)^{(\omega)}$ for some prime p, the corollary immediately follows from Theorems 3 and 4. □

The following problem was posed in [9, Problem 6]: describe all infinite Abelian groups G such that $\mathcal{FS}(G) \subset \mathcal{NR}(G)$. (We note that this inclusion is strict since $\mathcal{NR}(G)$ contains a countably infinite subgroup.) A solution to this problem is provided by the following theorem.

Theorem 5. *Let G be an infinite Abelian group. Then the following statements are equivalent:* 1. $\mathcal{FGS}(G) \subseteq \mathcal{NR}(G)$; 2. $\mathcal{FS}(G) \subset \mathcal{NR}(G)$; 3. *$G$ satisfies one of the following conditions: 1)* $\exp G = \infty$; *2)* $\exp G = m$ *is finite and G contains a subgroup of the form* $\mathbb{Z}(m)^{(\omega)}$.

We can reformulate Theorem 5 for bounded groups as follows. It is well known that a bounded group G has the form $G = \bigoplus_{p\in M} \bigoplus_{i=1}^{n_p} \mathbb{Z}(p^i)^{(k_{i,p})}$, where M is a finite set of prime numbers. Leading Ulm-Kaplansky invariants of G are the cardinal numbers $k_{n_p,p}, p \in M$.

Corollary 3. *All finite subgroups H of an infinite bounded Abelian group G belong to $\mathcal{NR}(G)$ if and only if all leading Ulm-Kaplansky invariants of G are infinite.*

The article is organized as follows. In Section 2 we prove some auxiliary lemmas that will be used to prove the main results. In Section 3 special T-sequences are constructed for some Abelian groups. These T-sequences are used to define the topologies with the desired property in Theorem 3. In the last Section 4 we prove Theorems 3, 4 and 5.

2. Auxiliary lemmas. Let us recall that a subset X of an Abelian group G is called *independent* provided that for every finite sequence $x_1, \dots, x_n$ of pairwise distinct elements of X and each sequence $m_1, \dots, m_n$ of integer numbers, if $m_1x_1 + \dots + m_nx_m = 0$ then $m_ix_i = 0$ for all $i \in \{1, \dots, n\}$.

Lemma 1. *Let $\{b_n\}_{n\in\omega}$ be an independent sequence of an Abelian group G. Then for every nonzero element g of G there is n_0 such that the set $\{g, b_{n_0}, b_{n_0+1}, \dots\}$ is independent.*

Proof. Set $H = \bigoplus_{n\in\omega}\langle b_n\rangle$. If the intersection $H\cap\langle g\rangle$ is trivial then one can take $n_0 = 0$. Otherwise, $H\cap\langle g\rangle$ is a subgroup of $\langle g\rangle$, hence $H\cap\langle g\rangle = \langle mg\rangle \neq 0$ for some $m \in \mathbb{N}$. The support of $mg \in \bigoplus_n\langle b_n\rangle$ is finite, so there exists k such that $mg \in \bigoplus_{n=0}^{k}\langle b_n\rangle$. Thus, $H\cap\langle g\rangle = \langle mg\rangle \subseteq \bigoplus_{n=0}^{k}\langle b_n\rangle$. Therefore, $\langle g\rangle\cap\bigoplus_{n=k+1}^{\infty}\langle b_n\rangle = 0$. Putting $n_0 = k+1$ we obtain that the set $\{g, b_{n_0}, b_{n_0+1}, \dots\}$ is independent. □

As usual, for an element g of an Abelian group G, we denote by $\langle g\rangle$ the subgroup of G generated by g.

For the proof of Theorem 4 we need the following lemma.

Lemma 2. *Let G be an infinite Abelian group and $e_1, \dots, e_q \in G$. Then the following assertions are equivalent:*

1. *G contains a subgroup of the form $\langle e_1\rangle^{(\omega)} \oplus \langle e_2\rangle^{(\omega)} \oplus \cdots \oplus \langle e_q\rangle^{(\omega)}$;*
2. *G contains a subgroup of the form $\langle e_i\rangle^{(\omega)}$ for every $1 \leqslant i \leqslant q$.*

Proof. We need to prove only the implication (2) ⇒ (1). It is easy to see that we can restrict ourselves to the case when e_i has a finite order n_i for every $1 \leqslant i \leqslant q$.

Let $p_1^{b_1}\dots p_l^{b_l}$ be the prime decomposition of the least common multiple of $n_1, \dots, n_q$. Since any $p_j^{b_j}$ is a divisor of some $n_{k(j)}$, by hypothesis, G contains a subgroup of the form

$$\bigoplus_{n=1}^{\infty} H_n^j, \quad \text{where } H_n^j \cong \mathbb{Z}(p_j^{b_j}).$$

Thus, G contains the following subgroup

$$\bigoplus_{i=1}^{q}\left(\bigoplus_{n=1}^{\infty} H^1_{nq+i} \oplus \bigoplus_{n=1}^{\infty} H^2_{nq+i} \oplus \cdots \oplus \bigoplus_{n=1}^{\infty} H^l_{nq+i}\right).$$

Evidently, the group $\bigoplus_{n=1}^{\infty} H^1_{nq+i} \oplus \bigoplus_{n=1}^{\infty} H^2_{nq+i} \oplus \cdots \oplus \bigoplus_{n=1}^{\infty} H^l_{nq+i}$ contains a subgroup of the form $\langle e_i\rangle^{(\omega)}$ for every $1 \leqslant i \leqslant q$. □

Let us consider the group $\mathbb{Z}(p^\infty)$ with discrete topology. Then $\mathbb{Z}(p^\infty)^\wedge = \Delta_p$ is the compact group of p-adic integers which elements are denoted by $x = (a_i), 0 \leqslant a_i < p$, and the identity is $\mathbf{1} = (1, 0, 0, \dots)$. By [10, Remark 10.6], $\langle\mathbf{1}\rangle$ is dense in Δ_p and, by [10, 25.2], $(\lambda, \mathbf{1}) = \exp\{2\pi i\cdot\lambda\}$ for every $\lambda \in \mathbb{Z}(p^\infty)$. Following [10, 10.4], we denote by $\Lambda_k, k \geqslant 1$, the set of all $\mathbf{x} = (x_0, \dots, x_{k-1}, x_k, \dots) \in \Delta_p$ such that $x_0 = \cdots = x_{k-1} = 0$ and put $\Lambda_0 = \Delta_p$. Note that Λ_k is just $p^k\Delta_p$.

A group G with the discrete topology is denoted by G_d. If H is a subgroup of $(G_d)^\wedge$ then $H^\perp := \{g \in G\colon (g, h) = 1\ \forall h \in H\}$. We use the following lemma to prove Theorem 3.

Lemma 3. *Let $G = \mathbb{Z}(p^\infty) + H$, where H is a finite group, endowed with the discrete topology. Let H_1 be a finite group such that $G = \mathbb{Z}(p^\infty) \oplus H_1$. Then there exist $k \geqslant 0$ and a finite set $S_0 \subset \langle\mathbf{1}\rangle \oplus H_1^\wedge$ such that $H^\perp = \langle S_0\rangle + \Lambda_k$. In particular, the finitely generated subgroup $\langle S_0 \cup \{p^k\mathbf{1}\}\rangle$ is dense in $H^\perp$.*

Proof. Let $H = \langle e_1\rangle \oplus \cdots \oplus \langle e_l\rangle \oplus \langle e_{l+1}\rangle \oplus \cdots \oplus \langle e_q\rangle$, where $o(e_i) = p^{w_i}$ for $1 \leqslant i \leqslant l$ and $o(e_i) = p_i^{w_i}, p_i \neq p$, for $l < i \leqslant q$. Then for some integer t we have

$$H_1 = \langle g_1\rangle \oplus \cdots \oplus \langle g_t\rangle \oplus \langle e_{l+1}\rangle \oplus \cdots \oplus \langle e_q\rangle,$$

where $o(g_i) = p^{r_i}$ for some natural number r_i and $e_i = a_0^i g_0 + a_1^i g_1 + \cdots + a_t^i g_t,\ 1 \leqslant i \leqslant l$, where $g_0 = \frac{1}{p^{r_0}} \in \mathbb{Z}(p^\infty)$, and $0 \leqslant a_j^i < p^{r_i}$ for every $0 \leqslant j \leqslant t$. Since H_1 is finite, we will identify H_1 with $H_1^\wedge$. Let $\omega = \mathbf{x} + \lambda_1 g_1 + \cdots + \lambda_t g_t + \lambda_{t+1} e_{l+1} + \cdots + \lambda_{t+q-l} e_q \in G^\wedge$, where $\mathbf{x} = (x_0, x_1, \dots) \in \Delta_p$, $0 \leqslant \lambda_i < p^{r_i}$ for $1 \leqslant i \leqslant t$ and $0 \leqslant \lambda_{t+j} < p_{l+j}^{w_{l+j}}$ for $1 \leqslant j \leqslant q-l$. By definition, $\omega \in H^\perp$ iff $(\omega, e_i) = 1$ for every $1 \leqslant i \leqslant q$. In particular, for every $1 \leqslant j \leqslant q-l$,

$$(\omega, e_{l+j}) = \frac{\lambda_{t+j}}{p_{l+j}^{w_{l+j}}} = 0 (\mathrm{mod}\, 1).$$

Since $\lambda_{t+j} < p_{l+j}^{w_{l+j}}$, we have $\lambda_{t+j} = 0$ for every $1 \leqslant j \leqslant q-l$. So, $\omega \in H^\perp$ if and only if it has the form $\omega = \mathbf{x} + \lambda_1 g_1 + \cdots + \lambda_t g_t$ and $(\omega, e_i) = 1, \forall 1 \leqslant i \leqslant l$. Thus, by [10, 25.2], $\omega \in H^\perp$ if and only if $\lambda_{t+j} = 0$, for every $1 \leqslant j \leqslant q-l$, and for any $1 \leqslant i \leqslant l$, $(\mathrm{mod}\, 1)$

$$\frac{a_0^i}{p^{r_0}} \left(x_0 + x_1 p + \cdots + x_{r_0 - 1} p^{r_0 - 1} \right) + \frac{\lambda_1 a_1^i}{p^{r_1}} + \cdots + \frac{\lambda_t a_t^i}{p^{r_t}} = 0. \tag{1}$$

Denote by S_0 the set of all $\omega \in H^\perp$ which have the form

$$\omega = \mathbf{x} + \lambda_1 g_1 + \cdots + \lambda_t g_t, \; \mathbf{x} = (x_0, x_1, \dots, x_{r_0 - 1}, 0 \dots),$$

where $\mathbf{x}$ and $\lambda_1, \dots, \lambda_t$ satisfy (1). By definition, $S_0 \subset H^\perp \cap (\langle \mathbf{1} \rangle \oplus H_1^\wedge)$ and for every $\omega \in H^\perp$ there is $\omega_0 \in S_0$ such that $\omega - \omega_0 = (0, \dots, 0_{r_0 - 1}, x_{r_0}, x_{r_0 + 1}, \dots) \in \Lambda_{r_0}$. Set $k = r_0$. Then $H^\perp \subseteq \langle S_0 \rangle + \Lambda_k$. The converse inclusion follows from (1). By [10, Remark 10.6], $\langle p^k \mathbf{1} \rangle$ is dense in Λ_k. So, $\langle S_0 \cup \{p^k \mathbf{1}\} \rangle$ is dense in $H^\perp$. □

3. Construction of T-sequences. In this section we construct T-sequences which will be needed for the proofs of the main results.

Following E. G. Zelenyuk and I. V. Protasov ([15], [16]), we say that a sequence $\mathbf{d} = \{d_n\}$ in a group G is a *T-sequence* if there is a Hausdorff group topology on G with respect to which d_n converges to zero. The group G equipped with the finest group topology with this property is denoted by $(G, \mathbf{d})$. We note also that, by [16, Theorem 2.3.11], the group $(G, \mathbf{d})$ is complete.

For a sequence $\{d_n\}$ and $k, m \in \mathbb{N}$, one defines ([15])

$$A(k,m) = \{ n_1 d_{r_1} + \cdots + n_s d_{r_s} : \; m \leqslant r_1 < \cdots < r_s,$$
$$n_1, n_2, \dots, n_s \in \mathbb{Z} \setminus \{0\}, \; \sum_{i=1}^s |n_i| \leqslant k+1 \} \cup \{0\}.$$

In this section we make extensive use of the following Protasov-Zelenyuk's criterion.

Theorem 6 ([15])**.** *A sequence $\{d_n\}$ of elements of an Abelian group G is a T-sequence if and only if, for every integer $k \geqslant 0$ and for each element $g \in G$ with $g \neq 0$, there is an integer m such that $g \notin A(k,m)$.*

For a prime p and $n \in \mathbb{N}$ we set $f_n = p^{n^3 - n^2} + \cdots + p^{n^3 - 2n} + p^{n^3 - n} + p^{n^3} \in \mathbb{Z}$. Then $f_n < 2p^{n^3} \leqslant p^{n^3+1}$. For $0 < r_1 < r_2 < \cdots < r_v$ and integers $l_1, l_2, \dots, l_v$ such that $\sum_{i=1}^v |l_i| \leqslant k+1$, we have

$$|l_1 f_{r_1} + l_2 f_{r_2} + \cdots + l_v f_{r_v}| < (k+1) f_{r_v} \leqslant (k+1) p^{r_v^3 + 1}. \tag{2}$$

Lemmas 4 and 5 are slight modifications of items (1) and (2) in the proof of [9, Theorem 1].

Lemma 4. *Let $G = \langle e_0\rangle \oplus \langle e_1\rangle \oplus \cdots \oplus \langle e_{q-1}\rangle$, where $\langle e_0\rangle \cong \mathbb{Z}$. Given a prime p and $\varepsilon_n \in \{-1, 0, 1\}$ for $n \geq 1$, the formulas*

$$d_{2n} = p^n e_0 \quad \text{and} \quad d_{2n-1} = f_n e_0 + \varepsilon_n e_{n(\mathrm{mod\ q})}$$

define a T-sequence $\{d_n\}$ in G.

Proof. Fix an integer $k \geqslant 0$ and an element $g \in G$ with $g \neq 0$. By Theorem 6, it suffices to prove that $g \notin A(k, m)$ for some $m \in \mathbb{N}$. Let $g = be_0 + a_1e_1 + \cdots + a_{q-1}e_{q-1}$, where $b \in \mathbb{Z}$, $0 \leqslant a_i < o(e_i)$ if $o(e_i) < \infty$ and $a_i \in \mathbb{Z}$ if $o(e_i) = \infty$. Let $t = (|b|+|a_1|+\cdots+|a_{q-1}|)(k+1)$ and $m = 20t$. We are going to check that $g \notin A(k, m)$. To accomplish this, we pick an arbitrarily $\sigma \in A(k, m)$ with $\sigma \neq 0$ and prove that $g \neq \sigma$. To this end, we prove that $|\phi_0| > b$, where ϕ_0 is the coefficient of e_0 in σ.

Since the sequence d_n is defined by two different subsequences, we have to consider some particular cases to estimate ϕ_0.

a) Assume that

$$\sigma = l_1 d_{2r_1} + l_2 d_{2r_2} + \cdots + l_s d_{2r_s} = (l_1 p^{r_1} + \cdots + l_s p^{r_s}) e_0 = p^{r_1} \cdot \sigma' \cdot e_0,$$

where $m \leqslant 2r_1 < 2r_2 < \cdots < 2r_s$ and $\sigma' \in \mathbb{Z}$. Since $\sigma' \neq 0$, we have $p^{r_1} > p^{5|b|} > |b|$, and $\sigma \neq g$.

b) Assume that $\sigma = l_1 d_{2r_1-1} + l_2 d_{2r_2-1} + \cdots + l_s d_{2r_s-1}$, where $m < 2r_1 - 1 < 2r_2 - 1 < \cdots < 2r_s - 1$ and the integers $l_1, l_2, \ldots, l_s$ are such that $l_s \neq 0$ and $\sum_{i=1}^{s} |l_i| \leqslant k + 1$. Then

$$\sigma = (l_1 f_{r_1} + \cdots + l_{s-1} f_{r_{s-1}} + l_s f_{r_s}) e_0 + l_1 \varepsilon_{r_1} e_{r_1(\mathrm{mod\ q})} + \cdots + l_s \varepsilon_{r_s} e_{r_s(\mathrm{mod\ q})}.$$

Since $n^3 < (n+1)^3 - (n+1)^2$ and $r_s > |b| + (k+1)$, by (2), we can estimate the coefficient ϕ_0 of e_0 in σ as follows

$$\begin{aligned} |\phi_0| &\geqslant |l_1 f_{r_1} + \cdots + l_{s-1} f_{r_{s-1}} + l_s f_{r_s}| - (k+1) > f_{r_s} - k \cdot p^{r_{s-1}^3+1} - (k+1) = \\ &= p^{r_s^3} + \left(p^{r_s^3 - r_s} + \cdots + p^{r_s^3 - r_s^2} - k \cdot p^{r_{s-1}^3+1} - k - 1\right) > p^{r_s^3} > |b|. \end{aligned}$$

Hence $\phi_0 \neq b$ and $\sigma \neq g$.

c) Assume that $\sigma = l_1 d_{2r_1-1} + l_2 d_{2r_2-1} + \cdots + l_s d_{2r_s-1} + l_{s+1} d_{2r_{s+1}} + \cdots + l_h d_{2r_h}$, where $0 < s < h$ and

$$m < 2r_1 - 1 < 2r_2 - 1 < \cdots < 2r_s - 1,$$

$$m \leqslant 2r_{s+1} < 2r_{s+2} < \cdots < 2r_h, \quad l_i \in \mathbb{Z} \setminus \{0\}, \sum_{i=1}^{h} |l_i| \leqslant k + 1.$$

Since the number of summands with different powers of p in f_{r_s} is $r_s + 1 > 10(k+1)$ and $h - s < k + 1$, by a simple pigeon-hole principle, there exists $r_s - 2 > i_0 > 2$ such that for every $1 \leqslant w \leqslant h - s$ we have

$$\text{either } r_{s+w} < r_s^3 - (i_0 + 2) r_s \text{ or } r_{s+w} > r_s^3 - (i_0 - 1) r_s.$$

The set of all w such that $r_{s+w} < r_s^3 - (i_0 + 2) r_s$ we denote by B (it can be empty or have the form $\{1, \ldots, \delta\}$ for some $1 \leqslant \delta \leqslant h - s$). Set $D = \{1, \ldots, h - s\} \setminus B$. Thus,

$$\begin{aligned} \sigma = {} & l_1 \varepsilon_{r_1} e_{r_1(\mathrm{mod\ q})} + \cdots + l_s \varepsilon_{r_s} e_{r_s(\mathrm{mod\ q})} + \left(l_1 f_{r_1} + \cdots + l_{s-1} f_{r_{s-1}}\right) e_0 + \\ & + \sum_{w \in B} l_{s+w} d_{2r_{s+w}} + \left(l_s p^{r_s^3 - r_s^2} + \cdots + l_s p^{r_s^3 - (i_0+2) r_s}\right) e_0 + \left(l_s p^{r_s^3 - (i_0+1) r_s} + l_s p^{r_s^3 - i_0 r_s}\right) e_0 + \\ & + \left(l_s p^{r_s^3 - (i_0-1) r_s} + \cdots + l_s p^{r_s^3}\right) e_0 + \sum_{w \in D} l_{s+w} d_{2r_{s+w}}. \end{aligned}$$

Denote the coefficients of e_0 in lines $1,\dots,5$ by $A_1,\dots,A_5$ respectively. Then $\phi_0 = A_1 + \dots + A_5$. We estimate $A_1,\dots,A_5$ as follows. For A_1 we have

$$|A_1| \leqslant |l_1| + \dots + |l_s| \leqslant k+1 < p^{k+1} < p^{r_s^3-(i_0+1)r_s}. \tag{3}$$

Since $l_s \neq 0$ and $kp < p^k p < p^{r_s}$, by (2), we have

$$|A_2| = \left|l_1 f_{r_1} + \dots + l_{s-1} f_{r_{s-1}}\right| \leqslant k \cdot p^{r_{s-1}^3+1} < p^{r_{s-1}^3+r_s} \leqslant p^{(r_s-1)^3+r_s} < p^{r_s^3-(i_0+1)r_s}. \tag{4}$$

Since $3(k+1) < r_s < p^{r_s}$, for A_3 we have

$$|A_3| = \left|\sum_{w\in B} l_{s+w} p^{r_{s+w}} + \left(l_s p^{r_s^3-r_s^2} + \dots + l_s p^{r_s^3-(i_0+2)r_s}\right)\right| < \sum_{w\in B} |l_{s+w}| p^{r_s^3-(i_0+2)r_s} + $$
$$+ |l_s| 2p^{r_s^3-(i_0+2)r_s} < 3(k+1)p^{r_s^3-(i_0+2)r_s} < p^{r_s^3-(i_0+1)r_s}. \tag{5}$$

For A_4 we have

$$p^{r_s^3-i_0 r_s} < |A_4| = |l_s| p^{r_s^3-(i_0+1)r_s} + |l_s| p^{r_s^3-i_0 r_s} < 2k \cdot p^{r_s^3-i_0 r_s}. \tag{6}$$

For A_5 we have

$$A_5 = l_s p^{r_s^3-(i_0-1)r_s} + \dots + l_s p^{r_s^3} + \sum_{w\in D} l_{s+w} p^{r_{s+w}} = p^{r_s^3-(i_0-1)r_s} \cdot \sigma'', \tag{7}$$

where $\sigma'' \in \mathbb{Z}$. We distinguish between two cases.

Case 1. $\sigma'' \neq 0$. By (3)–(7), we can estimate ϕ_0 from below as follows

$$|\phi_0| \geqslant |A_5| - (|A_1| + |A_2| + |A_3| + |A_4|) > p^{r_s^3-(i_0-1)r_s} - 3p^{r_s^3-(i_0+1)r_s} - 2kp^{r_s^3-i_0 r_s} >$$
$$> p^{r_s^3-(i_0-1)r_s} - (2k+3)p^{r_s^3-i_0 r_s} > p^{r_s^3-i_0 r_s} > p^{r_s^2} > p^{|b|} > |b|.$$

Hence $\phi_0 \neq b$ and $\sigma \neq g$.

Case 2. $\sigma'' = 0$. Then, by (3)–(5),

$$|\phi_0| \geqslant |A_4| - (|A_1| + |A_2| + |A_3|) > p^{r_s^3-i_0 r_s} - 3p^{r_s^3-(i_0+1)r_s} > p^{r_s^3-(i_0+1)r_s} > p^{r_s^2} > p^{|b|} > |b|.$$

Hence $\phi_0 \neq b$ and $\sigma \neq g$ too. □

In the following lemma we consider $\mathbb{Z}(p^\infty)$ as a subgroup of $(-\frac{1}{2}, \frac{1}{2}]$ by modulo 1. For the sake of clarity, $|x| (\mathrm{mod}\, 1)$ denotes the distance from a real number x to the nearest integer. Putting

$$\widetilde{f}_n = \frac{1}{p^{n^3-n^2}} + \dots + \frac{1}{p^{n^3-2n}} + \frac{1}{p^{n^3-n}} + \frac{1}{p^{n^3}} \in \mathbb{Z}(p^\infty),$$

we obtain ([11])

$$0 < \widetilde{f}_n = \frac{1}{p^{n^3-n^2}} + \dots + \frac{1}{p^{n^3-2n}} + \frac{1}{p^{n^3-n}} + \frac{1}{p^{n^3}} < \frac{n+1}{p^{n^3-n^2}} \to 0. \tag{8}$$

Lemma 5. *Let $G = \mathbb{Z}(p^\infty) + H$, where $H = \langle e_0\rangle \oplus \cdots \oplus \langle e_{q-1}\rangle$ is finite. Define*

$$d_{2n} = \frac{1}{p^n} \in \mathbb{Z}(p^\infty) \ \ \text{and} \ \ d_{2n-1} = \widetilde{f}_n + e_{n(\mathrm{mod}\ \mathrm{q})} \ \ \text{for} \ \ n \geqslant 1.$$

Then $\mathbf{d} = \{d_n\}$ is a T-sequence in G.

Proof. Let $k \geqslant 0$ be an integer and $g \in G$ with $g \neq 0$. Then $g = \frac{b}{p^z} + a_0 e_0 + \cdots + a_{q-1}e_{q-1}$, where $0 \leqslant a_i < o(e_i)$ and $\frac{b}{p^z} \in \mathbb{Z}(p^\infty)$. Let $\pi\colon G \to \mathbb{Z}(p^\infty)$ be the projection. Then $\pi\left(\langle g\rangle + H\right) = \langle \frac{1}{p^\beta}\rangle$.

Set $t = p(k+1)+\beta$ and $m = 20t$. By Theorem 6, it is enough to prove that $g \notin A(k,m)$. To achieve this, we take $\sigma \in A(k,m)\setminus\{0\}$ arbitrarily and show that $g \neq \sigma$. To this end, we prove two inequalities (mod 1):
1) $0 < |\pi(\sigma)|$ and 2) if $\pi(g) \neq 0$, then $|\pi(\sigma)| < |\pi(g)|$. This gives $\sigma \neq g$.

Since the sequence d_n is defined by the two different subsequences, we have to consider some particular cases to estimate $\pi(\sigma)$.

a) Assume that $\sigma = l_1 d_{2r_1} + l_2 d_{2r_2} + \cdots + l_s d_{2r_s}$, where $m \leqslant 2r_1 < 2r_2 < \cdots < 2r_s$. If $\pi(g) = 0$, then $\pi(\sigma) = \sigma \neq \pi(g)$. If $\pi(g) \neq 0$, then

$$0 < |\sigma| = |\pi(\sigma)| = |l_1 d_{2r_1} + l_2 d_{2r_2} + \cdots + l_s d_{2r_s}| \leqslant \sum_{i=1}^{s} \frac{|l_i|}{p^{r_i}} \leqslant \frac{k+1}{p^{r_1}} < \frac{k+1}{p^{k+1+\beta}} < \frac{1}{p^\beta} \leqslant |\pi(g)|.$$

So $\pi(\sigma) \neq \pi(g)$ and $\sigma \neq g$.

b) Assume that $\sigma = l_1 d_{2r_1-1} + l_2 d_{2r_2-1} + \cdots + l_s d_{2r_s-1}$, where $m < 2r_1 - 1 < 2r_2 - 1 < \cdots < 2r_s - 1$ and the integers $l_1, l_2, \dots, l_s$ are such that $l_s \neq 0$ and $\sum_{i=1}^{s} |l_i| \leqslant k+1$. Since $n^3 < (n+1)^3 - (n+1)^2$ and $r_s > 5p(k+1) + 5\beta$, we have

$$\pi(\sigma) = \frac{z'}{p^{r_s^3 - r_s}} + \frac{l_s}{p^{r_s^3}}, \ \text{where } z' \in \mathbb{Z}.$$

Since $|l_s| \leqslant k+1 < \frac{r_s}{p} < p^{r_s-1}$, we have the following: if $\pi(\sigma) = \frac{z''}{p^\alpha}$, $z'' \in \mathbb{Z}$, is an irreducible fraction then $\alpha > r_s^3 - r_s + 1 > 5\beta$. Hence $\pi(\sigma) \neq \pi(g)$ and $\sigma \neq g$.

c) Assume that $\sigma = l_1 d_{2r_1-1} + l_2 d_{2r_2-1} + \cdots + l_s d_{2r_s-1} + l_{s+1} d_{2r_{s+1}} + \cdots + l_h d_{2r_h}$, where $0 < s < h$ and

$$m < 2r_1 - 1 < 2r_2 - 1 < \cdots < 2r_s - 1,$$

$$m \leqslant 2r_{s+1} < 2r_{s+2} < \cdots < 2r_h, \quad l_i \in \mathbb{Z}\setminus\{0\}, \sum_{i=1}^{h} |l_i| \leqslant k+1.$$

Since the number of summands with different powers of p in $\widetilde{f}_{r_s}$ is $r_s + 1 > 10p(k+1)$ and $h - s < k+1$, by a simple pigeon-hole principle, there exists $r_s - 2 > i_0 > 2$ such that for every $1 \leqslant w \leqslant h-s$ we have

$$\text{either } r_{s+w} < r_s^3 - (i_0+2)r_s \text{ or } r_{s+w} > r_s^3 - (i_0-1)r_s.$$

The set of all w such that $r_{s+w} < r_s^3 - (i_0+2)r_s$ we denote by K (it can be empty or have the form $\{1,\dots,a\}$ for some $1 \leqslant a \leqslant h-s$). Set $L = \{1,\dots,h-s\}\setminus K$. Thus

$$\sigma = \left(l_1 e_{r_1(\mathrm{mod}\ \mathrm{q})} + \cdots + l_s e_{r_s(\mathrm{mod}\ \mathrm{q})}\right) + l_1\widetilde{f}_{r_1} + \cdots + l_{s-1}\widetilde{f}_{r_{s-1}} + \sum_{w\in K} l_{s+w} d_{2r_{s+w}} + \frac{l_s}{p^{r_s^3 - r_s^2}} + \cdots +$$

$$+ \frac{l_s}{p^{r_s^3-(i_0+2)r_s}} + \frac{l_s}{p^{r_s^3-(i_0+1)r_s}} + \frac{l_s}{p^{r_s^3-i_0 r_s}} + \frac{l_s}{p^{r_s^3-(i_0-1)r_s}} + \cdots + \frac{l_s}{p^{r_s^3}} + \sum_{w\in L} l_{s+w} d_{2r_{s+w}}.$$

The elements in the lines 1, 2 and 4 we denote by σ_1, σ_2 and σ_4 respectively. Since $n^3 < (n+1)^3 - (n+1)^2$ and $r_s > \beta$, the projection on $\mathbb{Z}(p^\infty)$ of every summand in lines 1 and 2 has the form $\frac{\delta}{p^\gamma}$, with $\gamma \leqslant r_s^3 - (i_0+2)r_s$ and $\delta \in \mathbb{Z}$. Thus,

$$\pi(\sigma_1+\sigma_2) = \frac{c}{p^{r_s^3-(i_0+2)r_s}}, \text{ for some } c \in \mathbb{Z}.$$

Hence

$$\pi(\sigma) = \frac{c}{p^{r_s^3-(i_0+2)r_s}} + \frac{l_s}{p^{r_s^3-(i_0+1)r_s}} + \frac{l_s}{p^{r_s^3-i_0 r_s}} + \pi(\sigma_4). \tag{9}$$

Since $r_s > 10p(k+1)$, then $\frac{1}{1-1/p^{r_s}} < \frac{1}{1-1/p^{10}} < \frac{1}{1-1/2^5} = \frac{32}{31}$ and $2k < p^{2k} < p^{r_s}$. Thus, we can estimate $\pi(\sigma_4)$ as follows:

$$|\pi(\sigma_4)| = \Big| \Big(\frac{l_s}{p^{r_s^3-(i_0-1)r_s}} + \dots + \frac{l_s}{p^{r_s^3}} \Big) + \sum_{w\in L} l_{s+w} \frac{1}{p^{r_{s+w}}} \Big| <$$

$$< \frac{|l_s|}{p^{r_s^3-(i_0-1)r_s}} \Big(1 + \frac{1}{p^{r_s}} + \frac{1}{p^{2r_s}} + \dots \Big) + \frac{1}{p^{r_s^3-(i_0-1)r_s+1}} \sum_{w\in L} |l_{s+w}| \leqslant \frac{|l_s|}{p^{r_s^3-(i_0-1)r_s}} \times$$

$$\times \frac{1}{1-\frac{1}{p^{r_s}}} + \frac{k}{p^{r_s^3-(i_0-1)r_s+1}} < \frac{1}{p^{r_s^3-(i_0-1)r_s}} \Big(k\frac{32}{31} + k\frac{1}{p} \Big) < \frac{2k}{p^{r_s^3-(i_0-1)r_s}} < \frac{1}{p^{r_s^3-i_0 r_s}}. \tag{10}$$

We distinguish between two cases.

Case 1. $\pi(\sigma_4) \neq 0$. By (10) we have the following. If $\pi(\sigma_4) = \frac{\widetilde{c}}{p^\alpha}$ is an irreducible fraction, then $\alpha > r_s^3 - i_0 r_s > 5\beta$. Thus, by (9), we also have

$$\pi(\sigma) = \frac{c''}{p^\alpha} \neq 0, \text{ where } c'' \in \mathbb{Z} \text{ and } (c'', p) = 1.$$

Since $\pi(g) \in \langle \frac{1}{p^\beta} \rangle$ and $\alpha > 5\beta$, we have $\pi(\sigma) \neq \pi(g)$ and $\sigma \neq g$.

Case 2. $\pi(\sigma_4) = 0$. Let $l_s = p^\psi \cdot l'_s$, where $(p, l'_s) = 1$ and $\psi < k < r_s$. Thus, by (9),

$$\pi(\sigma) = \frac{c}{p^{r_s^3-(i_0+2)r_s}} + \frac{l_s}{p^{r_s^3-(i_0+1)r_s}} + \frac{l_s}{p^{r_s^3-i_0 r_s}} = \frac{c''}{p^{r_s^3-i_0 r_s-\psi}},$$

where $c'' \in \mathbb{Z}$ and $(c'', p) = 1$. Since $r_s^3 - i_0 r_s - \psi > r_s^3 - (i_0+1)r_s > 5\beta$, we have $\pi(\sigma) \neq 0$ and $\pi(\sigma) \neq \pi(g)$. Thus $\sigma \neq g$. □

Put $S_0 = 0$ and $S_n = 1 + 2 + \dots + n$ for $n \in \mathbb{N}$.

Lemma 6. *Let q be an integer with $q \geqslant 2$. Then $(S_{n-1}+k)q + i \neq (S_{m-1}+l)q + j$ for every $m, n \geqslant 1$, $0 \leqslant i, j < q$, $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$ such that $(n, i, k) \neq (m, j, l)$.*

Proof. We have three cases:

(1) The case $n \neq m$. We may assume that $n \leqslant m-1$. Then for every $0 \leqslant i, j < q$ and $1 \leqslant k \leqslant n$ we have

$$(S_{n-1}+k)q + i \leqslant S_n q + (q-1) = (S_n+1)q - 1 < (S_{m-1}+1)q + j.$$

So $(S_{n-1}+k)q + i \neq (S_{m-1}+l)q + j$ for every $0 \leqslant i, j < q$, $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$.

(2) The case $n = m$ and $i \neq j$. It is clear that

$$(S_{n-1}+k)q + i \neq (S_{n-1}+l)q + j \text{ for every } 1 \leqslant k, l \leqslant n.$$

(3) The case $n = m$, $i = j$ and $k \neq l$. It is clear that $(S_{n-1}+k)q + i \neq (S_{n-1}+l)q + i$. □

As usual, $o(g)$ denotes the *order* of an element g of an Abelian group G.

In the following lemma we modify the construction of [15, Example 5] (or [16, Example 2.6.2]).

Lemma 7. *Let $H = \langle e_0\rangle \oplus \cdots \oplus \langle e_{q-1}\rangle$ and $G = H \oplus \bigoplus_{i=q}^{\infty} \langle e_i\rangle = \bigoplus_{i=0}^{\infty}\langle e_i\rangle$, where $u_i := o(e_i) < \infty$ for every $i \geqslant 0$. Define a sequence $\mathbf{d} = \{d_n\}_{n\geqslant 2q-1}$ as follows. For even indices we set*

$$d_{2q} = e_q,\quad d_{2(q+1)} = 2e_q, \dots , d_{2(q+u_q-2)} = (u_q - 1)e_q,\quad d_{2(q+u_q-1)} = e_{q+1},\ d_{2(q+u_q)} = 2e_{q+1}, \dots$$

For odd indices and for $0 \leqslant i < q$ and $n \geqslant 1$, we define

$$d_{2(nq+i)-1} = e_i + e_{(S_{n-1}+1)q+i} + e_{(S_{n-1}+2)q+i} + \cdots + e_{S_n q+i}.$$

Assume that one of the following two conditions holds:
a) there exists an integer $j_0 \geqslant 0$ such that $u_j = u_{j_0}$ for all integers $j \geq j_0$ and u_{j_0} is divided by every $u_0, \dots , u_{q-1}$, or b) $u_n \to \infty$.
Then $\mathbf{d} = \{d_n\}$ is a T-sequence in G.

Proof. Let $k \geqslant 0$ be an integer and $g \in G$ with $g \neq 0$. By Theorem 6, we have to show that there is $m \in \mathbb{N}$ such that $g \notin A(k, m)$.

Step 1. By construction, $d_{2n} = \lambda(n)e_{\mu(n)}$, where $1 \leqslant \lambda(n) < o(e_{\mu(n)})$ and $\mu(n) \to \infty$ at $n \to \infty$. Since also $(S_{n-1} + 1)q + i \to \infty$ at $n \to \infty$, we have the following: for every $j \geqslant q$ there exists $m \in \mathbb{N}$ such that $A(k, m) \subset H \oplus \bigoplus_{i=j}^{\infty}\langle e_i\rangle$. Thus,

$$\bigcap_{m=1}^{\infty} A(k, m) \subset \bigcap_{j\geqslant q} \Big(H \oplus \bigoplus_{i=j}^{\infty}\langle e_i\rangle\Big) = H.$$

So, the condition of the Protasov-Zelenyuk criterion holds for every $g \notin H$. (Note that a similar inclusion was proved in [11, Proposition 3.3] for another special case of T-sequence.)

By Step 1, it remains to check the Protasov-Zelenyuk criterion only for non-zero elements of H. Thus, in what follows, we assume that $g \in H$ and $g \neq 0$.

Note also that the summands of all the elements $d_{2(nq+i)-1} - e_i$ are independent, where $0 \leqslant i < q$ and $n \geqslant 1$. Indeed, this follows from Lemma 6 and the independence of the sequence $\{e_n\}$.

Step 2. Let $g \in A(k, 2m)$ for some natural m. Then g has the following representation

$$g = l_1 d_{2r_1-1} + l_2 d_{2r_2-1} + \cdots + l_s d_{2r_s-1} + l_{s+1} d_{2r_{s+1}} + l_{s+2} d_{2r_{s+2}} + \cdots + l_h d_{2r_h}, \tag{11}$$

where all summands are nonzero, $\sum_{i=1}^{h} |l_i| \leqslant k + 1$, $0 < s \leqslant h$ (by the construction of $\mathbf{d}$) and

$$2m < 2r_1 - 1 < 2r_2 - 1 < \cdots < 2r_s - 1,\quad 2m \leqslant 2r_{s+1} < 2r_{s+2} < \cdots < 2r_h.$$

Since all the summands of all the elements $d_{2(nq+i)-1} - e_i$ are independent and since $g \in H$, by the construction of the elements d_{2n} and (11), there is a subset Ω of the set $\{s+1, \dots , r_h\}$ such that

$$l_s d_{2r_s-1} + \sum_{w\in\Omega} l_w d_{2r_w} \in H. \tag{12}$$

Step 3. By Step 2, to prove the lemma it is enough to find m_0 such that (12) does not hold. We consider two cases a) and b) separately.

Assume that a) holds. Set $m_0 = 4q(j_0+1)(k+1)$. Then $d_{2r_s-1} - e_{r_s (\mathrm{mod}\, q)}$ contains exactly

$$t = \frac{1}{q}(r_s - r_s(\mathrm{mod}\, q)) > \frac{1}{q}(m_0 - q) \geqslant 4k + 3$$

independent summands of the form e_j with $j \geqslant (\frac{t(t-1)}{2} + 1) > m_0 > j_0$. Since $l_s d_{2r_s-1} \neq 0$ and u_{j_0} is divided by every $u_0, \dots , u_{q-1}$, we may assume that l_s is not divided by u_{j_0}. So, $l_s d_{2r_s-1}$ contains at least $4k + 3$ non-zero independent summands of the form $l_s e_j$ with $j > j_0$.

Since $|\Omega| \leqslant h-s \leqslant k$ and $l_w d_{2r_w}$ has the form $a_v e_v$, the conclusion of (12) does not hold. Thus, $g \notin A(k, 2m_0)$.

Assume that b) holds. Choose $j_0 > q$ such that $u_j > 2(k+1)$ for every $j > j_0$. Set $m_0 = 4j_0(q+1)(k+1)$. Then $d_{2r_s-1} - e_{r_s (\mathrm{mod}\, q)}$ contains at least $\frac{1}{q}(r_s - r_s(\mathrm{mod}\, q)) > \frac{1}{q}(m_0 - q) > 4j_0(k+1)$ summands that are multiples of e_j. So, since $|l_s| \leqslant k+1$, $l_s d_{2r_s-1}$ contains at least $3(k+1)$ non-zero independent summands of the form $l_s e_j$ with $j > j_0$.

Since $|\Omega| \leqslant h-s \leqslant k$ and $l_w d_{2r_w}$ has the form $a_v e_v$, the conclusion of (12) does not hold. Thus, $g \notin A(k, 2m_0)$. □

4. Proofs of Theorems 3, 4 and 5. Following [3], we say that a sequence $\mathbf{u} = \{u_n\}$ is a TB-sequence in a group G if there is a precompact Hausdorff group topology on G in which $u_n \to 0$.

Proof of Theorem 1. Let G be an infinite Abelian group. It is known ([6]) that G admits a non-trivial TB-sequence $\mathbf{u}$. As it was noted in [8], a sequence $\mathbf{u}$ is a TB-sequence if and only if it is a T-sequence and $(G, \mathbf{u})$ is maximally almost periodic. So $\mathbf{n}(G, \mathbf{u}) = 0$. Thus, G admits a complete non-discrete Hausdorff group topology with trivial von Neumann radical. □

Let X be an Abelian topological group and $\mathbf{u} = \{u_n\}$ a sequence of elements of $X^\wedge$. Following D. Dikranjan, C. Milan and A. Tonolo ([7]), we denote by $s_{\mathbf{u}}(X)$ the set of all $x \in X$ such that $(u_n, x) \to 1$.

A proof of the following lemma can be found, for example, in [16, Example 2.6.3].

Lemma 8. *Let $d_{2n} = \frac{1}{p^n} \in \mathbb{Z}(p^\infty)$ and $\widetilde{\mathbf{d}} = \{d_{2n}\}$. Then $x \in s_{\widetilde{\mathbf{d}}}(\Delta_p)$ if and only if there exists $m = m(x) \in \mathbb{Z}$ such that*

$$(\lambda, x) = \exp(2\pi i m \lambda) \quad \text{for all} \quad \lambda \in \mathbb{Z}(p^\infty). \tag{13}$$

In other words, $x \in s_{\widetilde{\mathbf{d}}}(\Delta_p)$ if and only if $x = m\mathbf{1}$ for some $m \in \mathbb{Z}$. In particular, $\mathrm{Cl}\left(s_{\widetilde{\mathbf{d}}}(\mathbb{Z}(p^\infty))\right) = \Delta_p$.

The following theorem is the algebraic part of [8, Theorem 4]. It shall be used to compute von Neumann kernels.

Theorem 7. *If $\mathbf{d} = \{d_n\}$ is a T-sequence of an Abelian group G then $\mathbf{n}(G, \mathbf{d}) = s_{\mathbf{d}}\left((G_d)^\wedge\right)^\perp$ algebraically.*

Another ingredient of the proof is the following reduction principle.

Lemma 9. *Let H be a subgroup of an Abelian group G. If there exists a subgroup G' of G containing H such that $H \in \mathcal{NR}(G')$ (or $H \in \mathcal{NRC}(G')$) then $H \in \mathcal{NR}(G)$ (respectively, $H \in \mathcal{NRC}(G)$).*

Proof. Since $H \in \mathcal{NR}(G')$, there exists a Hausdorff group topology τ' on G' such that $H = \mathbf{n}(G', \tau')$. Furthermore, if $H \in \mathcal{NRC}(G')$ then τ' can be chosen to be complete. Let τ be the group topology on G such that $G' \in \tau$ and (G', τ') is a subspace of (G, τ). We note that τ is complete whenever τ' is. Since (G', τ') is an open subgroup of (G, τ), one has $\mathbf{n}(G', \tau') = \mathbf{n}(G, \tau)$ (see also [8, Lemma 4] for a more general statement). This proves that $H = \mathbf{n}(G, \tau)$. □

Proof of Theorem 3. Our goal is to construct a T-sequence $\mathbf{d}$ in G satisfying

$$s_{\mathbf{d}}\left((G_d)^\wedge\right)^\perp = H. \tag{14}$$

Combining this with Theorem 7, we obtain that $\mathbf{n}(G, \mathbf{d}) = H$. Since $(G, \mathbf{d})$ is complete, this shows that $H \in \mathcal{NRC}(G)$.

The rest of the proof is split into the following four cases.
(1) H is infinite. **(2)** H is finite and G is not torsion. **(3)** H is finite, G is torsion but not reduced. **(4)** H is finite and G is both torsion and reduced.

Since H is finitely generated, it is a direct finite sum of cyclic groups.

(1) *H is infinite.* Then $H = \langle e_0\rangle \oplus \langle e_1\rangle \oplus \cdots \oplus \langle e_{q-1}\rangle$, where $\langle e_0\rangle \cong \mathbb{Z}$. Applying the reduction principle (Lemma 9), we may assume that $G = H$. Choose any prime p and let $\varepsilon_n = 1$ for all $n \in \mathbb{N}$. Let $\mathbf{d} = \{d_n\}$ be the T-sequence in G as in Lemma 4. To establish (14), it suffices to prove that $s_{\mathbf{d}}((G_d)^\wedge) = 0$. Let

$$\omega = x_0 + x_1 + \cdots + x_{q-1} \in (G_d)^\wedge, x_i \in \langle e_i\rangle^\wedge, \text{ and } (d_n, \omega) \to 1.$$

Then $(d_{2n}, \omega) = (p^n e_0, x_0) \to 1$. Hence $x_0 \in \mathbb{Z}(p^\infty)$ (see [2] or [4, Remark 3.8]). If $x_0 = \frac{\rho}{p^\tau}$, $\rho \in \mathbb{Z}, \tau > 0$, then for $n = qs + i > \tau$ we have $(d_{2(qs+i)-1}, \omega) = (e_i, x_i)$ for every $0 \leqslant i < q$. So $(d_{2(qs+i)-1}, \omega) \to 1$ only if $x_i = 0$ for every i. Hence $\omega = 0$.

(2) *H is finite and G is not torsion.* Fix $e_0 \in G$ such that $\langle e_0\rangle \cong \mathbb{Z}$. Since H is finite, $H \cap \langle e_0\rangle = 0$. Let $H = \langle e_1\rangle \oplus \cdots \oplus \langle e_{q-1}\rangle$ be a direct decomposition of H. Then $G' = \langle e_0\rangle \oplus \langle e_1\rangle \oplus \cdots \oplus \langle e_{q-1}\rangle \cong \mathbb{Z} \oplus H$ is a subgroup of G containing H. By Lemma 9, we may assume that $G = G'$.

Choose any prime p and set $\varepsilon_n = 1$ if $n(\text{mod q}) > 0$ and $\varepsilon_n = 0$ if $n(\text{mod q}) = 0$. Let $\mathbf{d} = \{d_n\}$ be the T-sequence in G as in Lemma 4. To establish (14), it suffices to show that $\mathrm{Cl}(s_{\mathbf{d}}(G_d^\wedge)) = \mathbb{Z}^\wedge = \mathbb{T}$. Let

$$\omega = x_0 + x_1 + \cdots + x_{q-1} \in (G_d)^\wedge, x_i \in \langle e_i\rangle^\wedge, \text{ and } (d_n, \omega) \to 1.$$

Then $(d_{2n}, \omega) = (p^n e_0, x_0) \to 1$. Hence $x_0 \in \mathbb{Z}(p^\infty)$ (see [2] or [4, Remark 3.8]). Let $x_0 = \frac{\rho}{p^\tau}, \rho \in \mathbb{Z}, \tau > 0$. Then for any $n = qs + i > \tau$ we have $(d_{2qs-1}, \omega) = 1$ if $i = 0$, and $(d_{2(qs+i)-1}, \omega) = (e_i, x_i)$ if $0 < i < q$. So $(d_{2n-1}, \omega) \to 1$ only if $x_i = 0$ for every $0 < i < q$. Thus $\omega = x_0$, where $x_0 \in \mathbb{Z}(p^\infty) \subset \mathbb{T}$. So $s_{\mathbf{d}}(G_d^\wedge) \subseteq \mathbb{Z}(p^\infty)$. Let us prove the converse inclusion. Let $\omega = x_0 = \frac{\rho}{p^\tau} \in \mathbb{Z}(p^\infty), \rho \in \mathbb{Z}, \tau > 0$. By the definition of d_m we have

$$(d_{2n}, x_0) = \exp\left\{2\pi i \frac{p^n \rho}{p^\tau}\right\}, \; (d_{2n-1}, x_0) = \exp\left\{2\pi i \frac{f_n \rho}{p^\tau}\right\}.$$

Thus, $(d_m, x_0) = 1$ for every $m > 2\tau$ and hence $s_{\mathbf{d}}(G_d^\wedge) \supseteq \mathbb{Z}(p^\infty)$.

Hence $s_{\mathbf{d}}(G_d^\wedge) = \mathbb{Z}(p^\infty)$ and $\mathrm{Cl}(s_{\mathbf{d}}(G_d^\wedge)) = \mathbb{T}$.

(3) *H is finite, G is torsion but not reduced.* Then G contains a subgroup isomorphic to $\mathbb{Z}(p^\infty)$ for some prime p. By Lemma 9, we may assume that $G \cong \mathbb{Z}(p^\infty) + H$. Let $H = \langle e_0\rangle \oplus \cdots \oplus \langle e_{q-1}\rangle$ be a direct decomposition of H. Let $\mathbf{d} = \{d_n\}$ be the T-sequence in G as in Lemma 5. To establish (14), it suffices to prove that $H^\perp = \mathrm{Cl}(s_{\mathbf{d}}((G_d)^\wedge))$.

Let us prove first that $s_{\mathbf{d}}((G_d)^\wedge) \subseteq H^\perp$. We use the notations from Lemma 3. Assume that

$$\omega = \mathbf{x}_0 + y \in s_{\mathbf{d}}((G_d)^\wedge), \text{ where } \mathbf{x}_0 \in \Delta_p, y \in H_1^\wedge.$$

Then $(d_{2n}, \omega) = (d_{2n}, \mathbf{x}_0) \to 1$. By (13), $\mathbf{x}_0 = m\mathbf{1}$ for some $m \in \mathbb{Z}$ and $(\lambda, \mathbf{x}_0) = \exp(2\pi i m\lambda)$, $\forall \lambda \in \mathbb{Z}(p^\infty)$. In particular, $(\widetilde{f}_n, \mathbf{x}_0) = \exp(2\pi i m\widetilde{f}_n)$ for every $n \geqslant 1$. By (8), we obtain that $(\widetilde{f}_n, \mathbf{x}_0) \to 1$. So, for every $0 \leqslant i < q$, we have $(s \to \infty)$

$$(d_{2(sq+i)-1},\omega)=(\widetilde{f}_{sq}+e_i,\omega)=(\widetilde{f}_{sq},\mathbf{x}_0)\cdot(e_i,\omega)\to(e_i,\omega)=1.$$

So $\omega\in\langle e_i\rangle^\perp$ for every $0\leqslant i<q$. Hence $\omega\in H^\perp$.

Let us show now the reverse inclusion $H^\perp\subseteq \mathrm{Cl}(s_{\mathbf{d}}((G_d)^\wedge))$. By Lemma 3, it is enough to prove that $\{S_0\cup\langle p^k\mathbf{1}\rangle\}\subset s_{\mathbf{d}}((G_d)^\wedge)$. Let $\omega\in S_0$. Then, by the construction of S_0, for $\omega=\mathbf{x}_0+y$ and $\mathbf{x}_0=(x_0,\dots,x_{k-1},0,\dots)=m\cdot\mathbf{1}$ we have

$$(d_{2n},\omega)=\exp\Big\{2\pi i\frac{1}{p^n}(x_0+\dots+x_{k-1}p^{k-1})\Big\}\to 1,\ \text{at } n\to\infty.$$

$$(d_{2(sq+i)-1},\omega)=(\widetilde{f}_{sq},\mathbf{x}_0)\cdot(e_i,\omega)=(\widetilde{f}_{sq},\mathbf{x}_0)=\exp\{2\pi i\widetilde{f}_{sq}m\}\to 1.$$

Hence $S_0\subset s_{\mathbf{d}}((G_d)^\wedge)$. For $p^k\mathbf{1}$ we obtain

$$(d_{2n},p^k\mathbf{1})=\exp\Big\{2\pi i\frac{1}{p^n}\cdot p^k\Big\}\to 1,\ \text{at } n\to\infty.$$

$$(d_{2(sq+i)-1},p^k\mathbf{1})=(\widetilde{f}_{sq},p^k\mathbf{1})=\exp\{2\pi i\widetilde{f}_{sq}p^k\}\to 1.$$

Thus, $H^\perp=\mathrm{Cl}(s_{\mathbf{d}}((G_d)^\wedge))$.

(4) *H is finite and G is both torsion and reduced.* Since G is not bounded, G contains an independent sequence $\{b_n\}$ of elements such that $o(b_n)\to\infty$. Let $H=\langle e_0\rangle\oplus\dots\oplus\langle e_{q-1}\rangle$ be a direct decomposition of H. Using q times Lemma 1, we can find $m\in\mathbb{N}$ such that the sequence $\{e_0,e_1,\dots,e_{q-1},b_m,b_{m+1},\dots\}$ is independent. Define $e_{q+k}=b_{m+k}$ for all integers $k\geq 0$. Clearly, $u_i:=o(e_i)<\infty$ for every $i\geqslant 0$ and $u_i\to\infty$. By Lemma 9, we may assume that

$$G=H\oplus\bigoplus_{i=q}^{\infty}\langle e_i\rangle=\bigoplus_{i=0}^{\infty}\langle e_i\rangle.$$

Then $(G_d)^\wedge=\prod_{i=0}^{\infty}\langle e_i\rangle$.

Let $\mathbf{d}=\{d_n\}$ be the T-sequence in G as in Lemma 7. To establish (14), it suffices to prove that

$$\mathrm{Cl}(s_{\mathbf{d}}((G_d)^\wedge))=\prod_{i=q}^{\infty}\langle e_i\rangle. \tag{15}$$

We modify the proof of [11, Proposition 3.3]. Let $\omega=(a_0,a_1,\dots)\in s_{\mathbf{d}}((G_d)^\wedge)$. By definition, there exists $N\in\mathbb{N}$ such that $|1-(d_{2n},\omega)|<0.1,\forall n>N$. Thus, there is $N_0>N$ such that $|1-(je_l,\omega)|=|1-(je_l,a_l)|<0.1,\forall j=1,\dots,u_l-1$, for every $l>N_0$. This means that $a_l=0$ for every $l>N_0$. So $\omega\in\bigoplus_{i=0}^{\infty}\langle e_i\rangle\subset(G_d)^\wedge$. Since $(d_{2(nq+i)-1},\omega)\to 1$ too and $(d_{2(nq+i)-1},\omega)=(e_i,a_i)$ for all sufficiently large n, we obtain that $a_i=0$ for any $i=0,\dots,q-1$. Thus $s_{\mathbf{d}}((G_d)^\wedge)\subseteq\bigoplus_{i=q}^{\infty}\langle e_i\rangle$. The converse inclusion is trivial. Hence $s_{\mathbf{d}}((G_d)^\wedge)=\bigoplus_{i=q}^{\infty}\langle e_i\rangle$ and it is dense in $\prod_{i=q}^{\infty}\langle e_i\rangle$. So, $\mathrm{Cl}(s_{\mathbf{d}}((G_d)^\wedge))=\prod_{i=q}^{\infty}\langle e_i\rangle$. □

Proof of Theorem 4. Let us prove the implication (1) ⇒ (2). Assume that G contains a subgroup of the form $H^{(\omega)}$. Let $H=\langle e_0^0\rangle\oplus\dots\oplus\langle e_0^q\rangle$ with $e_0^i\in G$. By our assumption, G contains a subgroup of the form $Y_0\oplus Y_1\oplus\dots\oplus Y_q$, where

$$Y_j=\bigoplus_{i=0}^{\infty}\langle e_i^j\rangle,\ 0\leqslant j\leqslant q,\ e_i^j\in G,$$

and the order of e_i^j is equal to u_j for every $i\geqslant 0$. By the reduction principle (Lemma 9), we may assume that $G=Y_0\oplus Y_1\oplus\dots\oplus Y_q$. Further, since the von Neumann radical of a product of topological groups is the product of their von Neumann radicals, it is enough to construct a Hausdorff group topology τ_j on Y_j such that $\mathbf{n}(Y_j,\tau_j)=\langle e_0^j\rangle$. So, we can restrict ourselves to the case $H = \langle e_0\rangle$ and $G = H \oplus \bigoplus_{i=1}^{\infty}\langle e_i\rangle = \bigoplus_{i=0}^{\infty}\langle e_i\rangle$, where the order of e_i is equal to u for every $i \geqslant 0$.

Let $\mathbf{d} = \{d_n\}$ be the T-sequence in G as in Lemma 7. As in the proof of Theorem 3, we only need to show that equality (14) holds. To this end, it is enough to prove that $\mathrm{Cl}(s_{\mathbf{d}}((G_d)^\wedge)) = \prod_{i=1}^{\infty}\langle e_i\rangle$. The proof of this equality is the same as the proof of equality (15) in item (4) of Theorem 3 (where one needs to take $q = 1$).

Implication (2) $\Rightarrow$ (3) is trivial.

Let us prove implication (3) $\Rightarrow$ (1). Let $H = \langle e_1\rangle \oplus \cdots \oplus \langle e_q\rangle$. Assume that (1) fails. By Lemma 2, there exists $1 \leqslant i_0 \leqslant q$ such that G does not contain a subgroup of the form $\langle e_{i_0}\rangle^{(\omega)}$. Set $n_{i_0} = o(e_{i_0})$. Let $n_{i_0} = p_1^{k_1}\dots p_l^{k_l}$ and $\exp G = p_1^{a_1}\dots p_l^{a_l}\cdot p_{l+1}^{a_{l+1}}\dots p_t^{a_t}$, where $p_1,\dots,p_t$ are distinct prime integers. For $1 \leqslant j \leqslant l$ we put $m_j = \exp G/p_j^{a_j-k_j+1}$. Set $\pi_j\colon G \to G$, $\pi_j(g) = m_j g$, and $G_j = \pi_j(G)$. Then $\pi_j(e_{i_0}) \neq 0$ for every $1 \leqslant j \leqslant l$.

(a) *Let us prove that there exists* $1 \leqslant j \leqslant l$ *such that* G_j *is finite.*

Assume for a contradiction that G_j is infinite for every j. Since $\exp G_j = p_j^{a_j-k_j+1}$, G_j contains a subgroup of the form

$$\bigoplus_{i=1}^{\infty}\langle\widetilde{b}_i\rangle,\ \text{where } \widetilde{b}_i \in G_j \text{ and } \langle\widetilde{b}_i\rangle \cong \mathbb{Z}(p_j).$$

Thus, for every $i \geqslant 1$ there exists an element $b_i \in G$ such that $o(b_i) = p_j^{k_j}$ and $\pi_j(b_i) = \widetilde{b}_i$. Indeed, if y is any element such that $\pi_j(y) = \widetilde{b}_i$, then we may put $b_i = c_j y$, where $c_j = \exp G/p_j^{a_j}$ (and $m_j = c_j\cdot p_j^{k_j-1}$).

Let us prove that the sequence $\{b_i\}$ is independent. Assuming the converse we obtain that

$$s_1 b_{i_1} + s_2 b_{i_2} + \cdots + s_w b_{i_w} = 0 \text{ and } s_r b_{i_r} \neq 0, 1 \leqslant r \leqslant w. \tag{16}$$

Let $s_r = p_j^{v_r}\cdot A_r$, where p and A_r are coprime. Set $v = \min\{v_1,\dots,v_w\}$. By our choice of b_i we have $v < k_j$. Thus, if we multiply equality (16) by $c_j\cdot p_j^{k_j-v-1}$ then we obtain

$$A_1 p_j^{v_1-v}\widetilde{b}_{i_1} + A_2 p_j^{v_2-v}\widetilde{b}_{i_2} + \cdots + A_w p_j^{v_w-v}\widetilde{b}_{i_w} = 0.$$

Since there exists r such that $v_r = v$ and $A_r p_j^{v_r-v}\widetilde{b}_{i_r} = A_r\widetilde{b}_{i_r} \neq 0$, we obtain that the elements $\widetilde{b}_i$ are dependent. Since the sequence $\{\widetilde{b}_i\}$ is independent, we obtain a contradiction.

Since the sequence $\{b_i\}$ is independent, G contains a subgroup of the form

$$\bigoplus_{i=1}^{\infty}\langle b_i\rangle,\ \text{where } \langle b_i\rangle \cong \mathbb{Z}(p_j^{k_j}),$$

for every $1 \leqslant j \leqslant l$. Since $p_1,\dots,p_l$ are coprime, G contains a subgroup of the form $\langle e_{i_0}\rangle^{(\omega)}$. This is a contradiction. Thus there exists $1 \leqslant j \leqslant l$ such that G_j is finite.

(b) *Let us prove that there is no Hausdorff group topology* τ *such that* $\mathbf{n}(G,\tau) = H$. (We repeat the arguments of D. Remus (see [5])).

Let τ be any Hausdorff group topology on G and let j be such that G_j is finite. Then $\mathrm{Ker}(\pi_j)$ is open and closed. So $\mathbf{n}(G,\tau) \subseteq \mathrm{Ker}(\pi_j)$. Since, $0 \neq \pi_j(e_{i_0}) \in H/\mathrm{Ker}(\pi_j)$, we obtain that $H \neq \mathbf{n}(G,\tau)$. □

Proof of Theorem 5. (1) is equivalent to (2) by Corollary 1.

Let us prove that (2) yields (3). If G does not satisfy condition (3) then $\exp G < \infty$. Let $\exp G = p_1^{a_1}p_2^{a_2}\dots p_t^{a_t}$, where $p_1,\dots,p_t$ are distinct prime integers. By Lemma 2, there exists $1 \leqslant i_0 \leqslant t$ such that G does not contain a subgroup of the form $\mathbb{Z}(p_{i_0}^{a_{i_0}})^{(\omega)}$. Set $H = \langle e_{i_0} \rangle$, where $o(e_{i_0}) = p_{i_0}^{a_{i_0}}$. Then H is finite and, by Theorem 4, $H \notin \mathcal{NR}(G)$. This is a contradiction. Thus, (2) yields (3).

Let us prove that (3) yields (1). If $\exp G = \infty$, the assertion follows from Theorem 3. If $\exp G < \infty$, the assertion follows from Theorem 4. □

Acknowledgement: I am deeply indebted to Professor Shakhmatov for numerous suggestions which essentially improve the exposition of the article. It is a pleasure to thank A. Leiderman.

REFERENCES

1. M. Ajtai, I. Havas, J. Komlós, Every group admits a bad topology, Stud. Pure Math., Memory of P. Turan, Basel–Boston, 1983, 21–34.
2. D.L. Armacost, The structure of locally compact Abelian groups, Monographs and Textbooks in Pure and Applied Mathematics, 68, Marcel Dekker, Inc., New York, 1981.
3. G. Barbieri, D. Dikranjan, C. Milan, H. Weber, *Answer to Raczkowski's question on convergent sequences of integers*, Topology Appl., **132** (2003), 89–101.
4. G. Barbieri, D. Dikranjan, C. Milan, H. Weber, *Topological torsion related to some sequences of integers*, Math. Nachr., **281** (2008), №7, 930–950.
5. W.W. Comfort, Problems on Topological Groups and Other Homogeneous Spaces, Open problems in topology, 314–347, North-Holland, 1990.
6. W.W. Comfort, S.U. Raczkowski, F. Trigos-Arrieta, *Making group topologies with, and without, convergent sequences*, Applied General Topology, **7** (2006), №1, 109–124.
7. D. Dikranjan, C. Milan, A. Tonolo, *A characterization of the maximally almost periodic Abelian groups*, J. Pure Appl. Algebra, **197** (2005), 23–41.
8. S.S. Gabriyelyan, *On T-sequences and characterized subgroups*, Topology Appl., **157** (2010), 2834–2843.
9. S.S. Gabriyelyan, *Characterization of almost maximally almost-periodic groups*, Topology Appl., **156** (2009), 2214–2219.
10. E. Hewitt, K.A. Ross, Abstract Harmonic Analysis, V.I, 2nd ed. Springer-Verlag, Berlin, 1979.
11. G. Lukács, *Almost maximally almost-periodic group topologies determined by T-sequences*, Topology Appl., **153** (2006), 2922–2932.
12. J. von Neumann, *Almost periodic functions in a group*, Trans. Amer. Math. Soc., **36** (1934), 445–492.
13. N. Noble, *k-groups and duality*, Trans. Amer. Math. Soc., **151** (1970), 551–561.
14. I.V. Protasov, Review of Ajtai, Havas and J. Komlós, *Zentralblatt für Matematik*, **535** (1983), 93.
15. I.V. Protasov, E.G. Zelenyuk, *Topologies on abelian groups*, Math. USSR Izv., **37** (1991), 445–460. Russian original: Izv. Akad. Nauk SSSR, **54** (1990), 1090–1107.
16. I.V. Protasov, E.G. Zelenyuk, Topologies on groups determined by sequences, Monograph Series, Math. Studies VNTL, Lviv, 1999.

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, Israel
saak@math.bgu.ac.il

Received 12.07.2012