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ON $\mathscr{H}$-COMPLETE TOPOLOGICAL SEMILATTICES

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In the paper we describe the structure of $\mathscr{AH}$-completions and $\mathscr{H}$-completions of the discrete semilattices $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$. We give an example of an $\mathscr{H}$-complete topological semilattice which is not $\mathscr{AH}$-complete. Also for an arbitrary infinite cardinal λ we construct an $\mathscr{H}$-complete topological semilattice of cardinality λ which has 2^λ many open-and-closed continuous homomorphic images which are not $\mathscr{H}$-complete topological semilattices. The constructed examples give a negative answer to Question 17 in the paper J. W. Stepp, *Algebraic maximal semilattices*, Pacific J. Math., **58** (1975), no.1, 243–248.

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Описывается структура $\mathscr{AH}$-пополнений и $\mathscr{H}$-пополнений дискретных полурешеток $(\mathbb{N}, \min)$ и $(\mathbb{N}, \max)$. Приводится пример $\mathscr{H}$-полной топологической полурешетки, не являющейся $\mathscr{AH}$-полной. Для произвольного бесконечного кардинала λ строится $\mathscr{H}$-полная топологическая полурешетка мощности λ, имеющая 2^λ открыто-замкнутых непрерывных гомоморфных образов, не являющихся $\mathscr{H}$-полными топологическими полурешетками. Построенные примеры дают отрицательный ответ на вопрос 17, сформулированный в работе J. W. Stepp, *Algebraic maximal semilattices*, Pacific J. Math., **58** (1975), №1, 243–248.

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [1, 3], and [4]. For a subset A of a topological space X by $\mathrm{cl}_X(A)$ we denote the closure of A in X. A filter $\mathscr{F}$ on a set S is called *free* if $\bigcap \mathscr{F} = \varnothing$.

A *semilattice* is a set endowed with a commutative idempotent associative operation. If E is a semilattice, then the semilattice operation on E determines the partial order $\leqslant$ on E

$$e \leqslant f \quad \text{if and only if} \quad ef = fe = e.$$

This order is called *natural.* An element e of a semilattice E is called *minimal* (*maximal*) if $f \leqslant e$ ($e \geqslant f$) implies $f = e$ for $f \in E$. A semilattice E is said to be *linearly ordered* or a *chain* if the natural order on E is linear.

If S is a topological space equipped with a continuous semigroup operation then S is called a *topological semigroup*. A *topological semilattice* is a topological semigroup which is algebraically a semilattice.

Let $\mathscr{TS}$ be a category whose objects are topological semigroups and morphisms are homomorphisms between topological semigroups. A topological semigroup $X \in \mathrm{Ob}\,\mathscr{TS}$ is called *$\mathscr{TS}$-complete* if for each object $Y \in \mathrm{Ob}\,\mathscr{TS}$ and a morphism $f\colon X \to Y$ of the category $\mathscr{TS}$ the image $f(X)$ is closed in Y.

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By a *$\mathscr{TS}$-completion* of a topological semigroup X we understand any $\mathscr{TS}$-complete topological semigroup $\tilde{X} \in \operatorname{Ob}\mathscr{TS}$ containing X as a dense subsemigroup. A $\mathscr{TS}$-completion $\widetilde{X}$ of X is called *universal* if each continuous homomorphism $h\colon X \to Y$ to a $\mathscr{TS}$-complete topological semigroup $Y \in \operatorname{Ob}\mathscr{TS}$ extends to a continuous homomorphism $\tilde{h}\colon \widetilde{X} \to Y$.

It is well-known that for the category $\mathscr{TG}$ of topological groups and their continuous homomorphisms, each object $G \in \operatorname{Ob}\mathscr{TG}$ has a $\mathscr{TG}$-completion and each $\mathscr{TG}$-completion of G is universal ([8]).

In the category of topological semigroups the situation is totally different. We show this on the example of the discrete topological semigroups $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$. We shall study $\mathscr{H}$-completions and $\mathscr{AH}$-completions of discrete topological semigroup $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$ in the category $\mathscr{AH}$ (resp. $\mathscr{H}$) whose objects are Hausdorff topological semigroups and morphisms are continuous homomorphisms (resp. isomorphic topological embeddings) between topological semigroups.

The notion of $\mathscr{H}$-completion was introduced by J. W. Stepp in [9], where he showed that for each locally compact topological semigroup S there exists an $\mathscr{H}$-complete topological semigroup T which contains S as a dense subsemigroup.

J. W. Stepp ([10]) proved that a discrete semilattice E is $\mathscr{H}$-complete if and only if any maximal chain in E is finite. In [6] O. Gutik and K. Pavlyk remarked that a topological semilattice is $\mathscr{H}$-complete ($\mathscr{AH}$-complete) if and only if it is $\mathscr{H}$-complete ($\mathscr{AH}$-complete) as a topological semigroup. In [7] O. Gutik and D. Repovš studied properties of linearly ordered $\mathscr{H}$-complete topological semilattices and proved the following characterization theorem.

Theorem 1 ([7, Theorem 2])**.** *A linearly ordered topological semilattice E is $\mathscr{H}$-complete if and only if the following conditions hold:*
(i) *E is complete;*
(ii) $x = \sup A$ *for* $A = {\downarrow}A$ *implies* $x \in \operatorname{cl}_E A$*;*
(iii) $x = \inf B$ *for* $B = {\uparrow}B$ *implies* $x \in \operatorname{cl}_E B$*.*

Also, in [7] O. Gutik and D. Repovš proved that each linearly ordered $\mathscr{H}$-complete topological semilattice is $\mathscr{AH}$-complete and showed that every linearly ordered semilattice is a dense subsemilattice of an $\mathscr{H}$-complete topological semilattice. In [2] I. Chuchman and O. Gutik proved that any $\mathscr{H}$-complete locally compact topological semilattice and any $\mathscr{H}$-complete topological weakly U-semilattice contain minimal idempotents.

In [10, Question 17] J. W. Stepp asked the following question: *Is each $\mathscr{H}$-complete topological semilattice $\mathscr{AH}$-complete?* In the present paper we answer this Stepp's question in the negative by constructing an example of an $\mathscr{H}$-complete topological semilattice which is not $\mathscr{AH}$-complete. Also we construct an $\mathscr{H}$-complete topological semilattice of arbitrary infinite cardinality λ which has 2^λ many open-and-closed continuous homomorphic images which are not $\mathscr{H}$-complete topological semilattices.

Let $\mathbb{N}$ denote the set of positive integers. For each free filter $\mathscr{F}$ on $\mathbb{N}$ consider the topological space $\mathbb{N}_{\mathscr{F}} = \mathbb{N} \cup \{\mathscr{F}\}$ in which all points $x \in \mathbb{N}$ are isolated while the sets $F \cup \{\mathscr{F}\}$, $F \in \mathscr{F}$, form a neighborhood base at the unique non-isolated point $\mathscr{F}$.

The semilattice operation min (resp., max) of $\mathbb{N}$ extends to a continuous semilattice operation min (resp., max) on $\mathbb{N}_{\mathscr{F}}$ such that $\min\{n, \mathscr{F}\} = \min\{\mathscr{F}, n\} = n$ and $\min\{\mathscr{F}, \mathscr{F}\} = \mathscr{F}$ (resp., $\max\{n, \mathscr{F}\} = \max\{\mathscr{F}, n\} = \mathscr{F} = \max\{\mathscr{F}, \mathscr{F}\}$) for all $n \in \mathbb{N}$. By $\mathbb{N}_{\mathscr{F},\min}$ (resp., $\mathbb{N}_{\mathscr{F},\max}$) we shall denote the topological space $\mathbb{N}_{\mathscr{F}}$ with the semilattice operation min (resp., max). Simple verifications show that $\mathbb{N}_{\mathscr{F},\min}$ and $\mathbb{N}_{\mathscr{F},\max}$ are topological semilattices.

Theorem 2. (i) *For each free filter* $\mathscr{F}$ *on* $\mathbb{N}$ *the topological semilattices* $\mathbb{N}_{\mathscr{F},\min}$ *and* $\mathbb{N}_{\mathscr{F},\max}$ *are* $\mathscr{AH}$*-complete.*
(ii) *Each* $\mathscr{H}$*-completion of the discrete semilattice* $(\mathbb{N},\min)$ *(resp.,* $(\mathbb{N},\max)$*) is topologically isomorphic to the topological semilattice* $\mathbb{N}_{\mathscr{F},\min}$ *(resp.,* $\mathbb{N}_{\mathscr{F},\max}$*) for some free filter* $\mathscr{F}$ *on* $\mathbb{N}$*.*
(iii) *The topological semilattice* $(\mathbb{N},\min)$ *(resp.,* $(\mathbb{N},\max)$*) has no universal* $\mathscr{AH}$*-completion.*

Proof. (i) By Theorem 1, we have that the topological semilattices $\mathbb{N}_{\mathscr{F},\min}$ and $\mathbb{N}_{\mathscr{F},\max}$ are $\mathscr{H}$-complete. Since $\mathbb{N}_{\mathscr{F},\min}$ and $\mathbb{N}_{\mathscr{F},\max}$ are linearly ordered semilattices, Theorem 3 of [7] implies that the topological semilattices $\mathbb{N}_{\mathscr{F},\min}$ and $\mathbb{N}_{\mathscr{F},\max}$ are $\mathscr{AH}$-complete.

(ii) We shall prove the statement for the semilattice $(\mathbb{N},\min)$. In the case of $(\mathbb{N},\max)$ the proof is similar. Let S be an $\mathcal{H}$-complete topological semilattice containing $(\mathbb{N},\min)$ as a dense subsemilattice. Since the closure of a linearly ordered subsemilattice in a Hausdorff topological semigroup is a linearly ordered topological semilattice (see [6, Corollary 19] and [7, Lemma 1]), we conclude that S is linearly ordered and $S\setminus\mathbb{N}$ is a singleton $\{a\}$. Then since $(\mathbb{N},\min)$ is a dense subsemilattice of S, the continuity of the semilattice operation in S implies that $a\cdot a=a$ and $a\cdot n=n\cdot a=n$ for any $n\in\mathbb{N}$. Let $\mathscr{B}(a)$ be the filter of neighborhoods of the point a in S. This filter induces the free filter $\mathscr{F}=\{F\subset\mathbb{N}: F\cup\{a\}\in\mathcal{B}(a)\}$. Then we can identify the topological semilattice S with $\mathbb{N}_{\mathscr{F},\min}$ by the topological isomorphism $f\colon S\to\mathbb{N}_{\mathscr{F},\min}$ such that $f(a)=\mathscr{F}$ and $f(n)=n$ for every $n\in\mathbb{N}$.

(iii) Suppose the contrary: there exists a universal $\mathscr{AH}$-completion S of the discrete semilattice $(\mathbb{N},\max)$. Then by statement (ii), the semilattice S can be identified with the semilattice $\mathbb{N}_{\mathscr{F},\max}$ for some free filter $\mathscr{F}$ on $\mathbb{N}$. Let $\mathscr{F}'$ be any free filter on $\mathbb{N}$ such that $\mathscr{F}'\not\subset\mathscr{F}$. Then the identity embedding $\mathrm{id}_{\mathbb{N}}\colon(\mathbb{N},\max)\to\mathbb{N}_{\mathscr{F}',\max}$ cannot be extended to a continuous homomorphism $h\colon\mathbb{N}_{\mathscr{F},\max}\to\mathbb{N}_{\mathscr{F}',\max}$, witnessing that the $\mathscr{AH}$-completion S of $(\mathbb{N},\min)$ is not universal. □

Later on, by $E_2=\{0,1\}$ we denote the discrete topological semilattice with the semilattice operation min.

Theorem 3. *Let* $\mathscr{F}$ *be a free filter on* $\mathbb{N}$ *and* $F\in\mathscr{F}$ *be a set with infinite complement* $\mathbb{N}\setminus F$*. Then the following statements hold:*
(i) *the closed subsemilattice* $E=(\mathbb{N}_{\mathscr{F},\min}\times\{0\})\cup((\mathbb{N}\setminus F)\times\{1\})$ *of the direct product* $\mathbb{N}_{\mathscr{F},\min}\times E_2$ *is* $\mathscr{H}$*-complete;*
(ii) *the subset* $I=\mathbb{N}_{\mathscr{F},\min}\times\{0\}$ *is an open-and-closed ideal in* E*, and the quotient semilattice* E/I *with the quotient topology is discrete and not* $\mathscr{H}$*-complete;*
(iii) *the semilattice* E *is not* $\mathscr{AH}$*-complete.*

Proof. (i) The definition of the topological semilattice $\mathbb{N}_{\mathscr{F},\min}\times E_2$ implies that E is a closed subsemilattice of $\mathbb{N}_{\mathscr{F},\min}\times E_2$. Suppose the contrary: the topological semilattice E is not $\mathscr{H}$-complete. Since the closure of a subsemilattice in a topological semigroup is a semilattice (see Corollary 19 of [6]), we conclude that there exists a topological semilattice S which contains E as a dense subsemilattice and $S\setminus E\neq\varnothing$. We fix an arbitrary $a\in S\setminus E$. Then for every open neighborhood $U(a)$ of the point a in S we have that the set $U(a)\cap E$ is infinite. By Theorem 2, the subspace $\mathbb{N}_{\mathscr{F},\min}\times\{0\}$ of E with the induced semilattice operation from E is an $\mathscr{H}$-complete topological semilattice. Therefore, there exists an open neighborhood $U(a)$ of the point a in S such that $U(a)\cap E\subseteq(\mathbb{N}\setminus F)\times\{1\}$ and hence the set $U(a)\cap((\mathbb{N}\setminus F)\times\{1\})$ is infinite.

Next we shall show that $a\cdot x=x$ for any $x\in E\setminus\{(\mathscr{F},0)\}$. Since the set $U(x)\cap((\mathbb{N}\setminus F)\times\{1\})$ is infinite, the continuity of the semilattice operation in E implies that $a\cdot x=x$ for any $x\in(\mathbb{N}\setminus F)\times\{1\}$. Now fix any point $y\in\mathbb{N}\times\{0\}\subset E$. By the definition of the semilattice operation on E, we can find a point $x_y\in(\mathbb{N}\setminus F)\times\{1\}$ with $x_y\cdot y=y$ and conclude that

$$a\cdot y=a\cdot(x_y\cdot y)=(a\cdot x_y)\cdot y=x_y\cdot y=y.$$

Since $(\mathscr{F},0)$ is a cluster point of the set $\mathbb{N}\times\{0\}$, the continuity of the semilattice operation implies that $a\cdot(\mathscr{F},0)=(\mathscr{F},0)$.

Since $W(\mathscr{F},0)=(F\cup\{\mathscr{F}\})\times\{0\}$ is a neighborhood of the point $(\mathscr{F},0)=a\cdot(\mathscr{F},0)$, the continuity of the semilattice operation yields the existence of neighborhoods $U(a)$ and $V(\mathscr{F},0)$ of the points a and $(\mathscr{F},0)$ in S such that $U(a)\cdot V(\mathscr{F},0)\subset W(\mathscr{F},0)$. Now choose any point $(n,1)\in U(a)\cap\big((\mathbb{N}\setminus F)\times\{1\}\big)$ and find a point $(m,0)\in V(\mathscr{F},0)$ such that $m\geq n$. Then

$$(n,0)=(n,1)\cdot(m,1)\in U(a)\cdot U(\mathscr{F},0)\subset W(\mathscr{F},0)=(F\cup\{\mathscr{F}\})\times\{0\},$$

which contradicts the choice of $n\in\mathbb{N}\setminus F$.

(ii) The definition of the semilattice E implies that $I=\mathbb{N}_{\mathscr{F},\min}\times\{0\}$ is an open-and-closed ideal in E. Then the quotient semilattice E/I (endowed with the quotient topology) is a discrete topological semilattice, topologically isomorphic to the discrete semilattice $(\mathbb{N},\min)$. By Theorem 1, the semilattice E/I is not $\mathscr{H}$-complete.

Statement (iii) follows from statement (ii). □

Corollary 1. *For a free filter $\mathscr{F}$ on $\mathbb{N}$, each closed subsemilattice of the semilattice $\mathbb{N}_{\mathscr{F},\min}\times E_2$ is $\mathscr{AH}$-complete if and only if $\mathscr{F}$ is the filter of cofinite subsets of $\mathbb{N}$.*

Proof. $(\Leftarrow)$ If $\mathscr{F}$ is the filter of cofinite subsets of $\mathbb{N}$, then the space $\mathbb{N}_{\mathscr{F},\min}\times E_2$ is compact. Then each closed subset of $\mathbb{N}_{\mathscr{F},\min}\times E_2$ is compact and hence each closed subsemilattice of the semilattice $\mathbb{N}_{\mathscr{F},\min}\times E_2$ is $\mathscr{AH}$-complete.

$(\Rightarrow)$ If $\mathscr{F}$ is a free filter on $\mathbb{N}$ containing a set $F\subseteq\mathbb{N}$ with the infinite complement $\mathbb{N}\setminus F$, then $E=\big(\mathbb{N}_{\mathscr{F},\min}\times\{0\}\big)\cup\big((\mathbb{N}\setminus F)\times\{1\}\big)$ is a closed subsemilattice of the topological semilattice $\mathbb{N}_{\mathscr{F},\min}\times E_2$ and Theorem 3 implies that E is not $\mathscr{AH}$-complete. □

The proof of the following theorem is similar to the proof of Theorem 3 with some simple modifications.

Theorem 4. *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ and $F\in\mathscr{F}$ be a set with infinite complement $\mathbb{N}\setminus F$. Then the following assertions hold:*

(i) the closed subsemilattice $E=(\mathbb{N}_{\mathscr{F},\max}\times\{0\})\cup((\mathbb{N}\setminus F)\times\{1\})$ of the direct product $\mathbb{N}_{\mathscr{F},\max}\times E_2$ is $\mathscr{H}$-complete;

(ii) the subset $I=\mathbb{N}_{\mathscr{F},\max}\times\{0\}$ is an open-and-closed ideal in E, and the quotient semilattice E/I with the quotient topology is discrete and not $\mathscr{H}$-complete;

(iii) the semilattice E is not $\mathscr{AH}$-complete.

The proof of the following corollary is similar to Corollary 1 and it follows from Theorem 4.

Corollary 2. *For a free filter $\mathscr{F}$ on $\mathbb{N}$, each closed subsemilattice of the semilattice $\mathbb{N}_{\mathscr{F},\max}\times E_2$ is $\mathscr{AH}$-complete if and only if $\mathscr{F}$ is the filter of cofinite subsets of $\mathbb{N}$.*

We remark that Theorems 3 and 4 give a negative answer on Question 17 from [10].

Also, Theorems 3 and 4 imply the following corollary.

Corollary 3. *There exists a countable locally compact $\mathscr{H}$-complete topological semilattice E with an open-and-closed ideal I such that I is an $\mathscr{AH}$-complete semilattice and the Rees quotient semigroup E/I with the quotient topology is not $\mathscr{H}$-complete.*

Remark 1. A Hausdorff partially ordered space X is called *$\mathscr{H}$-complete* if X is a closed subspace of every Hausdorff partially ordered space in which it is contained ([5]). A linearly ordered topological semilattice E is $\mathscr{H}$-complete if and only if E is an $\mathscr{H}$-complete partially ordered space ([5]). In [11] Yokoyama showed that a partially ordered space X without an infinite antichain is an $\mathscr{H}$-complete partially ordered space if and only if X is a directed complete and down-complete poset such that $\sup L$ and $\inf L$ are contained in the closure of L for any nonempty chain L in X. Theorems 3 and 4 imply that there exists an $\mathscr{H}$-complete topological semilattice without an infinite antichain which is not an $\mathscr{H}$-complete partially ordered space. Also Theorem 3 implies that there exists a countable $\mathscr{H}$-complete locally compact topological semilattice E without an infinite antichain which contains a maximal chain L which is not directed complete, and L does not have a maximal element.

Let λ be any infinite cardinal and let $0 \notin \lambda$. On the set $E_\lambda = \{0\} \cup \lambda$ endowed with the discrete topology we define the semilattice operation by the formula $x \cdot y = \begin{cases} x, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$

Theorem 5. *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ and $F \in \mathscr{F}$ be a set with infinite complement $\mathbb{N} \setminus F$. Then for each infinite cardinal λ the following statements hold :*
(i) the closed subsemilattice $E = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \lambda)$ of the direct product $\mathbb{N}_{\mathscr{F},\max} \times E_\lambda$ is $\mathscr{H}$-complete;
(ii) for each subset $\kappa \subset \lambda$ the subset $I_\kappa = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \kappa)$ is an open-and-closed ideal in E, and the quotient semilattice E/I_κ with the quotient topology is discrete and not $\mathscr{H}$-complete;
(iii) the semilattice E is not $\mathscr{AH}$-complete.

Proof. (i) Assuming that the topological semilattice E is not $\mathscr{H}$-complete, find a topological semilattice T containing E as a dense subsemilattice with non-empty complement $T \setminus E$. Fix any element $e \in T \setminus E$. By Theorem 1, the topological semilattice $E^0 = \mathbb{N}_{\mathscr{F},\max} \times \{0\}$ is $\mathscr{H}$-complete and hence is closed in T. Then there exists an open neighborhood $U(e)$ of the point e in T such that $U(e) \cap E^0 = \varnothing$. By the continuity of the semilattice operation in T, there exists an open neighborhood $V(e) \subseteq U(e)$ of the point e in T such that $V(e) \cdot V(e) \subseteq U(e)$. By Theorem 4, for each $a \in \lambda$, the subsemilattice $E_a = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \{a\})$ of the direct product $\mathbb{N}_{\mathscr{F},\max} \times E_\lambda$ is $\mathscr{H}$-complete and hence is closed in T. This implies that $V(e) \cap E_a \neq \varnothing$ for infinitely many points $a \in \lambda$, and hence $(V(e) \cdot V(e)) \cap E^0 \neq \varnothing$. This contradicts the choice of the neighborhood $U(e)$. The obtained contradiction implies that the topological semilattice E is $\mathscr{H}$-complete.

(ii) The definition of the semilattice E implies that $I_\kappa = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \kappa)$ is an open-and-closed ideal in E. Then we have that the quotient semilattice E/I_κ with the quotient topology is a discrete topological semilattice. Also, E/I_κ is topologically isomorphic to the orthogonal sum of λ infinitely many of $(\mathbb{N}, \max)$ with isolated zero. This implies that the semilattice E/I_κ is not $\mathscr{H}$-complete.

Statement (iii) follows from statement (ii). $\square$

Remark 2. The topological semilattices E and I_κ from Theorem 5 are metrizable locally compact spaces for each free countably generated filter $\mathscr{F}$ on $\mathbb{N}$ and any $\kappa \subset \lambda$.

Remark 3. It can be shown that continuous homomorphisms into the discrete semilattice $(\{0,1\}, \min)$ separate points of the topological semilattices E considered in Theorems 4 and 5.

Since for each subset $\kappa \subset \lambda$ the natural homomorphism $\pi\colon E \to E/I_\kappa$ is an open-and-closed map, Theorem 5 implies the following corollary.

Corollary 4. *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ containing a set $F \in \mathscr{F}$ with infinite complement $\mathbb{N} \setminus F$. Then for each infinite cardinal λ there exist 2^λ many continuous open-and-closed surjective homomorphic images of the topological semilattice*

$$E = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \lambda) \subset \mathbb{N}_{\mathscr{F},\max} \times E_\lambda,$$

which are not $\mathscr{H}$-complete.

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