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ON $\mathscr{H}$-COMPLETE TOPOLOGICAL SEMILATTICES

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In the paper we describe the structure of $\mathscr{AH}$-completions and $\mathscr{H}$-completions of the discrete semilattices $(\mathbb{N},\min)$ and $(\mathbb{N},\max)$. We give an example of an $\mathscr{H}$-complete topological semilattice which is not $\mathscr{AH}$-complete. Also for an arbitrary infinite cardinal λ we construct an $\mathscr{H}$-complete topological semilattice of cardinality λ which has 2^λ many open-and-closed continuous homomorphic images which are not $\mathscr{H}$-complete topological semilattices. The constructed examples give a negative answer to Question 17 in the paper J. W. Stepp, *Algebraic maximal semilattices*, Pacific J. Math., **58** (1975), no.1, 243–248.

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Описывается структура $\mathscr{AH}$-пополнений и $\mathscr{H}$-пополнений дискретных полурешеток $(\mathbb{N},\min)$ и $(\mathbb{N},\max)$. Приводится пример $\mathscr{H}$-полной топологической полурешетки, не являющейся $\mathscr{AH}$-полной. Для произвольного бесконечного кардинала λ строится $\mathscr{H}$-полная топологическая полурешетка мощности λ, имеющая 2^λ открыто-замкнутых непрерывных гомоморфных образов, не являющихся $\mathscr{H}$-полными топологическими полурешетками. Построенные примеры дают отрицательный ответ на вопрос 17, сформулированный в работе J. W. Stepp, *Algebraic maximal semilattices*, Pacific J. Math., **58** (1975), №1, 243–248.

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [1, 3], and [4]. For a subset A of a topological space X by $\operatorname{cl}_X(A)$ we denote the closure of A in X. A filter $\mathscr{F}$ on a set S is called *free* if $\bigcap\mathscr{F}=\varnothing$.

A *semilattice* is a set endowed with a commutative idempotent associative operation. If E is a semilattice, then the semilattice operation on E determines the partial order $\leqslant$ on E

$$e\leqslant f \quad \text{if and only if} \quad ef=fe=e.$$

This order is called *natural.* An element e of a semilattice E is called *minimal* (*maximal*) if $f\leqslant e$ ($e\geqslant f$) implies $f=e$ for $f\in E$. A semilattice E is said to be *linearly ordered* or a *chain* if the natural order on E is linear.

If S is a topological space equipped with a continuous semigroup operation then S is called a *topological semigroup*. A *topological semilattice* is a topological semigroup which is algebraically a semilattice.

Let $\mathscr{TS}$ be a category whose objects are topological semigroups and morphisms are homomorphisms between topological semigroups. A topological semigroup $X\in\operatorname{Ob}\mathscr{TS}$ is called *$\mathscr{TS}$-complete* if for each object $Y\in\operatorname{Ob}\mathscr{TS}$ and a morphism $f\colon X\to Y$ of the category $\mathscr{TS}$ the image $f(X)$ is closed in Y.

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By a *$\mathscr{TS}$-completion* of a topological semigroup X we understand any $\mathscr{TS}$-complete topological semigroup $\tilde{X} \in \operatorname{Ob}\mathscr{TS}$ containing X as a dense subsemigroup. A $\mathscr{TS}$-completion $\widetilde{X}$ of X is called *universal* if each continuous homomorphism $h\colon X \to Y$ to a $\mathscr{TS}$-complete topological semigroup $Y \in \operatorname{Ob}\mathscr{TS}$ extends to a continuous homomorphism $\tilde{h}\colon \widetilde{X} \to Y$.

It is well-known that for the category $\mathscr{TG}$ of topological groups and their continuous homomorphisms, each object $G \in \operatorname{Ob}\mathscr{TG}$ has a $\mathscr{TG}$-completion and each $\mathscr{TG}$-completion of G is universal ([8]).

In the category of topological semigroups the situation is totally different. We show this on the example of the discrete topological semigroups $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$. We shall study $\mathscr{H}$-completions and $\mathscr{AH}$-completions of discrete topological semigroup $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$ in the category $\mathscr{AH}$ (resp. $\mathscr{H}$) whose objects are Hausdorff topological semigroups and morphisms are continuous homomorphisms (resp. isomorphic topological embeddings) between topological semigroups.

The notion of $\mathscr{H}$-completion was introduced by J. W. Stepp in [9], where he showed that for each locally compact topological semigroup S there exists an $\mathscr{H}$-complete topological semigroup T which contains S as a dense subsemigroup.

J. W. Stepp ([10]) proved that a discrete semilattice E is $\mathscr{H}$-complete if and only if any maximal chain in E is finite. In [6] O. Gutik and K. Pavlyk remarked that a topological semilattice is $\mathscr{H}$-complete ($\mathscr{AH}$-complete) if and only if it is $\mathscr{H}$-complete ($\mathscr{AH}$-complete) as a topological semigroup. In [7] O. Gutik and D. Repovš studied properties of linearly ordered $\mathscr{H}$-complete topological semilattices and proved the following characterization theorem.

Theorem 1 ([7, Theorem 2])**.** *A linearly ordered topological semilattice E is $\mathscr{H}$-complete if and only if the following conditions hold:*
(i) *E is complete;*
(ii) $x = \sup A$ *for* $A = {\downarrow}A$ *implies* $x \in \operatorname{cl}_E A$*;*
(iii) $x = \inf B$ *for* $B = {\uparrow}B$ *implies* $x \in \operatorname{cl}_E B$*.*

Also, in [7] O. Gutik and D. Repovš proved that each linearly ordered $\mathscr{H}$-complete topological semilattice is $\mathscr{AH}$-complete and showed that every linearly ordered semilattice is a dense subsemilattice of an $\mathscr{H}$-complete topological semilattice. In [2] I. Chuchman and O. Gutik proved that any $\mathscr{H}$-complete locally compact topological semilattice and any $\mathscr{H}$-complete topological weakly U-semilattice contain minimal idempotents.

In [10, Question 17] J. W. Stepp asked the following question: *Is each $\mathscr{H}$-complete topological semilattice $\mathscr{AH}$-complete?* In the present paper we answer this Stepp's question in the negative by constructing an example of an $\mathscr{H}$-complete topological semilattice which is not $\mathscr{AH}$-complete. Also we construct an $\mathscr{H}$-complete topological semilattice of arbitrary infinite cardinality λ which has 2^λ many open-and-closed continuous homomorphic images which are not $\mathscr{H}$-complete topological semilattices.

Let $\mathbb{N}$ denote the set of positive integers. For each free filter $\mathscr{F}$ on $\mathbb{N}$ consider the topological space $\mathbb{N}_{\mathscr{F}} = \mathbb{N} \cup \{\mathscr{F}\}$ in which all points $x \in \mathbb{N}$ are isolated while the sets $F \cup \{\mathscr{F}\}$, $F \in \mathscr{F}$, form a neighborhood base at the unique non-isolated point $\mathscr{F}$.

The semilattice operation min (resp., max) of $\mathbb{N}$ extends to a continuous semilattice operation min (resp., max) on $\mathbb{N}_{\mathscr{F}}$ such that $\min\{n, \mathscr{F}\} = \min\{\mathscr{F}, n\} = n$ and $\min\{\mathscr{F}, \mathscr{F}\} = \mathscr{F}$ (resp., $\max\{n, \mathscr{F}\} = \max\{\mathscr{F}, n\} = \mathscr{F} = \max\{\mathscr{F}, \mathscr{F}\}$) for all $n \in \mathbb{N}$. By $\mathbb{N}_{\mathscr{F},\min}$ (resp., $\mathbb{N}_{\mathscr{F},\max}$) we shall denote the topological space $\mathbb{N}_{\mathscr{F}}$ with the semilattice operation min (resp., max). Simple verifications show that $\mathbb{N}_{\mathscr{F},\min}$ and $\mathbb{N}_{\mathscr{F},\max}$ are topological semilattices.

Theorem 2. *(i) For each free filter $\mathscr{F}$ on $\mathbb{N}$ the topological semilattices $\mathbb{N}_{\mathscr{F},\min}$ and $\mathbb{N}_{\mathscr{F},\max}$ are $\mathscr{AH}$-complete.*
(ii) Each $\mathscr{H}$-completion of the discrete semilattice $(\mathbb{N},\min)$ (resp., $(\mathbb{N},\max)$) is topologically isomorphic to the topological semilattice $\mathbb{N}_{\mathscr{F},\min}$ (resp., $\mathbb{N}_{\mathscr{F},\max}$) for some free filter $\mathscr{F}$ on $\mathbb{N}$.
(iii) The topological semilattice $(\mathbb{N},\min)$ (resp., $(\mathbb{N},\max)$) has no universal $\mathscr{AH}$-completion.

Proof. (i) By Theorem 1, we have that the topological semilattices $\mathbb{N}_{\mathscr{F},\min}$ and $\mathbb{N}_{\mathscr{F},\max}$ are $\mathscr{H}$-complete. Since $\mathbb{N}_{\mathscr{F},\min}$ and $\mathbb{N}_{\mathscr{F},\max}$ are linearly ordered semilattices, Theorem 3 of [7] implies that the topological semilattices $\mathbb{N}_{\mathscr{F},\min}$ and $\mathbb{N}_{\mathscr{F},\max}$ are $\mathscr{AH}$-complete.

(ii) We shall prove the statement for the semilattice $(\mathbb{N},\min)$. In the case of $(\mathbb{N},\max)$ the proof is similar. Let S be an $\mathcal{H}$-complete topological semilattice containing $(\mathbb{N},\min)$ as a dense subsemilattice. Since the closure of a linearly ordered subsemilattice in a Hausdorff topological semigroup is a linearly ordered topological semilattice (see [6, Corollary 19] and [7, Lemma 1]), we conclude that S is linearly ordered and $S\setminus\mathbb{N}$ is a singleton $\{a\}$. Then since $(\mathbb{N},\min)$ is a dense subsemilattice of S, the continuity of the semilattice operation in S implies that $a\cdot a=a$ and $a\cdot n=n\cdot a=n$ for any $n\in\mathbb{N}$. Let $\mathscr{B}(a)$ be the filter of neighborhoods of the point a in S. This filter induces the free filter $\mathscr{F}=\{F\subset\mathbb{N}: F\cup\{a\}\in\mathcal{B}(a)\}$. Then we can identify the topological semilattice S with $\mathbb{N}_{\mathscr{F},\min}$ by the topological isomorphism $f\colon S\to\mathbb{N}_{\mathscr{F},\min}$ such that $f(a)=\mathscr{F}$ and $f(n)=n$ for every $n\in\mathbb{N}$.

(iii) Suppose the contrary: there exists a universal $\mathscr{AH}$-completion S of the discrete semilattice $(\mathbb{N},\max)$. Then by statement (ii), the semilattice S can be identified with the semilattice $\mathbb{N}_{\mathscr{F},\max}$ for some free filter $\mathscr{F}$ on $\mathbb{N}$. Let $\mathscr{F}'$ be any free filter on $\mathbb{N}$ such that $\mathscr{F}'\not\subset\mathscr{F}$. Then the identity embedding $\mathrm{id}_{\mathbb{N}}\colon(\mathbb{N},\max)\to\mathbb{N}_{\mathscr{F}',\max}$ cannot be extended to a continuous homomorphism $h\colon\mathbb{N}_{\mathscr{F},\max}\to\mathbb{N}_{\mathscr{F}',\max}$, witnessing that the $\mathscr{AH}$-completion S of $(\mathbb{N},\min)$ is not universal. □

Later on, by $E_2=\{0,1\}$ we denote the discrete topological semilattice with the semilattice operation min.

Theorem 3. *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ and $F\in\mathscr{F}$ be a set with infinite complement $\mathbb{N}\setminus F$. Then the following statements hold:*
(i) the closed subsemilattice $E=(\mathbb{N}_{\mathscr{F},\min}\times\{0\})\cup((\mathbb{N}\setminus F)\times\{1\})$ of the direct product $\mathbb{N}_{\mathscr{F},\min}\times E_2$ is $\mathscr{H}$-complete;
(ii) the subset $I=\mathbb{N}_{\mathscr{F},\min}\times\{0\}$ is an open-and-closed ideal in E, and the quotient semilattice E/I with the quotient topology is discrete and not $\mathscr{H}$-complete;
(iii) the semilattice E is not $\mathscr{AH}$-complete.

Proof. (i) The definition of the topological semilattice $\mathbb{N}_{\mathscr{F},\min}\times E_2$ implies that E is a closed subsemilattice of $\mathbb{N}_{\mathscr{F},\min}\times E_2$. Suppose the contrary: the topological semilattice E is not $\mathscr{H}$-complete. Since the closure of a subsemilattice in a topological semigroup is a semilattice (see Corollary 19 of [6]), we conclude that there exists a topological semilattice S which contains E as a dense subsemilattice and $S\setminus E\neq\varnothing$. We fix an arbitrary $a\in S\setminus E$. Then for every open neighborhood $U(a)$ of the point a in S we have that the set $U(a)\cap E$ is infinite. By Theorem 2, the subspace $\mathbb{N}_{\mathscr{F},\min}\times\{0\}$ of E with the induced semilattice operation from E is an $\mathscr{H}$-complete topological semilattice. Therefore, there exists an open neighborhood $U(a)$ of the point a in S such that $U(a)\cap E\subseteq(\mathbb{N}\setminus F)\times\{1\}$ and hence the set $U(a)\cap((\mathbb{N}\setminus F)\times\{1\})$ is infinite.

Next we shall show that $a\cdot x = x$ for any $x \in E\setminus\{(\mathscr{F},0)\}$. Since the set $U(x)\cap((\mathbb{N}\setminus F)\times\{1\})$ is infinite, the continuity of the semilattice operation in E implies that $a\cdot x = x$ for any $x \in (\mathbb{N}\setminus F)\times\{1\}$. Now fix any point $y \in \mathbb{N}\times\{0\} \subset E$. By the definition of the semilattice operation on E, we can find a point $x_y \in (\mathbb{N}\setminus F)\times\{1\}$ with $x_y\cdot y = y$ and conclude that

$$a\cdot y = a\cdot(x_y\cdot y) = (a\cdot x_y)\cdot y = x_y\cdot y = y.$$

Since $(\mathscr{F},0)$ is a cluster point of the set $\mathbb{N}\times\{0\}$, the continuity of the semilattice operation implies that $a\cdot(\mathscr{F},0) = (\mathscr{F},0)$.

Since $W(\mathscr{F},0) = (F\cup\{\mathscr{F}\})\times\{0\}$ is a neighborhood of the point $(\mathscr{F},0) = a\cdot(\mathscr{F},0)$, the continuity of the semilattice operation yields the existence of neighborhoods $U(a)$ and $V(\mathscr{F},0)$ of the points a and $(\mathscr{F},0)$ in S such that $U(a)\cdot V(\mathscr{F},0) \subset W(\mathscr{F},0)$. Now choose any point $(n,1) \in U(a)\cap\big((\mathbb{N}\setminus F)\times\{1\}\big)$ and find a point $(m,0) \in V(\mathscr{F},0)$ such that $m \geq n$. Then

$$(n,0) = (n,1)\cdot(m,1) \in U(a)\cdot U(\mathscr{F},0) \subset W(\mathscr{F},0) = (F\cup\{\mathscr{F}\})\times\{0\},$$

which contradicts the choice of $n \in \mathbb{N}\setminus F$.

(ii) The definition of the semilattice E implies that $I = \mathbb{N}_{\mathscr{F},\min}\times\{0\}$ is an open-and-closed ideal in E. Then the quotient semilattice E/I (endowed with the quotient topology) is a discrete topological semilattice, topologically isomorphic to the discrete semilattice $(\mathbb{N},\min)$. By Theorem 1, the semilattice E/I is not $\mathscr{H}$-complete.

Statement (iii) follows from statement (ii). □

Corollary 1. *For a free filter $\mathscr{F}$ on $\mathbb{N}$, each closed subsemilattice of the semilattice $\mathbb{N}_{\mathscr{F},\min}\times E_2$ is $\mathscr{AH}$-complete if and only if $\mathscr{F}$ is the filter of cofinite subsets of $\mathbb{N}$.*

Proof. ($\Leftarrow$) If $\mathscr{F}$ is the filter of cofinite subsets of $\mathbb{N}$, then the space $\mathbb{N}_{\mathscr{F},\min}\times E_2$ is compact. Then each closed subset of $\mathbb{N}_{\mathscr{F},\min}\times E_2$ is compact and hence each closed subsemilattice of the semilattice $\mathbb{N}_{\mathscr{F},\min}\times E_2$ is $\mathscr{AH}$-complete.

($\Rightarrow$) If $\mathscr{F}$ is a free filter on $\mathbb{N}$ containing a set $F \subseteq \mathbb{N}$ with the infinite complement $\mathbb{N}\setminus F$, then $E = \big(\mathbb{N}_{\mathscr{F},\min}\times\{0\}\big)\cup\big((\mathbb{N}\setminus F)\times\{1\}\big)$ is a closed subsemilattice of the topological semilattice $\mathbb{N}_{\mathscr{F},\min}\times E_2$ and Theorem 3 implies that E is not $\mathscr{AH}$-complete. □

The proof of the following theorem is similar to the proof of Theorem 3 with some simple modifications.

Theorem 4. *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ and $F \in \mathscr{F}$ be a set with infinite complement $\mathbb{N}\setminus F$. Then the following assertions hold:*
(i) the closed subsemilattice $E = (\mathbb{N}_{\mathscr{F},\max}\times\{0\})\cup((\mathbb{N}\setminus F)\times\{1\})$ of the direct product $\mathbb{N}_{\mathscr{F},\max}\times E_2$ is $\mathscr{H}$-complete;
(ii) the subset $I = \mathbb{N}_{\mathscr{F},\max}\times\{0\}$ is an open-and-closed ideal in E, and the quotient semilattice E/I with the quotient topology is discrete and not $\mathscr{H}$-complete;
(iii) the semilattice E is not $\mathscr{AH}$-complete.

The proof of the following corollary is similar to Corollary 1 and it follows from Theorem 4.

Corollary 2. *For a free filter $\mathscr{F}$ on $\mathbb{N}$, each closed subsemilattice of the semilattice $\mathbb{N}_{\mathscr{F},\max}\times E_2$ is $\mathscr{AH}$-complete if and only if $\mathscr{F}$ is the filter of cofinite subsets of $\mathbb{N}$.*

We remark that Theorems 3 and 4 give a negative answer on Question 17 from [10].

Also, Theorems 3 and 4 imply the following corollary.

Corollary 3. *There exists a countable locally compact $\mathscr{H}$-complete topological semilattice E with an open-and-closed ideal I such that I is an $\mathscr{AH}$-complete semilattice and the Rees quotient semigroup E/I with the quotient topology is not $\mathscr{H}$-complete.*

Remark 1. A Hausdorff partially ordered space X is called *$\mathscr{H}$-complete* if X is a closed subspace of every Hausdorff partially ordered space in which it is contained ([5]). A linearly ordered topological semilattice E is $\mathscr{H}$-complete if and only if E is an $\mathscr{H}$-complete partially ordered space ([5]). In [11] Yokoyama showed that a partially ordered space X without an infinite antichain is an $\mathscr{H}$-complete partially ordered space if and only if X is a directed complete and down-complete poset such that $\sup L$ and $\inf L$ are contained in the closure of L for any nonempty chain L in X. Theorems 3 and 4 imply that there exists an $\mathscr{H}$-complete topological semilattice without an infinite antichain which is not an $\mathscr{H}$-complete partially ordered space. Also Theorem 3 implies that there exists a countable $\mathscr{H}$-complete locally compact topological semilattice E without an infinite antichain which contains a maximal chain L which is not directed complete, and L does not have a maximal element.

Let λ be any infinite cardinal and let $0 \notin \lambda$. On the set $E_\lambda = \{0\} \cup \lambda$ endowed with the discrete topology we define the semilattice operation by the formula $x \cdot y = \begin{cases} x, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$

Theorem 5. *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ and $F \in \mathscr{F}$ be a set with infinite complement $\mathbb{N} \setminus F$. Then for each infinite cardinal λ the following statements hold :*
(i) the closed subsemilattice $E = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \lambda)$ of the direct product $\mathbb{N}_{\mathscr{F},\max} \times E_\lambda$ is $\mathscr{H}$-complete;
(ii) for each subset $\kappa \subset \lambda$ the subset $I_\kappa = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \kappa)$ is an open-and-closed ideal in E, and the quotient semilattice E/I_κ with the quotient topology is discrete and not $\mathscr{H}$-complete;
(iii) the semilattice E is not $\mathscr{AH}$-complete.

Proof. (i) Assuming that the topological semilattice E is not $\mathscr{H}$-complete, find a topological semilattice T containing E as a dense subsemilattice with non-empty complement $T \setminus E$. Fix any element $e \in T \setminus E$. By Theorem 1, the topological semilattice $E^0 = \mathbb{N}_{\mathscr{F},\max} \times \{0\}$ is $\mathscr{H}$-complete and hence is closed in T. Then there exists an open neighborhood $U(e)$ of the point e in T such that $U(e) \cap E^0 = \varnothing$. By the continuity of the semilattice operation in T, there exists an open neighborhood $V(e) \subseteq U(e)$ of the point e in T such that $V(e) \cdot V(e) \subseteq U(e)$. By Theorem 4, for each $a \in \lambda$, the subsemilattice $E_a = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \{a\})$ of the direct product $\mathbb{N}_{\mathscr{F},\max} \times E_\lambda$ is $\mathscr{H}$-complete and hence is closed in T. This implies that $V(e) \cap E_a \neq \varnothing$ for infinitely many points $a \in \lambda$, and hence $(V(e) \cdot V(e)) \cap E^0 \neq \varnothing$. This contradicts the choice of the neighborhood $U(e)$. The obtained contradiction implies that the topological semilattice E is $\mathscr{H}$-complete.

(ii) The definition of the semilattice E implies that $I_\kappa = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \kappa)$ is an open-and-closed ideal in E. Then we have that the quotient semilattice E/I_κ with the quotient topology is a discrete topological semilattice. Also, E/I_κ is topologically isomorphic to the orthogonal sum of λ infinitely many of $(\mathbb{N}, \max)$ with isolated zero. This implies that the semilattice E/I_κ is not $\mathscr{H}$-complete.

Statement (iii) follows from statement (ii). □

Remark 2. The topological semilattices E and I_κ from Theorem 5 are metrizable locally compact spaces for each free countably generated filter $\mathscr{F}$ on $\mathbb{N}$ and any $\kappa \subset \lambda$.

Remark 3. It can be shown that continuous homomorphisms into the discrete semilattice $(\{0,1\}, \min)$ separate points of the topological semilattices E considered in Theorems 4 and 5.

Since for each subset $\kappa \subset \lambda$ the natural homomorphism $\pi\colon E \to E/I_\kappa$ is an open-and-closed map, Theorem 5 implies the following corollary.

Corollary 4. *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ containing a set $F \in \mathscr{F}$ with infinite complement $\mathbb{N} \setminus F$. Then for each infinite cardinal λ there exist 2^λ many continuous open-and-closed surjective homomorphic images of the topological semilattice*

$$E = (\mathbb{N}_{\mathscr{F},\max} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \lambda) \subset \mathbb{N}_{\mathscr{F},\max} \times E_\lambda,$$

which are not $\mathscr{H}$-complete.

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REFERENCES

1. J.H. Carruth, J.A. Hildebrant, R.J. Koch, The theory of topological semigroups, Vol. I, Marcel Dekker, Inc., New York and Basel, 1983; Vol. II, Marcel Dekker, Inc., New York and Basel, 1986.
2. I. Chuchman, O. Gutik, *On H-closed topological semigroups and semilattices*, Algebra Discrete Math., (2007), №1, 13–23.
3. R. Engelking, General topology, 2nd ed., Heldermann, Berlin, 1989.
4. G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, D.S. Scott, Continuous lattices and domains, Cambridge Univ. Press, Cambridge, 2003.
5. O. Gutik, D. Pagon, D. Repovš, *On chains in H-closed topological pospaces*, Order, **27** (2010), №1, 69–81.
6. O. Gutik, K. Pavlyk, *Topological Brandt λ-extensions of absolutely H-closed topological inverse semigroups*, Visnyk Lviv. Univ. Ser. Mekh.-Mat., **61** (2003), 98–105.
7. O. Gutik, D. Repovš, *On linearly ordered H-closed topological semilattices*, Semigroup Forum, **77** (2008), №3, 474–481.
8. D.A. Raikov, *On a completion of topological groups*, Izv. Akad. Nauk SSSR, **10** (1946), №6, 513–528. (in Russian)
9. J.W. Stepp, *A note on maximal locally compact semigroups*, Proc. Amer. Math. Soc., **20** (1969), 251–253.
10. J. W. Stepp, *Algebraic maximal semilattices*, Pacific J. Math., **58** (1975), №1, 243–248.
11. T. Yokoyama, *On completeness of H-closed pospaces*, arXiv:1004.3038v1.

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