УДК 519.51

K. D. Protasova

PRETHICK SUBSETS AND PARTITIONS OF METRIC SPACES

K. D. Protasova. *Prethick subsets and partitions of metric spaces*, Mat. Stud. **38** (2012), 115–117.

A subset A of a metric space (X,d) is called thick if, for every $r>0$, there is $a\in A$ such that $B_d(a,r)\subseteq A$, where $B_d(a,r)=\{x\in X\colon d(x,a)\le r\}$. We show that if (X,d) is unbounded and has no asymptotically isolated balls then, for each $r>0$, there exists a partition $X=X_1\cup X_2$ such that $B_d(X_1,r)$ and $B_d(X_2,r)$ are not thick.

К. Д. Протасова. *Предтолстые подмножества и разбиения метрических пространств* // Мат. Студії. – 2012. – Т.38, №2. – C.115–117.

Подмножество A метрического пространства (X,d) называется толстым, если для любого $r>0$ существует элемент $a\in A$ такой, что $B_d(a,r)\subseteq A$, где $B_d(a,r)=\{x\in X\colon d(x,a)\le r\}$. Доказано, что если (X,d) неограниченно и не имеет асимптотически изолированных шаров, то для любого $r>0$ существует разбиение $X=X_1\cup X_2$ такое, что подмножества $B_d(X_1,r)$ и $B_d(X_2,r)$ не являются толстыми.

Given a metric space (X,d) and $x\in X$, $A\subseteq X$, $r\in\mathbb{R}^+$, $\mathbb{R}^+=\{r\in\mathbb{R}\colon r>0\}$ let

$$B(x,r)=\{y\in X\colon d(x,y)\le r\},\ B(A,r)=\bigcup_{a\in A}B(a,r).$$

A subset A of X is called

- *large* if $X=B(A,r)$ for some $r\in\mathbb{R}^+$;
- *small* if $L\setminus A$ is large for each large subset L;
- *thick* if, for each $r\in\mathbb{R}^+$, there is $a\in A$ such that $B(a,r)\subseteq A$;
- *r-prethick* if $B(A,r)$ is thick;
- *prethick* if A is r-prethick for some $r\in\mathbb{R}^+$.

We note that A is small if and only if A is not prethick, A is thick if and only if $X\setminus A$ is not large. If X is bounded (i.e., $X=B(x,r)$ for some $x\in X$ and $r\in\mathbb{R}^+$), each nonempty subset of X is large and prethick, if A is thick then $A=X$.

In what follows, all metric spaces are supposed to be unbounded.

By [2, Theorem 11.2], the family of all small subsets of X is an ideal in the Boolean algebra of all subsets of X. It follows that if X is finitely partitioned $X=X_1\cup\ldots\cup X_n$ then at least one of the cells X_i is prethick.

In this note, we give a complete answer to the following question: *Given a metric X and $n\in\mathbb{N}$, does there exist $r=r(X,n)$, $r\in\mathbb{R}^+$ such that, for each n-partition of X, at least*

2010 *Mathematics Subject Classification:* 05D05, 20A05.
Keywords: metric space, thick and prethick subsets, asymptotically isolated balls.

one of the partition cells is r-prethick? An analogous problem in the realm of G-spaces and groups was considered in [1].

We use the following definition from [2]. For $r > 0$, a metric space X has isolated r-balls if, for each $t > r$, there is $x \in X$ such that $B(x,t) \setminus B(x,r) = \varnothing$. If X has asymptotically isolated r-balls for some $r > 0$, we say that X has *asymptotically isolated balls.*

A partition $X = X_1 \cup \dots \cup X_n$ is called r-meager, if each cell X_i is not r-prethick.

Theorem 1. *For a metric space X, the following statements hold:*

(i) if X has asymptotically isolated r-balls, then for any n-partition $X = X_1 \cup \dots \cup X_n$, at least one of the cells X_i is r-prethick;

(ii) if X has no asymptotically isolated balls, then for each $r > 0$, there exists an r-meager 2-partition of X.

Proof. (i) We choose a sequence $(x_n)_{n\in\omega}$ in X and an increasing sequence $(k_n)_{n\in\omega}$ of positive integers such that $B(x_n, k_n) \setminus B(x_n, r) = \varnothing$. Then we pick a cell X_i of the partition containing infinitely many members of $(x_n)_{n\in\omega}$, and note that $B(X_i, r)$ is thick.

(ii) We take $t > 2r$ such that $B(x,t) \setminus B(x, 2r) \neq \varnothing$ for each $x \in X$. Using the Zorn lemma, we choose a subset $Y \subset X$ such that

(1) $B(y,t) \cap B(y',t) = \varnothing$ for all distinct $y, y' \in Y$;

(2) for each $x \in X$, there is $y \in Y$ such that $B(x,t) \cap B(y,t) \neq \varnothing$.

We put $X_1 = \cup_{y\in Y} B(y,r)$, $X_2 = X \setminus X_1$. By (1), (2) and the choice of t, the subsets $X \setminus B(X_1, r)$ and $X \setminus B(X_2, r)$ are large. Hence, X_1 and X_2 are not r-prethick. □

Remark 1. A metric space (X,d) is called *coarsely geodesic* if there are $\varepsilon > 0$ and a function $f\colon [0,\infty) \to \mathbb{N}$ such that any points $x, y \in X$ can be linked by a sequence of points $x = x_0, \dots, x_n = y$ of length $n \le f(d(x,y))$ such that $d(x_i, x_{i+1}) \le \varepsilon$ for all $i < n$.

Each connected graph with the set of vertices V can be considered as a coarsely geodesic metric space (V,d), where d is the path metric on V. By [4, 5.1.1], each coarsely geodesic metric space is coarsely equivalent to some connected graph.

It is easy to see that a coarsely geodesic metric space (X,d) has no asymptotically isolated balls, but in the proof of (ii) the corresponding subsets X_1, X_2 can be chosen more constructively. We fix $x_0 \in X$, take an arbitrary $s > r + t$, t is chosen from the definition of a geodesic space, and put

$$X_1 = \bigcup_{s\in\omega} (B(x_0, 2s+1) \setminus B(x_0, 2s)),\ \ X_2 = X \setminus X_1.$$

Remark 2. We can generalize Theorem for balleans instead of metric spaces. Recall [4] that a *ball structure* B is a triple (X, P, B), where X, P are non-empty sets and, for every $x \in X$ and $\alpha \in P$, $B(x,\alpha)$ is a subset of X which is called a *ball of radius α around x*. The set X is called the *support* of B, P is called the *set of radii.*

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x,\alpha) = \{y \in X \colon x \in B(y,\alpha)\},\ \ B(A,\alpha) = \bigcup_{a\in A} B(a,\alpha).$$

A ball structure $\mathcal{B}$ is called a *ballean* if

- $\forall\,\alpha,\beta \in P\;\exists\,\alpha',\beta' \in P\;\forall\,x \in X\big(B(x,\alpha) \subseteq B^*(x,\alpha')$ and $B^*(x,\beta) \subseteq B(x,\beta')\big)$;
- $\forall\,\alpha,\beta \in P\;\exists\,\gamma \in P\;\forall\,x \in X\big(B(B(x,\alpha),\beta) \subseteq B(x,\gamma)\big)$.

A ballean B is connected (bounded) if for any $x, y \in X$ there is $\alpha \in P$ such that $y \in B(x,\alpha)$ ($X = B(x,\alpha)$ for some $x \in X$, $\alpha \in P$).

We use a natural preordering α on P defined by $\alpha \prec \beta$ if $B(x,\alpha) \subseteq B(x,\beta)$ for every $x \in X$.

Each metric space (X,d) defines the ballean $(X, \mathbb{R}^+, B_\alpha)$. Clearly, all the definitions from this note can be literally rewritten for balleans instead of metric spaces. Moreover, the same can be done with the proof of Theorem 1 for all connected unbounded balleans.

REFERENCES

1. T. Banakh, I.V. Protasov, S. Slobodianiuk, *Subamenable groups and their partitions*, preprint (http://arxiv.org/abs/1210.5804).
2. T. Banakh, I. Zarichnyi, *The coarse characterization of homogeneous ultrametric space*, Groups, Geometry and Dynamics, **5** (2011), 691–728.
3. I. Protasov, T. Banakh, Ball Structures and Colorings of Groups and Graphs, Math. Stud. Monogr. Ser., V.11, VNTL Publisher, Lviv, 2003.
4. I. Protasov, M. Zarichnyi, General Asymptology, Math. Stud. Monogr. Ser. V.12, VNTL Publisher, Lviv, 2007.

Department of Cybernetics, Kyiv University
islab@unicyb.kiev.ua

Received 01.09.2012