УДК 517.956

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GLOBAL CLASSICAL SOLVABILITY OF A PROBLEM WITH NONLOCAL CONDITIONS FOR A DEGENERATE HYPERBOLIC SYSTEM OF THE FIRST ORDER EQUATIONS

R. V. Andrusyak, V. M. Kyrylych, O. V. Peliushkevych. *Global classical solvability of a problem with nonlocal conditions for degenerate hyperbolic system of the first order equations*, Mat. Stud. **38** (2012), 80–92.

Using the method of characteristics and the Banach fixed point theorem we established the existence and uniqueness of a global classical (smooth) solution to an initial-boundary value problem with nonlocal boundary conditions for a hyperbolic integro-differential system involving equations without time derivative of unknown functions.

Р. В. Андрусяк, В. М. Кирилич, О. В. Пелюшкевич. *Глобальная классическая разрешимость задачи с нелокальными условиями для вырожденной гиперболической системы уравнений первого порядка* // Мат. Студії. – 2012. – Т.38, №1. – С.80–92.

С использованием метода характеристик и теорема Банаха о неподвижной точке установлены условия существования и единственности глобального классического (гладкого) решения смешанной задачи с нелокальными краевыми условиями для гиперболической интегро-дифференциальной системы, при этом часть уравнений системы не содержит производной по времени от искомых функций.

Introduction. Hyperbolic equations and systems are modeling wave phenomena of natural science, particularly such systems appear in gas dynamics, hydrodynamics, shallow-water theory, biological population theory, optimal control etc. ([1]–[3]).

Traditionally we consider hyperbolic systems that are solved with respect to the time derivative of unknown functions. An initial-boundary value problem for this system is reduced to the operator equation $u = \mathcal{A}u$, where an operator $\mathcal{A}$ is defined on elements u of some metric space. Solvability of the problem is generally established on a diminished time interval, moreover the interval smallness provide a contractive property of the operator $\mathcal{A}$ or $\mathcal{A}(\mathcal{A})$.

Degenerate hyperbolic systems (in the sense of paper [7]) are more complicated for research. These systems can be represented for example in the form

$$\begin{cases} \frac{\partial u}{\partial t} + A(x,t)\frac{\partial u}{\partial x} = f(x,t,u,v), \\ \frac{\partial v}{\partial x} = g(x,t,u,v), \end{cases}$$

where u, v are column vectors of the decision functions, A is a matrix, f, g are column vectors ([4, 5]). Changing the unknown functions, we reduce the system to a simpler form

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} + \Lambda(x,t)\frac{\partial \tilde{u}}{\partial x} = \tilde{f}(x,t,\tilde{u},\tilde{v}), \\ \frac{\partial \tilde{v}}{\partial x} = \tilde{g}(x,t,\tilde{u},\tilde{v}), \end{cases}$$

2010 *Mathematics Subject Classification:* 35L50.
Keywords: hyperbolic system, method of characteristics, Banach theorem, fixed point.

where $\tilde{u}$, $\tilde{v}$ are new decision functions, Λ is a diagonal matrix with the eigenvalues of A on the diagonal.

The problem for such a system can be similarly reduced to an operator equation but the time interval smallness is not sufficient for a contractive property of the corresponding operator with respect to the uniform metric.

In this paper we consider an initial-boundary value problem for a degenerate hyperbolic system with integral terms, where integration is performed with respect to the time and space variables. The problem is involved by nonlocal boundary conditions with integral terms. The main result of our research concerns classical solvability of the problem without any restrictions on the time interval magnitude. The finding of a problem's classical solution is reduced to the finding of an operator's fixed point, moreover globality of the solution was found thanks to a specially selected metric with weight functions ([6, 7]).

Statement of a problem. In the domain $\Pi = \{(x,t)\colon 0 < x < l,\ 0 < t < T\}$ we consider a linear hyperbolic system with integral terms. Moreover, some equations of the system do not contain the time derivative of unknown functions

$$\frac{\partial u_i}{\partial t} + \lambda_i(x,t)\frac{\partial u_i}{\partial x} =$$
$$= \sum_{j=1}^{m}\left(a_{ij}(x,t)u_j(x,t) + \int_0^t A^1_{ij}(x,t,\sigma)u_j(x,\sigma)d\sigma + \int_0^x A^2_{ij}(x,t,z)u_j(z,t)dz\right) +$$
$$+ \sum_{j=1}^{n}\left(b_{ij}(x,t)v_j(x,t) + \int_0^t B^1_{ij}(x,t,\sigma)v_j(x,\sigma)d\sigma + \int_0^x B^2_{ij}(x,t,z)v_j(z,t)dz\right) +$$
$$+ f_i(x,t), \quad i \in \{1,\dots,m\}, \tag{1}$$

$$\frac{\partial v_i}{\partial x} = \sum_{j=1}^{m}\left(c_{ij}(x,t)u_j(x,t) + \int_0^t C^1_{ij}(x,t,\sigma)u_j(x,\sigma)d\sigma + \int_0^x C^2_{ij}(x,t,z)u_j(z,t)dz\right) +$$
$$+ \sum_{j=1}^{n}\left(d_{ij}(x,t)v_j(x,t) + \int_0^t D^1_{ij}(x,t,\sigma)v_j(x,\sigma)d\sigma + \int_0^x D^2_{ij}(x,t,z)v_j(z,t)dz\right) +$$
$$+ g_i(x,t), \quad i \in \{1,\dots,n\}. \tag{2}$$

Suppose u_i is subject to initial conditions

$$u_i(x,0) = q_i(x), \quad 0 \le x \le l,\ i \in \{1,\dots,m\}. \tag{3}$$

Assume that $\operatorname{sgn}\lambda_i(0,t)$, $\operatorname{sgn}\lambda_i(l,t)$ are constant for every $t \in [0,T]$. Let us define sets of indices $I_0 = \{i \in \{1,2,\dots,m\}\colon \lambda_i(0,t) > 0\}$, $I_l = \{i \in \{1,2,\dots,m\}\colon \lambda_i(l,t) < 0\}$, which contain r_0 and r_l elements respectively. Then we impose boundary conditions as follows

$$\sum_{j=1}^{m}\left(\gamma^0_{ij}(t)u_j(0,t) + \gamma^l_{ij}(t)u_j(l,t) + \int_0^t \left(\Gamma^0_{ij}(t,\tau)u_j(0,\tau) + \Gamma^l_{ij}(t,\tau)u_j(l,\tau)\right)d\tau\right) +$$
$$+ \sum_{j=1}^{n}\left(\psi_{ij}(t)v_j(0,t) + \int_0^t \left(\Psi^0_{ij}(t,\tau)v_j(0,\tau) + \Psi^l_{ij}(t,\tau)v_j(l,\tau)\right)d\tau\right) = \delta_i(t),$$
$$i \in \{1,\dots,n+r_0+r_l\}. \tag{4}$$

Definition. Suppose $u \in (C^1(\overline{\Pi}))^m$, $v \in (C^{1,0}(\overline{\Pi}))^n$; then a pair of functions $w = (u, v)$ satisfing system (1), (2) and conditions (3), (4) is called a *classical solution of problem* (1)–(4).

Equivalent integral systems. Assume $\lambda_i \in C(\overline{\Pi}) \cap \mathrm{Lip}_x(\overline{\Pi})$. We denote by $\varphi_i(t; x_0, t_0)$ the solution of the Cauchy problem

$$\frac{dx}{dt} = \lambda_i(x,t), \quad (x,t) \in \overline{\Pi}, \quad x(t_0) = x_0, \quad (x_0, t_0) \in \overline{\Pi}.$$

Note these solutions are characteristics of system (1), (2). Moreover, this system has also characteristics $t = t_0$, where $t_0 \in [0, T]$. Suppose the integral curve $x = \varphi_i(t; x_0, t_0)$, $t \le t_0$ reaches the boundary of $\overline{\Pi}$ at $t = \chi_i(x_0, t_0)$.

Integrating (1), (2) along the corresponding characteristics, we obtain the following system of integral equations

$$\begin{aligned} u_i(x,t) = u_i\Big(\varphi_i(\chi_i(x,t); x,t), \chi_i(x,t)\Big) + \int_{\chi_i(x,t)}^{t} \Bigg(\sum_{j=1}^{m} \Bigg(a_{ij}(\varphi_i(\tau; x,t), \tau) u_j(\varphi_i(\tau; x,t), \tau) + \\ + \int_0^{\tau} A_{ij}^1(\varphi_i(\tau; x,t), \tau, \sigma) u_j(\varphi_i(\tau; x,t), \sigma) d\sigma + \int_0^{\varphi_i(\tau; x,t)} A_{ij}^2(\varphi_i(\tau; x,t), \tau, z) u_j(z, \tau) dz \Bigg) + \\ + \sum_{j=1}^{n} \Bigg(b_{ij}(\varphi_i(\tau; x,t), \tau) v_j(\varphi_i(\tau; x,t), \tau) + \int_0^{\tau} B_{ij}^1(\varphi_i(\tau; x,t), \tau, \sigma) v_j(\varphi_i(\tau; x,t), \sigma) d\sigma + \\ + \int_0^{\varphi_i(\tau; x,t)} B_{ij}^2(\varphi_i(\tau; x,t), \tau, z) v_j(z, \tau) dz \Bigg) + f_i(\varphi_i(\tau; x,t), \tau) \Bigg) d\tau, \quad i \in \{1, \dots, m\}, \end{aligned} \tag{5}$$

$$\begin{aligned} v_i(x,t) = v_i(0,t) + \int_0^x \Bigg(\sum_{j=1}^{m} \Bigg(c_{ij}(y,t) u_j(y,t) + \int_0^t C_{ij}^1(y,t,\sigma) u_j(y,\sigma) d\sigma + \\ + \int_0^y C_{ij}^2(y,t,z) u_j(z,t) dz \Bigg) + \sum_{j=1}^{n} \Bigg(d_{ij}(y,t) v_j(y,t) + \int_0^t D_{ij}^1(y,t,\sigma) v_j(y,\sigma) d\sigma + \\ + \int_0^y D_{ij}^2(y,t,z) v_j(z,t) dz \Bigg) + g_i(y,t) \Bigg) dy, \quad i \in \{1, \dots, n\}. \end{aligned} \tag{6}$$

Suppose we can rewrite boundary conditions (4) as follows

$$\begin{aligned} u_i(0,t) = \sum_{j \notin I_0} \mu_{ij}^1(t) u_j(0,t) + \sum_{j \notin I_l} \nu_{ij}^1(t) u_j(l,t) + \sum_{j=1}^{m} \int_0^t \Big(G_{ij}^1(t,\tau) u_j(0,\tau) + \\ + H_{ij}^1(t,\tau) u_j(l,\tau) \Big) d\tau + \sum_{j=1}^{n} \int_0^t \Big(F_{ij}^1(t,\tau) v_j(0,\tau) + K_{ij}^1(t,\tau) v_j(l,\tau) \Big) d\tau + \omega_i^1(t), \ i \in I_0, \end{aligned} \tag{7}$$

$$\begin{aligned} u_i(l,t) = \sum_{j \notin I_0} \mu_{ij}^2(t) u_j(0,t) + \sum_{j \notin I_l} \nu_{ij}^2(t) u_j(l,t) + \sum_{j=1}^{m} \int_0^t \Big(G_{ij}^2(t,\tau) u_j(0,\tau) + \\ + H_{ij}^2(t,\tau) u_j(l,\tau) \Big) d\tau + \sum_{j=1}^{n} \int_0^t \Big(F_{ij}^2(t,\tau) v_j(0,\tau) + K_{ij}^2(t,\tau) v_j(l,\tau) \Big) d\tau + \omega_i^2(t), \ i \in I_l, \end{aligned} \tag{8}$$

$$v_i(0,t) = \sum_{j\notin I_0} \mu^3_{ij}(t)u_j(0,t) + \sum_{j\notin I_l} \nu^3_{ij}(t)u_j(l,t) + \sum_{j=1}^{m}\int_0^t \Big(G^3_{ij}(t,\tau)u_j(0,\tau) + H^3_{ij}(t,\tau)u_j(l,\tau)\Big)d\tau +$$
$$+\sum_{j=1}^{n}\int_0^t \Big(F^3_{ij}(t,\tau)v_j(0,\tau) + K^3_{ij}(t,\tau)v_j(l,\tau)\Big)d\tau + \omega^3_i(t),\ i\in\{1,\dots,n\}. \qquad (9)$$

To be precise, suppose $I_0 = \{i_1,\dots,i_{r_0}\}$, $I_l = \{j_1,\dots,j_{r_l}\}$, and $N = n + r_0 + r_l$, then introduce the matrix

$$\Gamma(t) = \begin{pmatrix} \gamma^0_{1i_1}(t) & \dots & \gamma^0_{1i_{r_0}}(t) & \gamma^l_{1j_1}(t) & \dots & \gamma^l_{1j_{r_l}}(t) & \psi_{11}(t) & \dots & \psi_{1n}(t) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma^0_{Ni_1}(t) & \dots & \gamma^0_{Ni_{r_0}}(t) & \gamma^l_{Nj_1}(t) & \dots & \gamma^l_{Nj_{r_l}}(t) & \psi_{N1}(t) & \dots & \psi_{Nn}(t) \end{pmatrix}.$$

Further, if $\det\Gamma(t)\neq 0$, then conditions (4) can be rewritten in form (7)–(9).

For any $u\in (C^1(\overline{\Pi}))^m$, $v\in (C^{1,0}(\overline{\Pi}))^n$ we define an operator $\mathcal{B} = (\mathcal{B}_1,\dots,\mathcal{B}_m)$ in the following way. For $(x,t)\in\overline{\Pi}$ such that $\chi_i(x,t) = 0$, we put $\mathcal{B}_i[w](x,t) = q_i(\varphi_i(0;x,t))$. But in the case $\varphi_i(\chi_i(x,t);x,t) = 0$, we define

$$\mathcal{B}_i[w](x,t) = \sum_{j\notin I_0}\mu^1_{ij}(\chi_i(x,t))u_j(0,\chi_i(x,t)) + \sum_{j\notin I_l}\nu^1_{ij}(\chi_i(x,t))u_j(l,\chi_i(x,t)) +$$
$$+\sum_{j=1}^{m}\int_0^{\chi_i(x,t)} \Big(G^1_{ij}(\chi_i(x,t),\tau)u_j(0,\tau) + H^1_{ij}(\chi_i(x,t),\tau)u_j(l,\tau)\Big)d\tau +$$
$$+\sum_{j=1}^{n}\int_0^{\chi_i(x,t)} \Big(F^1_{ij}(\chi_i(x,t),\tau)v_j(0,\tau) + K^1_{ij}(\chi_i(x,t),\tau)v_j(l,\tau)\Big)d\tau + \omega^1_i(\chi_i(x,t)).$$

Likewise if $\varphi_i(\chi_i(x,t);x,t) = l$, we put

$$\mathcal{B}_i[w](x,t) = \sum_{j\notin I_0}\mu^2_{ij}(\chi_i(x,t))u_j(0,\chi_i(x,t)) + \sum_{j\notin I_l}\nu^2_{ij}(\chi_i(x,t))u_j(l,\chi_i(x,t)) +$$
$$+\sum_{j=1}^{m}\int_0^{\chi_i(x,t)} \Big(G^2_{ij}(\chi_i(x,t),\tau)u_j(0,\tau) + H^2_{ij}(\chi_i(x,t),\tau)u_j(l,\tau)\Big)d\tau +$$
$$+\sum_{j=1}^{n}\int_0^{\chi_i(x,t)} \Big(F^2_{ij}(\chi_i(x,t),\tau)v_j(0,\tau) + K^2_{ij}(\chi_i(x,t),\tau)v_j(l,\tau)\Big)d\tau + \omega^2_i(\chi_i(x,t)).$$

By construction of the operator $\mathcal{B}$, the equality

$$u_i\Big(\varphi_i(\chi_i(x,t);x,t),\chi_i(x,t)\Big) = \mathcal{B}_i[w](x,t), \quad i\in\{1,\dots,m\}, \quad (x,t)\in\overline{\Pi}$$

is equivalent to initial conditions (3) and boundary conditions (7), (8). Taking into account the previous equality and conditions (9), we rewrite system (5), (6) in the form

$$u_i(x,t) = \mathcal{B}_i[w](x,t) + \int_{\chi_i(x,t)}^{t} \Bigg(\sum_{j=1}^{m}\Bigg(a_{ij}(\varphi_i(\tau;x,t),\tau)u_j(\varphi_i(\tau;x,t),\tau) +$$
$$+\int_0^{\tau} A^1_{ij}(\varphi_i(\tau;x,t),\tau,\sigma)u_j(\varphi_i(\tau;x,t),\sigma)d\sigma + \int_0^{\varphi_i(\tau;x,t)} A^2_{ij}(\varphi_i(\tau;x,t),\tau,z)u_j(z,\tau)dz\Bigg) +$$

$$+\sum_{j=1}^{n}\Bigg(b_{ij}(\varphi_i(\tau;x,t),\tau)v_j(\varphi_i(\tau;x,t),\tau)+\int_0^{\tau}B^1_{ij}(\varphi_i(\tau;x,t),\tau,\sigma)v_j(\varphi_i(\tau;x,t),\sigma)d\sigma+$$

$$+\int_0^{\varphi_i(\tau;x,t)}B^2_{ij}(\varphi_i(\tau;x,t),\tau,z)v_j(z,\tau)dz\Bigg)+f_i(\varphi_i(\tau;x,t),\tau)\Bigg)d\tau,\quad i\in\{1,\dots,m\},\tag{10}$$

$$v_i(x,t)=\sum_{j\notin I_0}\mu^3_{ij}(t)u_j(0,t)+\sum_{j\notin I_l}\nu^3_{ij}(t)u_j(l,t)+\sum_{j=1}^{m}\int_0^t\Big(G^3_{ij}(t,\tau)u_j(0,\tau)+$$

$$+H^3_{ij}(t,\tau)u_j(l,\tau)\Big)d\tau+\sum_{j=1}^{n}\int_0^t\Big(F^3_{ij}(t,\tau)v_j(0,\tau)+K^3_{ij}(t,\tau)v_j(l,\tau)\Big)d\tau+\omega^3_i(t)+$$

$$+\int_0^x\Bigg(\sum_{j=1}^{m}\Bigg(c_{ij}(y,t)u_j(y,t)+\int_0^tC^1_{ij}(y,t,\sigma)u_j(y,\sigma)d\sigma+\int_0^yC^2_{ij}(y,t,z)u_j(z,t)dz\Bigg)+$$

$$+\sum_{j=1}^{n}\Bigg(d_{ij}(y,t)v_j(y,t)+\int_0^tD^1_{ij}(y,t,\sigma)v_j(y,\sigma)d\sigma+\int_0^yD^2_{ij}(y,t,z)v_j(z,t)dz\Bigg)+g_i(y,t)\Bigg)dy,$$

$$i\in\{1,\dots,n\}.\tag{11}$$

Thus system of functional integral equations (10), (11) is equivalent to system (1), (2) with conditions (3), (7)–(9) in the class of smooth functions.

Compatibility conditions. Consider boundary conditions (7)–(8) at $t=0$, consequently we obtain a system of relations in unknowns $u_i(0,0)$ and $u_i(l,0)$. Therefore, using initial conditions (3), we have zero-order compatibility conditions

$$\begin{aligned}q_i(0)&=\sum_{j\notin I_0}\mu^1_{ij}(0)q_j(0)+\sum_{j\notin I_l}\nu^1_{ij}(0)q_j(l)+\omega^1_i(0),\quad i\in I_0,\\ q_i(l)&=\sum_{j\notin I_0}\mu^2_{ij}(0)q_j(0)+\sum_{j\notin I_l}\nu^2_{ij}(0)q_j(l)+\omega^2_i(0),\quad i\in I_l.\end{aligned}\tag{12}$$

Differentiating conditions (7)–(8) at $t=0$, we get equalities in unknowns $u_i(0,0)$, $u_i(l,0)$, $\frac{\partial}{\partial t}u_i(0,0)$, $\frac{\partial}{\partial t}u_i(l,0)$, $v_i(0,0)$, and $v_i(l,0)$

$$\frac{\partial}{\partial t}u_i(0,0)=\sum_{j\notin I_0}\Big(\frac{\partial}{\partial t}\mu^1_{ij}(0)u_j(0,0)+\mu^1_{ij}(0)\frac{\partial}{\partial t}u_j(0,0)\Big)+$$

$$+\sum_{j\notin I_l}\Big(\frac{\partial}{\partial t}\nu^1_{ij}(0)u_j(l,0)+\nu^1_{ij}(0)\frac{\partial}{\partial t}u_j(l,0)\Big)+\sum_{j=1}^{m}\Big(G^1_{ij}(0,0)u_j(0,0)+H^1_{ij}(0,0)u_j(l,0)\Big)+$$

$$+\sum_{j=1}^{n}\Big(F^1_{ij}(0,0)v_j(0,0)+K^1_{ij}(0,0)v_j(l,0)\Big)+\omega^1_i(0),\quad i\in I_0;\tag{13}$$

$$\frac{\partial}{\partial t}u_i(l,0)=\sum_{j\notin I_0}\Big(\frac{\partial}{\partial t}\mu^2_{ij}(0)u_j(0,0)+\mu^2_{ij}(0)\frac{\partial}{\partial t}u_j(0,0)\Big)+$$

$$+\sum_{j\notin I_l}\Big(\frac{\partial}{\partial t}\nu^2_{ij}(0)u_j(l,0)+\nu^2_{ij}(0)\frac{\partial}{\partial t}u_j(l,0)\Big)+\sum_{j=1}^{m}\Big(G^2_{ij}(0,0)u_j(0,0)+H^2_{ij}(0,0)u_j(l,0)\Big)+$$

$$+\sum_{j=1}^{n}\Big(F^2_{ij}(0,0)v_j(0,0)+K^2_{ij}(0,0)v_j(l,0)\Big)+\omega^2_i(0),\quad i\in I_l.\tag{14}$$

From initial conditions (3) it follows that $u_i(0,0) = q_i(0)$ and $u_i(l,0) = q_i(l)$. Using boundary conditions (9), we establish

$$v_i(0,0) = \sum_{j\notin I_0} \mu^3_{ij}(0)q_j(0) + \sum_{j\notin I_l} \nu^3_{ij}(0)q_j(l) + \omega^3_i(0), \quad i \in \{1,\dots,n\}. \tag{15}$$

Further, rewrite system (2) fixing $t = 0$

$$\begin{aligned}\frac{\partial v_i(x,0)}{\partial x} &= \sum_{j=1}^{m}\left(c_{ij}(x,0)q_j(x) + \int_0^x C^2_{ij}(x,0,z)q_j(z)dz\right) + \\ &+ \sum_{j=1}^{n}\left(d_{ij}(x,0)v_j(x,0) + \int_0^x D^2_{ij}(x,0,z)v_j(z,0)dz\right) + g_i(x,0) = \\ &= \sum_{j=1}^{n}\left(d_{ij}(x,0)v_j(x,0) + \int_0^x D^2_{ij}(x,0,z)v_j(z,0)dz\right) + G_i(x), \quad i \in \{1,\dots,n\},\end{aligned} \tag{16}$$

where $G_i(x)$ is determined from the initial data. Let $d_{ij}(x,0)$, $D^2_{ij}(x,0,z)$, $G_i(x)$ be continuous functions on the corresponding domains, then the Cauchy problem for system (16) with initial conditions (15) has a unique solution, which is denoted by $v^0_i(x)$. From system (1) at $t = 0$ we follow

$$\begin{aligned}\frac{\partial u_i(x,0)}{\partial t} &= -\lambda_i(x,0)\frac{\partial q_i(x)}{\partial x} + \sum_{j=1}^{m}\left(a_{ij}(x,0)q_j(x) + \int_0^x A^2_{ij}(x,0,z)q_j(z)dz\right) + \\ &+ \sum_{j=1}^{n}\left(b_{ij}(x,0)v^0_j(x) + \int_0^x B^2_{ij}(x,0,z)v^0_j(z)dz\right) + f_i(x,0), \quad i \in \{1,\dots,m\}.\end{aligned}$$

Thus the right-hand sides of the obtained equalities are known functions, which are denoted by $Q_i(x)$. Using the introduced notation, equalities (13), (14) can be rewritten in the form

$$\begin{aligned}Q_i(0) = &\sum_{j\notin I_0}\left(\tfrac{\partial}{\partial t}\mu^1_{ij}(0)q_j(0) + \mu^1_{ij}(0)Q_j(0)\right) + \sum_{j\notin I_l}\left(\tfrac{\partial}{\partial t}\nu^1_{ij}(0)q_j(l) + \nu^1_{ij}(0)Q_j(l)\right) + \sum_{j=1}^{m}\Big(G^1_{ij}(0,0)\times \\ &\times q_j(0) + H^1_{ij}(0,0)q_j(l)\Big) + \sum_{j=1}^{n}\left(F^1_{ij}(0,0)v^0_j(0) + K^1_{ij}(0,0)v^0_j(l)\right) + \omega^1_i(0), \ i \in I_0,\end{aligned} \tag{17}$$

$$\begin{aligned}Q_i(l) = &\sum_{j\notin I_0}\left(\tfrac{\partial}{\partial t}\mu^2_{ij}(0)q_j(0) + \mu^2_{ij}(0)Q_j(0)\right) + \sum_{j\notin I_l}\left(\tfrac{\partial}{\partial t}\nu^2_{ij}(0)q_j(l) + \nu^2_{ij}(0)Q_j(l)\right) + \sum_{j=1}^{m}\Big(G^2_{ij}(0,0)\times \\ &\times q_j(0) + H^2_{ij}(0,0)q_j(l)\Big) + \sum_{j=1}^{n}\left(F^2_{ij}(0,0)v^0_j(0) + K^2_{ij}(0,0)v^0_j(l)\right) + \omega^2_i(0), \quad i \in I_l.\end{aligned} \tag{18}$$

Thus equations (17), (18) are first-order compatibility conditions.

Global solvability theorem. Let us define domains

$$\Delta_1 = [0,l]\times[0,T]\times[0,T], \quad \Delta_2 = [0,l]\times[0,T]\times[0,l], \quad \Delta_3 = [0,T]\times[0,T].$$

Theorem. *Suppose the following conditions hold:*

1) λ_i, $\frac{\partial}{\partial x}\lambda_i$, a_{ij}, $\frac{\partial}{\partial x}a_{ij}$, b_{ij}, $\frac{\partial}{\partial x}b_{ij}$, f_i, $\frac{\partial}{\partial x}f_i$, c_{ij}, d_{ij}, $g_i \in C(\overline{\Pi})$;
2) A^1_{ij}, $\frac{\partial}{\partial x}A^1_{ij}$, B^1_{ij}, $\frac{\partial}{\partial x}B^1_{ij}$, C^1_{ij}, $D^1_{ij} \in C(\Delta_1)$;
3) A^2_{ij}, $\frac{\partial}{\partial x}A^2_{ij}$, B^2_{ij}, $\frac{\partial}{\partial x}B^2_{ij}$, C^2_{ij}, $D^2_{ij} \in C(\Delta_2)$;
4) q_i, $\frac{\partial}{\partial x}q_i \in C[0,l]$;
5) γ^0_{ij}, $\frac{\partial}{\partial t}\gamma^0_{ij}$, γ^l_{ij}, $\frac{\partial}{\partial t}\gamma^l_{ij}$, ψ_{ij}, $\frac{\partial}{\partial t}\psi_{ij}$, δ_i, $\frac{\partial}{\partial t}\delta_i \in C[0,T]$;
6) Γ^0_{ij}, $\frac{\partial}{\partial t}\Gamma^0_{ij}$, Γ^l_{ij}, $\frac{\partial}{\partial t}\Gamma^l_{ij}$, Ψ^0_{ij}, $\frac{\partial}{\partial t}\Psi^0_{ij}$, Ψ^l_{ij}, $\frac{\partial}{\partial t}\Psi^l_{ij} \in C(\Delta_3)$;
7) $\lambda_i(0,t)$ *and* $\lambda_i(l,t)$ *maintain sign on* $[0,T]$;
8) $\det\Gamma(t) \neq 0$ *for all* $t \in [0,T]$;
9) *zero-order and first-order compatibility conditions* (12), (17), (18).

Then there exists a unique classical solution of problem (1)–(4).

Proof. Consider the metric space $\mathcal{Q}$ that consists of pairs $w = (u,v)$ such that $u \in (C^1(\overline{\Pi}))^m$, $v \in (C^{1,0}(\overline{\Pi}))^n$. Besides, u_i is subject to (3), and satisfies conditions $\frac{\partial u_i(x,0)}{\partial t} = Q_i(x)$, $v_i(x,0) = v^0_i(x)$. Let us define a space metric by the formula

$$\rho(w^1,w^2) = \max\Big\{\max_{i,x,t}\left|u^1_i(x,t) - u^2_i(x,t)\right|\alpha_i(x)e^{-at},\ \max_{i,x,t}\left|v^1_i(x,t) - v^2_i(x,t)\right|\beta_i(x)e^{-at},$$
$$\max_{i,x,t}\left|\frac{\partial}{\partial x}u^1_i(x,t) - \frac{\partial}{\partial x}u^2_i(x,t)\right|\gamma_i(x)e^{-at},\ \max_{i,x,t}\left|\frac{\partial}{\partial t}u^1_i(x,t) - \frac{\partial}{\partial t}u^2_i(x,t)\right|\gamma_i(x)e^{-at},$$
$$\max_{i,x,t}\left|\frac{\partial}{\partial x}v^1_i(x,t) - \frac{\partial}{\partial x}v^2_i(x,t)\right|\delta_i(x)e^{-at}\Big\}, \qquad (19)$$

where a constant $a \geq 1$ and positive functions α_i, β_j, γ_i, δ_i defined on $[0,l]$ will be chosen later on.

We introduce an operator $\mathcal{A}$ on the space $\mathcal{Q}$ in the following way. Let $w \in \mathcal{Q}$, then $\mathcal{A}[w] = (\mathcal{A}^1_1[w],\ldots,\mathcal{A}^1_m[w], \mathcal{A}^2_1[w],\ldots,\mathcal{A}^2_n[w])$, where the functions $\mathcal{A}^1_i[w]$, $\mathcal{A}^2_i[w]$ are defined by the right sides of functional integral system (10), (11), i.e.

$$\mathcal{A}^1_i[w](x,t) = \mathcal{B}_i[w](x,t) + \int_{\chi_i(x,t)}^t\Bigg(\sum_{j=1}^m\bigg(a_{ij}(\varphi_i(\tau;x,t),\tau)u_j(\varphi_i(\tau;x,t),\tau)+$$
$$+\int_0^\tau A^1_{ij}(\varphi_i(\tau;x,t),\tau,\sigma)u_j(\varphi_i(\tau;x,t),\sigma)d\sigma + \int_0^{\varphi_i(\tau;x,t)} A^2_{ij}(\varphi_i(\tau;x,t),\tau,z)u_j(z,\tau)dz\bigg)+$$
$$+\sum_{j=1}^n\bigg(b_{ij}(\varphi_i(\tau;x,t),\tau)v_j(\varphi_i(\tau;x,t),\tau) + \int_0^\tau B^1_{ij}(\varphi_i(\tau;x,t),\tau,\sigma)v_j(\varphi_i(\tau;x,t),\sigma)d\sigma+$$
$$+\int_0^{\varphi_i(\tau;x,t)} B^2_{ij}(\varphi_i(\tau;x,t),\tau,z)v_j(z,\tau)dz\bigg) + f_i(\varphi_i(\tau;x,t),\tau)\Bigg)d\tau, \quad i \in \{1,\ldots,m\},$$

$$\mathcal{A}^2_i[w](x,t) = \sum_{j\notin I_0}\mu^3_{ij}(t)u_j(0,t) + \sum_{j\notin I_l}\nu^3_{ij}(t)u_j(l,t) + \sum_{j=1}^m\int_0^t\Big(G^3_{ij}(t,\tau)u_j(0,\tau)+$$
$$+H^3_{ij}(t,\tau)u_j(l,\tau)\Big)d\tau + \sum_{j=1}^n\int_0^t\Big(F^3_{ij}(t,\tau)v_j(0,\tau) + K^3_{ij}(t,\tau)v_j(l,\tau)\Big)d\tau + \omega^3_i(t)+$$

$$+\int_0^x \left(\sum_{j=1}^m \left(c_{ij}(y,t)u_j(y,t) + \int_0^t C^1_{ij}(y,t,\sigma)u_j(y,\sigma)d\sigma + \int_0^y C^2_{ij}(y,t,z)u_j(z,t)dz\right)+\right.$$
$$\left.+\sum_{j=1}^n \left(d_{ij}(y,t)v_j(y,t) + \int_0^t D^1_{ij}(y,t,\sigma)v_j(y,\sigma)d\sigma + \int_0^y D^2_{ij}(y,t,z)v_j(z,t)dz\right) + g_i(y,t)\right) dy,$$
$$i \in \{1,\dots,n\}.$$

Thus, finding a classical solution of problem (1)–(4) is reduced to finding a fixed point of the operator $\mathcal{A}$ in the space $\mathcal{Q}$. Let us remark that $\mathcal{A}[w] \in \mathcal{Q}$ whenever $w \in \mathcal{Q}$. It follows from the smoothness conditions imposed on given data, the smoothness of φ_i, χ_i, as well as the compatibility conditions. In the sequel, we choose weighting functions of the space metric so that the operator $\mathcal{A}$ is contractive.

Suppose the absolute values of functions λ_i, a_{ij}, $\frac{\partial}{\partial x}a_{ij}$, b_{ij}, $\frac{\partial}{\partial x}b_{ij}$, c_{ij}, d_{ij}, A^1_{ij}, $\frac{\partial}{\partial x}A^1_{ij}$, B^1_{ij}, $\frac{\partial}{\partial x}B^1_{ij}$, C^1_{ij}, D^1_{ij} A^2_{ij}, $\frac{\partial}{\partial x}A^2_{ij}$, B^2_{ij}, $\frac{\partial}{\partial x}B^2_{ij}$, C^2_{ij}, D^2_{ij}, μ^k_{ij}, $\frac{\partial}{\partial t}\mu^k_{ij}$, ν^k_{ij}, $\frac{\partial}{\partial t}\nu^k_{ij}$, G^k_{ij}, $\frac{\partial}{\partial t}G^k_{ij}$, H^k_{ij}, $\frac{\partial}{\partial t}H^k_{ij}$, F^k_{ij}, $\frac{\partial}{\partial t}F^k_{ij}$, K^k_{ij}, $\frac{\partial}{\partial t}K^k_{ij}$, φ_i, $\frac{\partial}{\partial x}\varphi_i$, χ_i, $\frac{\partial}{\partial x}\chi_i$ are bounded by a constant L.

The definition of the metric implies the following inequalities

$$|u^1_i(x,t) - u^2_i(x,t)| \le \frac{\rho(w^1,w^2)}{\alpha_i(x)}e^{at}, \quad |v^1_i(x,t) - v^2_i(x,t)| \le \frac{\rho(w^1,w^2)}{\beta_i(x)}e^{at},$$
$$\left|\frac{\partial}{\partial x}u^1_i(x,t) - \frac{\partial}{\partial x}u^2_i(x,t)\right| \le \frac{\rho(w^1,w^2)}{\gamma_i(x)}e^{at}, \quad \left|\frac{\partial}{\partial t}u^1_i(x,t) - \frac{\partial}{\partial t}u^2_i(x,t)\right| \le \frac{\rho(w^1,w^2)}{\gamma_i(x)}e^{at},$$
$$\left|\frac{\partial}{\partial x}v^1_i(x,t) - \frac{\partial}{\partial x}v^2_i(x,t)\right| \le \frac{\rho(w^1,w^2)}{\delta_i(x)}e^{at},$$

where w^1, $w^2 \in \mathcal{Q}$. These inequalities will be used to obtain the required estimates. To simplify formulas, we denote the difference $s^1 - s^2$ by $\Delta_k s^k$.

For $(x,t) \in \overline{\Pi}$ such that $\varphi_i(\chi_i(x,t);x,t) = 0$, we get the following estimate

$$\left|\Delta_k \mathcal{B}_i[w^k](x,t)\right| \alpha_i(x)e^{-at} =$$
$$= \left|\sum_{j\notin I_0} \mu^1_{ij}(\chi_i(x,t))\Delta_k u^k_j(0,\chi_i(x,t)) + \sum_{j\notin I_l} \nu^1_{ij}(\chi_i(x,t))\Delta_k u^k_j(l,\chi_i(x,t)) + \right.$$
$$+\sum_{j=1}^m \int_0^{\chi_i(x,t)} \Big(G^1_{ij}(\chi_i(x,t),\tau)\Delta_k u^k_j(0,\tau) + H^1_{ij}(\chi_i(x,t),\tau)\Delta_k u^k_j(l,\tau)\Big)d\tau +$$
$$\left.+\sum_{j=1}^n \int_0^{\chi_i(x,t)} \Big(F^1_{ij}(\chi_i(x,t),\tau)\Delta_k v^k_j(0,\tau) + K^1_{ij}(\chi_i(x,t),\tau)\Delta_k v^k_j(l,\tau)\Big)d\tau\right| \alpha_i(x)e^{-at} \le$$
$$\le (n+m)L\left(\max_{\substack{i\in I_0\\ j\notin I_0}} |\Delta_k u^k_j(0,\chi_i(x,t))|\alpha_i(x)e^{-at} + \max_{\substack{i\in I_0\\ j\notin I_l}} |\Delta_k u^k_j(l,\chi_i(x,t))|\alpha_i(x)e^{-at} + \right.$$
$$+\int_0^t e^{-at}\left(\max_{i,j} |\Delta_k u^k_j(0,\tau)|\alpha_i(x) + \max_{i,j} |\Delta_k u^k_j(l,\tau)|\alpha_i(x) + \right.$$
$$\left.\left.+\max_{i,j} |\Delta_k v^k_j(0,\tau)|\alpha_i(x) + \max_{i,j} |\Delta_k v^k_j(l,\tau)|\alpha_i(x)\right)d\tau\right)\rho(w^1,w^2) \le$$

$$\leq (n+m)L\left(\max_{\substack{i\in I_0\\ j\notin I_0}}\frac{\alpha_i(x)e^{a(\chi_i(x,t)-t)}}{\alpha_j(0)}+\max_{\substack{i\in I_0\\ j\notin I_l}}\frac{\alpha_i(x)e^{a(\chi_i(x,t)-t)}}{\alpha_j(l)}+\right.$$
$$\left.+\int_0^t e^{a(\tau-t)}d\tau\left(\max_{i,j}\frac{\alpha_i(x)}{\alpha_j(0)}+\max_{i,j}\frac{\alpha_i(x)}{\alpha_j(l)}+\max_{i,j}\frac{\alpha_i(x)}{\beta_j(0)}+\max_{i,j}\frac{\alpha_i(x)}{\beta_j(l)}\right)\right)\rho(w^1,w^2)\leq$$
$$\leq (n+m)L\left(\max_{\substack{i\in I_0\\ j\notin I_0}}\frac{\alpha_i(x)e^{a(\chi_i(x,t)-t)}}{\alpha_j(0)}+\max_{\substack{i\in I_0\\ j\notin I_l}}\frac{\alpha_i(x)e^{a(\chi_i(x,t)-t)}}{\alpha_j(l)}+\right.$$
$$\left.+\frac{2}{a}\left(\max_{i,j,s}\frac{\alpha_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\alpha_i(x)}{\beta_j(s)}\right)\right)\rho(w^1,w^2).$$

In the case $\varphi_i(\chi_i(x,t);x,t)=l$, we have a similar estimate. Finally, we derive a general estimate

$$\left|\Delta_k\mathcal{B}_i[w^k](x,t)\right|\alpha_i(x)e^{-at}\leq (n+m)L\left(\max_{\substack{i\in I_0\\ j\notin I_0}}\frac{\alpha_i(x)e^{a(\chi_i(x,t)-t)}}{\alpha_j(0)}+\max_{\substack{i\in I_0\\ j\notin I_l}}\frac{\alpha_i(x)e^{a(\chi_i(x,t)-t)}}{\alpha_j(l)}+\right.$$
$$\left.+\max_{\substack{i\in I_l\\ j\notin I_0}}\frac{\alpha_i(x)e^{a(\chi_i(x,t)-t)}}{\alpha_j(0)}+\max_{\substack{i\in I_l\\ j\notin I_l}}\frac{\alpha_i(x)e^{a(\chi_i(x,t)-t)}}{\alpha_j(l)}+\frac{2}{a}\left(\max_{i,j,s}\frac{\alpha_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\alpha_i(x)}{\beta_j(s)}\right)\right)\rho(w^1,w^2).$$

Likewise, we obtain

$$\left|\Delta_k\mathcal{A}^1_i[w^k](x,t)\right|\alpha_i(x)e^{-at}\leq\left|\Delta_k\mathcal{B}_i[w^k](x,t)\right|\alpha_i(x)e^{-at}+$$
$$+(n+m)L\left(\int_0^t e^{a(\tau-t)}d\tau+\int_0^t d\tau\int_0^\tau e^{a(\sigma-t)}d\sigma+\int_0^t e^{a(\tau-t)}d\tau\int_0^{\varphi_i(\tau;x,t)}dz\right)\times$$
$$\times\left(\max_{i,j,s}\frac{\alpha_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\alpha_i(x)}{\beta_j(s)}\right)\rho(w^1,w^2)\leq\left|\Delta_k\mathcal{B}_i[w^k](x,t)\right|\alpha_i(x)e^{-at}+$$
$$+(n+m)(2+l)\frac{L}{a}\left(\max_{i,j,s}\frac{\alpha_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\alpha_i(x)}{\beta_j(s)}\right)\rho(w^1,w^2).$$

And we also get

$$\left|\Delta_k\mathcal{A}^2_i[w](x,t)\right|\beta_i(x)e^{-at}\leq (n+m)L\left(\max_{\substack{i\\ j\notin I_0}}\frac{\beta_i(x)}{\alpha_j(0)}+\max_{\substack{i\\ j\notin I_l}}\frac{\beta_i(x)}{\alpha_j(l)}+\right.$$
$$+\int_0^t e^{a(\tau-t)}d\tau\left(\max_{i,j}\frac{\beta_i(x)}{\alpha_j(0)}+\max_{i,j}\frac{\beta_i(x)}{\alpha_j(l)}+\max_{i,j}\frac{\beta_i(x)}{\beta_j(0)}+\max_{i,j}\frac{\beta_i(x)}{\beta_j(l)}\right)+$$
$$+\int_0^x\left(\max_{i,j}\frac{\beta_i(x)}{\alpha_j(y)}+\max_{i,j}\frac{\beta_i(x)}{\beta_j(y)}\right)dy+\int_0^t e^{a(\sigma-t)}d\sigma\int_0^x\left(\max_{i,j}\frac{\beta_i(x)}{\alpha_j(y)}+\max_{i,j}\frac{\beta_i(x)}{\beta_j(y)}\right)dy+$$
$$\left.+\int_0^x\int_0^y\left(\max_{i,j}\frac{\beta_i(x)}{\alpha_j(z)}+\max_{i,j}\frac{\beta_i(x)}{\beta_j(z)}\right)dzdy\right)\rho(w^1,w^2)\leq$$
$$\leq (n+m)L\left((4+2l+l^2)\max_{i,j,s}\frac{\beta_i(x)}{\alpha_j(s)}+\frac{2}{a}\max_{i,j,s}\frac{\beta_i(x)}{\beta_j(s)}+\right.$$

$$+ 2\int_0^x \max_{i,j} \frac{\beta_i(x)}{\beta_j(y)} dy + \int_0^x \int_0^y \max_{i,j} \frac{\beta_i(x)}{\beta_j(z)} dz dy \Big) \rho(w^1, w^2).$$

For $(x,t) \in \overline{\Pi}$ such that $\varphi_i(\chi_i(x,t); x, t) = 0$, the following estimate holds

$$\left| \Delta_k \frac{\partial}{\partial x} \mathcal{B}_i[w^k](x,t) \right| \gamma_i(x) e^{-at} =$$

$$= \Bigg| \Bigg(\sum_{j \notin I_0} \left(\frac{\partial}{\partial t} \mu_{ij}^1(\chi_i(x,t)) \Delta_k u_j^k(0, \chi_i(x,t)) + \mu_{ij}^1(\chi_i(x,t)) \Delta_k \frac{\partial}{\partial t} u_j^k(0, \chi_i(x,t)) \right) +$$

$$+ \sum_{j \notin I_l} \left(\frac{\partial}{\partial t} \nu_{ij}^1(\chi_i(x,t)) \Delta_k u_j^k(l, \chi_i(x,t)) + \nu_{ij}^1(\chi_i(x,t)) \Delta_k \frac{\partial}{\partial t} u_j^k(l, \chi_i(x,t)) \right) +$$

$$+ \sum_{j=1}^m \Bigg(G_{ij}^1(\chi_i(x,t), \chi_i(x,t)) \Delta_k u_j^k(0, \chi_i(x,t)) + H_{ij}^1(\chi_i(x,t), \chi_i(x,t)) \Delta_k u_j^k(l, \chi_i(x,t)) +$$

$$+ \int_0^{\chi_i(x,t)} \left(\frac{\partial}{\partial t} G_{ij}^1(\chi_i(x,t), \tau) \Delta_k u_j^k(0,\tau) + \frac{\partial}{\partial t} H_{ij}^1(\chi_i(x,t), \tau) \Delta_k u_j^k(l, \tau) \right) d\tau \Bigg) +$$

$$+ \sum_{j=1}^n \Bigg(F_{ij}^1(\chi_i(x,t), \chi_i(x,t)) \Delta_k v_j^k(0, \chi_i(x,t)) + K_{ij}^1(\chi_i(x,t), \chi_i(x,t)) \Delta_k v_j^k(l, \chi_i(x,t)) +$$

$$+ \int_0^{\chi_i(x,t)} \left(\frac{\partial}{\partial t} F_{ij}^1(\chi_i(x,t), \tau) \Delta_k v_j^k(0,\tau) \frac{\partial}{\partial t} K_{ij}^1(\chi_i(x,t), \tau) \Delta_k v_j^k(l, \tau) \right) d\tau \Bigg) \frac{\partial}{\partial x} \chi_i(x,t) \Bigg| \times$$

$$\times \gamma_i(x) e^{-at} \le (n+m) L^2 \Bigg(\max_{\substack{i \in I_0 \\ j \notin I_0}} \frac{\gamma_i(x) e^{a(\chi_i(x,t) - t)}}{\alpha_j(0)} + \max_{\substack{i \in I_0 \\ j \notin I_l}} \frac{\gamma_i(x) e^{a(\chi_i(x,t) - t)}}{\alpha_j(l)} +$$

$$+ \max_{\substack{i \in I_0 \\ j \notin I_0}} \frac{\gamma_i(x) e^{a(\chi_i(x,t) - t)}}{\gamma_j(0)} + \max_{\substack{i \in I_0 \\ j \notin I_l}} \frac{\gamma_i(x) e^{a(\chi_i(x,t) - t)}}{\gamma_j(l)} +$$

$$+ \left(1 + \int_0^t e^{a(\tau - t)} d\tau \right) \left(\max_{i,j} \frac{\gamma_i(x)}{\alpha_j(0)} + \max_{i,j} \frac{\gamma_i(x)}{\alpha_j(l)} + \max_{i,j} \frac{\gamma_i(x)}{\beta_j(0)} + \max_{i,j} \frac{\gamma_i(x)}{\beta_j(l)} \right) \Bigg) \rho(w^1, w^2).$$

Clearly, we get a similar estimate in the case $\varphi_i(\chi_i(x,t); x, t) = l$. Therefore we derive a general estimate

$$\left| \Delta_k \frac{\partial}{\partial x} \mathcal{B}_i[w^k](x,t) \right| \gamma_i(x) e^{-at} \le (n+m) L^2 \Bigg(\max_{\substack{i \in I_0 \\ j \notin I_0}} \frac{\gamma_i(x) e^{a(\chi_i(x,t) - t)}}{\gamma_j(0)} + \max_{\substack{i \in I_0 \\ j \notin I_l}} \frac{\gamma_i(x) e^{a(\chi_i(x,t) - t)}}{\gamma_j(l)} +$$

$$+ \max_{\substack{i \in I_l \\ j \notin I_0}} \frac{\gamma_i(x) e^{a(\chi_i(x,t) - t)}}{\gamma_j(0)} + \max_{\substack{i \in I_l \\ j \notin I_l}} \frac{\gamma_i(x) e^{a(\chi_i(x,t) - t)}}{\gamma_j(l)} + 8 \left(\max_{i,j,s} \frac{\gamma_i(x)}{\alpha_j(s)} + \max_{i,j,s} \frac{\gamma_i(x)}{\beta_j(s)} \right) \Bigg) \rho(w^1, w^2).$$

As before, we have

$$\left| \Delta_k \frac{\partial}{\partial x} \mathcal{A}_i^1[w^k](x,t) \right| \gamma_i(x) e^{-at} \le \left| \Delta_k \frac{\partial}{\partial x} \mathcal{B}_i[w^k](x,t) \right| \gamma_i(x) e^{-at} +$$

$$+ (n+m) L^2 \left(\max_{i,j,s} \frac{\gamma_i(x)}{\alpha_j(s)} + \int_0^t e^{a(\sigma - t)} d\sigma \max_{i,j,s} \frac{\gamma_i(x)}{\alpha_j(s)} + \int_0^l \max_{i,j} \frac{\gamma_i(x)}{\alpha_j(z)} dz + \right.$$

$$+\max_{i,j,s}\frac{\gamma_i(x)}{\beta_j(s)}+\int_0^t e^{a(\sigma-t)}d\sigma\max_{i,j,s}\frac{\gamma_i(x)}{\beta_j(s)}+\int_0^l\max_{i,j}\frac{\gamma_i(x)}{\beta_j(z)}dz+$$
$$+\int_0^t e^{a(\tau-t)}d\tau\left(2\max_{i,j,s}\frac{\gamma_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\gamma_j(s)}+2\max_{i,j,s}\frac{\gamma_i(x)}{\beta_j(s)}++\max_{i,j,s}\frac{\gamma_i(x)}{\delta_j(s)}\right)+$$
$$+\int_0^t d\tau\int_0^\tau e^{a(\sigma-t)}d\sigma\left(\max_{i,j,s}\frac{\gamma_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\gamma_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\beta_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\delta_j(s)}\right)+$$
$$+\int_0^t e^{a(\tau-t)}d\tau\int_0^l\left(\max_{i,j}\frac{\gamma_i(x)}{\alpha_j(z)}+\max_{i,j}\frac{\gamma_i(x)}{\beta_j(z)}\right)dz\Bigg)\rho(w^1,w^2)\le\left|\Delta_k\frac{\partial}{\partial x}\mathcal{B}_i[w^k](x,t)\right|\gamma_i(x)e^{-at}+$$
$$+(n+m)L^2\left(\left(5+2l\right)\left(\max_{i,j,s}\frac{\gamma_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\beta_j(s)}\right)+\frac{2}{a}\left(\max_{i,j,s}\frac{\gamma_i(x)}{\gamma_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\delta_j(s)}\right)\right)\rho(w^1,w^2).$$

Taking into account the equality

$$\frac{\partial}{\partial t}\mathcal{A}_i^1[w](x,t)=-\lambda_i(x,t)\frac{\partial}{\partial x}\mathcal{A}_i^1[w](x,t)+$$
$$+\sum_{j=1}^m\left(a_{ij}(x,t)u_j(x,t)+\int_0^t A_{ij}^1(x,t,\sigma)u_j(x,\sigma)d\sigma+\int_0^x A_{ij}^2(x,t,z)u_j(z,t)dz\right)+$$
$$+\sum_{j=1}^n\left(b_{ij}(x,t)v_j(x,t)+\int_0^t B_{ij}^1(x,t,\sigma)v_j(x,\sigma)d\sigma+\int_0^x B_{ij}^2(x,t,z)v_j(z,t)dz\right)+f_i(x,t),$$

we deduce

$$\left|\Delta_k\frac{\partial}{\partial t}\mathcal{A}_i^1[w^k](x,t)\right|\gamma_i(x)e^{-at}\le L\left|\Delta_k\frac{\partial}{\partial x}\mathcal{A}_i^1[w^k](x,t)\right|\gamma_i(x)e^{-at}+$$
$$+(n+m)(2+l)L\left(\max_{i,j,s}\frac{\gamma_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\beta_j(s)}\right)\rho(w^1,w^2).$$

In the same way, we establish the estimate

$$\left|\Delta_k\frac{\partial}{\partial x}\mathcal{A}_i^2[w^k](x,t)\right|\delta_i(x)e^{-at}\le(n+m)(2+l)L\left(\max_{i,j,s}\frac{\delta_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\delta_i(x)}{\beta_j(s)}\right)\rho(w^1,w^2).$$

Let $\max\limits_{i,x,t}|\lambda_i(x,t)|\neq 0$, introduce the notation $\mu=(\max\limits_{i,x,t}|\lambda_i(x,t)|)^{-1}$. If $\varphi_i(\chi_i(x,t);x,t)=0$, then we get $\chi_i(x,t)\le t-\mu x$. Similarly, assuming that $\varphi_i(\chi_i(x,t);x,t)=l$, we establish $\chi_i(x,t)\le t-\mu(l-x)$ ([7]). Otherwise if $\lambda_i(x,t)=0$ for all i,x,t, then $\mathcal{B}_i[w](x,t)=q_i(\varphi_i(0;x,t))$, therefore we need not have the notation of constant μ. By the last estimates we have the inequality

$$\rho\left(\mathcal{A}[w^1],\mathcal{A}[w^2]\right)\le C_1\max_x\left(\max_{\substack{i\in I_0\\ j\notin I_0}}\frac{\alpha_i(x)e^{-a\mu x}}{\alpha_j(0)}+\max_{\substack{i\in I_0\\ j\notin I_l}}\frac{\alpha_i(x)e^{-a\mu x}}{\alpha_j(l)}+\max_{\substack{i\in I_l\\ j\notin I_0}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\alpha_j(0)}+\right.$$
$$+\max_{\substack{i\in I_l\\ j\notin I_l}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\alpha_j(l)}+\max_{\substack{i\in I_0\\ j\notin I_0}}\frac{\gamma_i(x)e^{-a\mu x}}{\gamma_j(0)}+\max_{\substack{i\in I_0\\ j\notin I_l}}\frac{\gamma_i(x)e^{-a\mu x}}{\gamma_j(l)}+\max_{\substack{i\in I_l\\ j\notin I_0}}\frac{\gamma_i(x)e^{-a\mu(l-x)}}{\gamma_j(0)}+$$

$$+\max_{\substack{i\in I_l\\ j\notin I_l}}\frac{\gamma_i(x)e^{-a\mu(l-x)}}{\gamma_j(l)}+\frac{1}{a}\left(\max_{i,j,s}\frac{\alpha_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\alpha_i(x)}{\beta_j(s)}+\max_{i,j,s}\frac{\beta_i(x)}{\beta_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\gamma_j(s)}+\right.$$
$$\left.+\max_{i,j,s}\frac{\gamma_i(x)}{\delta_j(s)}\right)+\max_{i,j,s}\frac{\beta_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\gamma_i(x)}{\beta_j(s)}+\max_{i,j,s}\frac{\delta_i(x)}{\alpha_j(s)}+\max_{i,j,s}\frac{\delta_i(x)}{\beta_j(s)}+$$
$$\left.+\int_0^x\max_{i,j}\frac{\beta_i(x)}{\beta_j(y)}dy+\int_0^x\int_0^y\max_{i,j}\frac{\beta_i(x)}{\beta_j(z)}dzdy\right)\rho(w^1,w^2),$$

where C_1 is some positive constant, which is determined from the initial data.

Let us choose the weighting functions of the space metric so that $\mathcal{A}$ is a contractive operator on $\mathcal{Q}$

$$\alpha_i(x)=\begin{cases}e^{px(l-x)}, & i\in I_0\cup I_l,\\ e^{px}, & i\in I_0\setminus I_l,\\ e^{p(l-x)}, & i\in I_l\setminus I_0,\\ e^{pl}, & i\notin I_0\cup I_l,\end{cases}\qquad \gamma_i(x)=\varepsilon^2\alpha_i(x),$$
$$\beta_i(x)=\varepsilon e^{-px},\quad \delta_i(x)=\varepsilon^2\beta_i(x),\quad i\in\{1,\dots,n\},$$

where $0<\varepsilon\le 1$, $p\ge 1$ are some parameters. Using the chosen weighting functions, we estimate the contraction coefficient of $\mathcal{A}$.

Suppose the following conditions hold $p\le a\mu,\ p\,l\le a\mu$, then we have

$$\max_x\max_{\substack{i\in I_0\\ j\notin I_0}}\frac{\alpha_i(x)e^{-a\mu x}}{\alpha_j(0)}=\max_x\max_{\substack{i\in I_0\\ j\notin I_l}}\frac{\alpha_i(x)e^{-a\mu x}}{\alpha_j(l)}=$$
$$=\max_x\max_{\substack{i\in I_0\\ j\notin I_0}}\frac{\gamma_i(x)e^{-a\mu x}}{\gamma_j(0)}=\max_x\max_{\substack{i\in I_0\\ j\notin I_l}}\frac{\gamma_i(x)e^{-a\mu x}}{\gamma_j(l)}=$$
$$=\max_x\max_{i\in I_0}\frac{\alpha_i(x)e^{-a\mu x}}{e^{pl}}\le\max_x\max\left\{e^{px(l-x)-a\mu x-pl},\ e^{px-a\mu x-pl}\right\}\le e^{-pl},$$
$$\max_x\max_{\substack{i\in I_l\\ j\notin I_0}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\alpha_j(0)}=\max_x\max_{\substack{i\in I_l\\ j\notin I_l}}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{\alpha_j(l)}=$$
$$=\max_x\max_{\substack{i\in I_l\\ j\notin I_0}}\frac{\gamma_i(x)e^{-a\mu(l-x)}}{\gamma_j(0)}=\max_x\max_{\substack{i\in I_l\\ j\notin I_l}}\frac{\gamma_i(x)e^{-a\mu(l-x)}}{\gamma_j(l)}=$$
$$=\max_x\max_{i\in I_l}\frac{\alpha_i(x)e^{-a\mu(l-x)}}{e^{pl}}\le\max_x\max\left\{e^{px(l-x)-a\mu(l-x)-pl},\ e^{p(l-x)-a\mu(l-x)-pl}\right\}\le e^{-pl}.$$

In addition, we deduce the following inequalities

$$\max_{i,j,s,x}\frac{\beta_i(x)}{\alpha_j(s)}\le\varepsilon,\quad \max_{i,j,s,x}\frac{\delta_i(x)}{\alpha_j(s)}\le\varepsilon^2,\quad \max_{i,j,s,x}\frac{\delta_i(x)}{\beta_j(s)}\le\varepsilon e^{pl},$$
$$\max_{i,j,s,x}\frac{\gamma_i(x)}{\alpha_j(s)}\le\varepsilon^2\max\left\{e^{pl},e^{p\frac{l^2}{4}}\right\},\quad \max_{i,j,s,x}\frac{\gamma_i(x)}{\beta_j(s)}\le\varepsilon e^{pl}\max\left\{e^{pl},e^{p\frac{l^2}{4}}\right\}.$$
$$\max_x\int_0^x\max_{i,j}\frac{\beta_i(x)}{\beta_j(y)}dy\le\max_x\int_0^x e^{p(y-x)}dy\le\frac{1}{p},$$
$$\max_x\int_0^x\int_0^y\max_{i,j}\frac{\beta_i(x)}{\beta_j(z)}dzdy\le\max_x\int_0^x\int_0^y e^{p(z-x)}dzdy\le\frac{1}{p^2}.$$

Finally, we obtain

$$\rho(A[w^1], A[w^2]) \le C_2 \left(e^{-pl} + \frac{1}{p} + \varepsilon \left(1 + e^{C_3 p}\right) + \right.$$

$$\left. + \frac{1}{a}\left(\max_{i,j,s,x} \frac{\alpha_i(x)}{\alpha_j(s)} + \max_{i,j,s,x} \frac{\alpha_i(x)}{\beta_j(s)} + \max_{i,j,s,x} \frac{\beta_i(x)}{\beta_j(s)} + \max_{i,j,s,x} \frac{\gamma_i(x)}{\gamma_j(s)} + \max_{i,j,s,x} \frac{\gamma_i(x)}{\delta_j(s)} \right)\right) \rho(w^1, w^2),$$

where C_2, C_3 are some positive constants, which are determined from the initial data. Fix p^* so large that $e^{-p^* l} + \frac{1}{p^*} < \frac{1}{3C_2}$, and $\varepsilon^* > 0$ so small that $\varepsilon^* \left(1 + e^{C_3 p^*}\right) < \frac{1}{3C_2}$. Denote by α_i^*, β_i^*, γ_i^*, δ_i^* the corresponding functions α_i, β_i, γ_i, δ_i under the fixed parameters p^*, ε^*. Finally, we choose the parameter a large enough so that $p^* \le a\mu$, $p^* l \le a\mu$,

$$\frac{1}{a}\left(\max_{i,j,s,x} \frac{\alpha_i^*(x)}{\alpha_j^*(s)} + \max_{i,j,s,x} \frac{\alpha_i^*(x)}{\beta_j^*(s)} + \max_{i,j,s,x} \frac{\beta_i^*(x)}{\beta_j^*(s)} + \max_{i,j,s,x} \frac{\gamma_i^*(x)}{\gamma_j^*(s)} + \max_{i,j,s,x} \frac{\gamma_i^*(x)}{\delta_j^*(s)} \right) < \frac{1}{3C_2}.$$

Then $\mathcal{A}$ is a contractive operator on $\mathcal{Q}$ with the chosen metric.

Thus, by the Banach theorem, there exists a unique fixed point of the operator $\mathcal{A}$ in $\mathcal{Q}$. This point is a classical solution of problem (1)–(4). □

Remark. Suppose theorem's conditions hold on the time interval $[0, +\infty)$; then there exists an unique classical solution of problem (1)–(4) in the set $\overline{\Pi}_\infty = \{(x,t)\colon 0 \le x \le l,\, 0 \le t < +\infty\}$.

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Received 27.01.2012
Revised 5.06.2012