УДК 517.95

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INITIAL-BOUNDARY PROBLEMS FOR SYSTEMS OF A HIGH ORDER DOUBLY NONLINEAR PARABOLIC EQUATIONS WITH VARIABLE EXPONENT OF NONLINEARITY

T. M. Bokalo. *Initial-boundary problems for systems of a high order doubly nonlinear parabolic equations with variable exponent of nonlinearity*, Mat. Stud. **38** (2012), 68–79.

Consider a mixed problem for a class of system of a high order doubly nonlinear parabolic equations with variable exponent of nonlinearity. This problem is considered in generalized Lebesgue-Sobolev spaces. As a result, we reached a condition of the existence of a solution. We use here Galerkin's procedure.

Т. Н. Бокало. *Начально-краевые задачи для систем параболических уравнений высокого порядка с двойной нелинейностью и переменным показателем нелинейности* // Мат. Студії. – 2012. – Т.38, №1. – C.68–79.

Рассмотрена смешанная задача для классов систем параболических уравнений высокого порядка с двойной нелинейностью и переменным показателем нелинейности. Эта задача рассматривается в обобщенных пространствах Лебега-Соболева. Как результат, получено условие существования решения. Тут используется метод Галёркина.

Thin liquid films are important in biophysics, physics, and engineering, as well as in natural settings. They can be composed of common liquids such as water, oil, or complex mixtures of components. Let us consider length scales in the x direction.

The fluid motion is described by the equations in two dimensions of the form ([1, p. 936])

$$\begin{cases} \rho(u_t+uu_x+wu_z)=-p_x+\mu u_{xx}-\varphi_x, \\ \rho(w_t+uw_x+ww_z)=-p_z+\mu w_{xx}-\varphi_z, \\ u_x+w_z=0. \end{cases}$$

If we switch to a new dimensionless parameters and applied some additional assumptions, this problem is reduced to an equation with respect to a function h of the form

$$h_t+h_{xxxx}+h_{xx}=f(x,t). \tag{1}$$

In [3, p. 1034] it is shown that the movement of the considered liquid can be described by a system of equations, much more precisely than by (1). A model example of such a system is

$$\begin{cases} n_t-n_{xx}+h_{xx}=F(x,t), \\ h_t+h_{xxxx}+h_{xx}+n_{xx}=f(x,t), \end{cases} \tag{2}$$

2010 *Mathematics Subject Classification:* 35D02, 35G02, 35K02.
Keywords: system of doubly nonlinear equations, existence theorem, uniqueness theorem, Galerkin procedure.

where dimensionless parameter n is the concentration of a liquid.

Particulary, A. N. Tikhonov and A. A. Samarskii ([4, p. 184]) state that the diffusion process in a porous medium, which is described by the linear equation $n_t - L\, n_{xx} = 0$, can be better described by the nonlinear equation $(\varphi(n))_t - L\, n_{xx} = 0$, where φ is some nonlinear function, for instance $\varphi(n) = n^\alpha n + n$, $\alpha > 0$.

Thus instead of (2) we study the model system

$$\begin{cases} (n^\alpha n + n)_t - n_{xx} + h_{xx} = F(x,t), \\ h_t + h_{xxxx} + h_{xx} + n_{xx} = f(x,t), \end{cases} \tag{3}$$

which drops into a class of systems of equations we consider.

1. Problem statement. Let $T > 0$ be an arbitrary number, Ω be a bounded domain, $Q_{t_1,t_2} = \Omega \times (t_1, t_2)$, $0 \le t_1 < t_2 \le T$, $\Omega_\tau = \{(x,t)\colon x \in \Omega, t = \tau\}$, $\tau \in [0,T]$,

$$L_+^\infty(\Omega) = \Big\{ v \in L^\infty(\Omega)\colon \operatorname{ess}\inf_{x\in\Omega} v(x) > 1 \Big\}.$$

We define $s_0 \equiv \operatorname{ess}\inf_{x\in\Omega} s(x)$, $s^0 \equiv \operatorname{ess}\sup_{x\in\Omega} s(x)$ for all $s \in L_+^\infty(\Omega)$, and by s' we denote the element from $L_+^\infty(\Omega)$ satisfying $\frac{1}{s(x)} + \frac{1}{s'(x)} = 1$ for almost all $x \in \Omega$. The symbols $L^{s(x)}(\Omega)$ and $L^{s(x)}(Q_{0,T})$ stand for the generalized Lebesgue spaces ([5–7]).

Let $r, s, q \in L_+^\infty(\Omega)$, and

$$V_1 = H_0^1(\Omega) \cap L^{q(x)}(\Omega),\ V_2 = H_0^2(\Omega) \cap L^{s(x)}(\Omega),$$
$$U_1(Q_{0,T}) = L^2(0,T; H_0^1(\Omega)) \cap L^{q(x)}(Q_{0,T}),\ U_2(Q_{0,T}) = L^2(0,T; H_0^2(\Omega)) \cap L^{s(x)}(Q_{0,T}),$$
$$U_3(Q_{0,T}) = L^2(0,T; H_0^2(\Omega)) \cap L^{r(x)}(Q_{0,T}) \cap L^{q(x)}(Q_{0,T}) \cap L^{s(x)}(Q_{0,T}).$$

We consider the following problem

$$(\mathcal{P}u)_t - \sum_{i,j=1}^n (a_{ij}(x,t) u_{x_i})_{x_j} + \sum_{i=1}^n z_i(x,t) u_{x_i} + d(x,t) u + g(x,t) |u|^{q(x)-2} u +$$
$$+ \sum_{i,j=1}^n b_{ij}(x,t) v_{x_i x_j} + \sum_{i=1}^n l_i(x,t) v_{x_i} + h(x,t) v = f(x,t), \tag{4}$$

$$v_t + \sum_{i,j,k,l=1}^n (A_{ijkl}(x,t) v_{x_i x_j})_{x_k x_l} + \sum_{i,j,k=1}^n (L_{ijk}(x,t) v_{x_i x_j})_{x_k} + \sum_{i,j=1}^n B_{ij}(x,t) v_{x_i x_j} +$$
$$+ \sum_{i=1}^n Z_i(x,t) v_{x_i} + D(x,t) v + G(x,t) |v|^{s(x)-2} v + \sum_{i,j=1}^n (E_{ij}(x,t) u_{x_i})_{x_j} +$$
$$+ \sum_{i=1}^n H_i(x,t) u_{x_i} + I(x,t) u = F(x,t), \tag{5}$$

$$u|_{\partial\Omega\times[0,T]} = v|_{\partial\Omega\times[0,T]} = \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega\times[0,T]} = 0, \tag{6}$$

$$u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \tag{7}$$

where $\mathcal{P}u = \mathcal{R}u + u$, $\mathcal{R}u = \frac{1}{r(x)-1} |u|^{r(x)-2} u$. Assume the measurable coefficients of (4), (5) satisfy the following conditions:

(a): $a_{ij}, (a_{ij})_t \in L^\infty(Q_{0,T})$, $a_0|\xi|^2 \le \sum\limits_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \le a^0|\xi|^2$ for every $\xi \in \mathbb{R}^n$ and a. e. $x \in \Omega$, where $a_0, a^0 > 0$;

(b): $|b_{ij}(x,t)| \le b^0$ for every $i,j \in \{1,2,\dots,n\}$ and a. e. $(x,t) \in Q_{0,T}$, where $b^0 > 0$;

(d): $|d(x,t)| \le d^0$ for a. e. $(x,t) \in Q_{0,T}$, where $d^0 > 0$;

(g): $g, g_t \in L^\infty(Q_{0,T})$, $0 < g_0 \le g(x,t) \le g^0$ for a. e. $(x,t) \in Q_{0,T}$;

(h): $|h(x,t)| < h^0$ for a. e. $(x,t) \in Q_{0,T}$, where $h^0 > 0$;

(l): $|l_i(x,t)| < l^0$ for every $i \in \{1,2,\dots,n\}$ and a. e. $(x,t) \in Q_{0,T}$, where $l^0 > 0$;

(z): $|z_i(x,t)| \le z^0$ for every $i \in \{1,2,\dots,n\}$ and a. e. $(x,t) \in Q_{0,T}$, where $z^0 > 0$;

(A): $A_0 \sum\limits_{i,j=1}^n |\eta_{ij}|^2 \le \sum\limits_{i,j,k,l=1}^n A_{ijkl}(x,t)\eta_{ij}\eta_{kl} \le A^0 \sum\limits_{i,j=1}^n |\eta_{ij}|^2$ for every $\eta \in \mathbb{R}^n \times \mathbb{R}^n$ and a. e. $(x,t) \in Q_{0,T}$, where $A_0, A^0 > 0$;

(B): $|B_{ij}(x,t)| \le B^0$ for every $i,j \in \{1,2,\dots,n\}$ and a. e. $(x,t) \in Q_{0,T}$, where $B^0 > 0$;

(D): $|D(x,t)| \le D^0$ for a. e. $(x,t) \in Q_{0,T}$, where $D^0 > 0$;

(E): $|E_{ij}(x,t)| \le E^0$ for every $i,j \in \{1,2,\dots,n\}$ and a. e. $(x,t) \in Q_{0,T}$, where $E^0 > 0$;

(G): $G(x,t) \ge G_0 > 0$ for a. e. $(x,t) \in Q_{0,T}$;

(H): $|H_i(x,t)| \le H^0$ for every $i \in \{1,2,\dots,n\}$ and a. e. $(x,t) \in Q_{0,T}$, where $H^0 > 0$;

(I): $|I(x,t)| \le I^0$ for a. e. $(x,t) \in Q_{0,T}$, where $I^0 > 0$;

(L): $|L_{ijk}(x,t)| \le L^0$ for every $i,j,k \in \{1,2,\dots,n\}$ and a. e. $(x,t) \in Q_{0,T}$, where $L^0 > 0$;

(Z): $|Z_i(x,t)| \le Z^0$ for every $i \in \{1,2,\dots,n\}$ and a. e. $(x,t) \in Q_{0,T}$, where $Z^0 > 0$;

(F): $f, f_t, F, F_t \in L^2(Q_{0,T})$.

To simplify the estimations we define

$$[u,v]_Q = \int_{Q_{0,T}} u(x,t)v(x,t)\,dxdt, \quad (y,z)_\Omega = \int_\Omega y(x)z(x)\,dx.$$

Definition 1. A pair of functions (u,v) such that

$(u,v) \in U_1(Q_{0,T}) \times U_2(Q_{0,T})$, $u \in L^{r(x)}(Q_{0,T})$, $u_t, (\mathcal{R}u)_t \in L^2(Q_{0,T})$, $v_t \in L^2(0,T;H^{-2}(\Omega))$ is called a *weak solution of problem* (4)–(7), if it satisfies (7) and the following equalities

$$[(\mathcal{R}u)_t + u_t, w]_Q + \sum_{i,j=1}^n [a_{ij}u_{x_i}, w_{x_j}]_Q + \sum_{i=1}^n [z_i u_{x_i}, w]_Q + [du, w]_Q + [g|u|^{q(x)-2}u, w]_Q +$$
$$+ \sum_{i,j=1}^n [b_{ij}v_{x_ix_j}, w]_Q + \sum_{i=1}^n [l_i v_{x_i}, w]_Q + [hv, w]_Q = [f, w]_Q, \tag{8}$$

$$[v_t, w]_Q + \sum_{i,j,k,l=1}^n [A_{ijkl}v_{x_ix_j}, w_{x_kx_l}]_Q + \sum_{i,j,k=1}^n [L_{ijk}v_{x_ix_j}, w_{x_k}]_Q + \sum_{i,j=1}^n [B_{ij}v_{x_ix_j}, w]_Q +$$
$$+ \sum_{i=1}^n [Z_i v_{x_i}, w]_Q + [Du, w]_Q + [G|v|^{s(x)-2}v, w]_Q - \sum_{i,j=1}^n [E_{ij}u_{x_i}, w_{x_j}]_Q +$$
$$+ \sum_{i=1}^n [Hu_{x_i}, w]_Q + [Iu, w]_Q = [F, w]_Q, \tag{9}$$

for every $w \in U_3(Q_{0,T})$.

Assume, the next condition fulfills

(T): $4a_0\widetilde{A}^0 < M_\Omega(E^0n)^2$, $A_0^2 > M_\Omega(L^0n)^2n$, $\widetilde{A}^0 = \frac{A_0^2 - M_\Omega(L^0)^2n^3}{2A_0}$,
where M_Ω is taken from Friedrichs' inequality (Lemma 1.36 [13, p. 50]),

$$\int_\Omega |v|^2 dx \le M_\Omega \int_\Omega |\nabla v|^2 dx \quad \text{for every } v \in H_0^1(\Omega), \tag{10}$$

n is a space dimension parameter, a_0, A_0, E^0, L^0 from **(a)**, **(A)**, **(E)**, **(L)**, respectively.

Proposition 1. *If condition (T) is carried out, then there exist $\eta_1, \eta_2 > 0$, such that we have*

$$a_0 - \frac{E^0n}{2}\eta_1 > 0, \quad A_0 - \frac{L^0n}{2}\eta_2 - \frac{M_\Omega L^0 n^2}{2}\frac{1}{\eta_2} - \frac{M_\Omega E^0 n}{2}\frac{1}{\eta_1} > 0, \tag{11}$$

where a_0, A_0, E^0, L^0, n, M_Ω are as stated above.

Proof. Define $\widetilde{A}(\eta_2) = A_0 - \frac{L^0n}{2}\eta_2 - \frac{M_\Omega L^0 n^2}{2}\frac{1}{\eta_2}$. Let us find $\eta_2 = \eta_2^0 > 0$, such that $\widetilde{A}^0 \stackrel{\text{def}}{=} \widetilde{A}(\eta_2^0) > 0$. Since

$$\widetilde{A}(\eta_2) = A_0 - \frac{L^0n}{2}\eta_2 - \frac{M_\Omega L^0 n^2}{2\eta_2} = \frac{2A_0\eta_2 - L^0n\eta_2^2 - M_\Omega L^0n^2}{2\eta_2} = \frac{-L^0n\eta_2^2 + 2A_0\eta_2 - M_\Omega L^0 n^2}{2\eta_2},$$

we are going to find out the abscissa of the culmination point of the parabola registered in the numerator of the expression above (such a point is $\eta_2 = \eta_2^0$) $\eta_2^0 = \frac{A_0}{L^0n}$. Obviously, $\eta_2^0 > 0$. Thus, taken into consideration condition **(T)** we have

$$\widetilde{A}^0 = A_0 - \frac{L^0n}{2}\frac{A_0}{L^0n} - \frac{M_\Omega L^0 n^2}{2}\frac{L^0n}{A_0} = \frac{A_0}{2} - \frac{M_\Omega (L^0)^2 n^3}{2A_0} = \frac{A_0^2 - M_\Omega(L^0)^2n^3}{2A_0} > 0.$$

Now we get

$$\frac{M_\Omega E^0 n}{2\widetilde{A}^0} < \eta_1 < \frac{2a_0}{E^0n}.$$

Hence, from **(T)** we obtain $\frac{M_\Omega E^0 n}{2\widetilde{A}^0} < \frac{2a_0}{E^0n}$. So such $\eta_1 > 0$ exists and therefore (11) is to be fulfilled. □

Theorem 1. If $u_0 \in V_1 \cap L^{r(x)}(\Omega)$, $v_0 \in V_2$, conditions **(a)**–**(F)**, **(T)** hold and $r_0 > 2$, then problem (4)–(7) has a weak solution.

We use the Galerkin procedure to proof the theorem. The proof is located in a separate section just only for convenience of our investigation process.

Systems of nonlinear parabolic equations with variable exponent of nonlinearity is widely studied, particularly (see [2]). As mentioned above, the system of equations of form (3) has been occurred in [3]. The solvability of some doubly nonlinear parabolical equations by elliptic regularization method is shown in [8, 9]. A model example of the equations is

$$|u|^{r(x)-2}u_t - (|u_x|^{p-2}u_x)_x = f.$$

Moreover, in [9] one can find a proof of solution existence to a homogenous mixed problem for the model equation $(|u|^{r(x)-2}u + u)_t - (|u_x|^{p-2}u_x)_x = f$. Other nonlinear equations in Lebesgue and Sobolev spaces were investigated in [10–12]. Conditions of solution existence and uniqueness of correspondent parabolic variational inequalities are mentioned in [6, 7].

2. Proof of the main result.

1. Let ω^1, ω^2, $\ldots$, ω^m, $\ldots$ be a basis in $H_0^2(\Omega)$. A solution to (4)–(7) we start to search in the form

$$u^m(x,t)=\sum_{\mu=1}^{m}\varphi_\mu^m(t)\omega^\mu(x),\quad v^m(x,t)=\sum_{\mu=1}^{m}\psi_\mu^m(t)\omega^\mu(x),\quad (x,t)\in Q_{0,T},$$

where functions $\varphi_\mu^m,\psi_\mu^m(\mu\in\{1,2,\ldots,m\})$ are solutions to the following Cauchy problem

$$((\mathcal{P}u^m(t))_t,\omega^\mu)_\Omega+\sum_{i,j=1}^{n}(a_{ij}u_{x_i}^m(t),\omega_{x_j}^\mu)_\Omega+\sum_{i=1}^{n}(z_iu_{x_i}^m(t),\omega^\mu)_\Omega+(du^m(t),\omega^\mu)_\Omega+$$
$$+(g|u^m(t)|^{q(x)-2}u^m(t),\omega^\mu)_\Omega+\sum_{i,j=1}^{n}(b_{ij}v_{x_ix_j}^m(t),\omega^\mu)_\Omega+\sum_{i=1}^{n}(l_iv_{x_i}^m(t),\omega^\mu)_\Omega+$$
$$+(hv^m(t),\omega^\mu)_\Omega=(f(t),\omega^\mu)_\Omega,\tag{12}$$
$$(v_t^m(t),\omega^\mu)_\Omega+\sum_{i,j,k,l=1}^{n}(A_{ijkl}v_{x_ix_j}^m(t),\omega_{x_kx_l}^\mu)_\Omega+\sum_{i,j,k=1}^{n}(L_{ijk}v_{x_ix_j}^m(t),\omega_{x_k}^\mu)_\Omega+$$
$$+\sum_{i,j=1}^{n}(B_{ij}v_{x_ix_j}^m(t),\omega^\mu)_\Omega+\sum_{i=1}^{n}(Z_iv_{x_i}^m(t),\omega^\mu)_\Omega+(Dv^m(t),\omega^\mu)_\Omega+$$
$$+(G|v^m(t)|^{s(x)-2}v^m(t),\omega^\mu)_\Omega-\sum_{i,j=1}^{n}(E_{ij}u_{x_i}^m(t),\omega_{x_j}^\mu)_\Omega+\sum_{i=1}^{n}(H_iu_{x_i}^m(t),\omega^\mu)_\Omega+$$
$$+(Iu^m(t),\omega)_\Omega=(F(t),\omega^\mu)_\Omega,\quad t\in(0,T),\tag{13}$$
$$\varphi_j^m(0)=\alpha_j^m,\quad \psi_j^m(0)=\beta_j^m,\quad j\in\{1,2,\ldots,m\},\tag{14}$$

where $\alpha_1^m,\ldots,\alpha_m^m,\beta_1^m,\ldots,\beta_m^m\in\mathbb{R}$ are such numbers that

$$u_0^m=\sum_{\mu=1}^{m}\alpha_\mu^m\omega^\mu\underset{m\to\infty}{\longrightarrow}u_0\quad\text{stongly in } V_1\cap L^{r(x)}(\Omega),\quad v_0^m=\sum_{\mu=1}^{m}\beta_\mu^m\omega^\mu\underset{m\to\infty}{\longrightarrow}v_0\quad\text{strongly in } V_2.$$

Additionally, note that

$$u^m\Big|_{t=0}=u_0^m,\quad v^m\Big|_{t=0}=v_0^m.\tag{15}$$

The system of ordinary differential equations (12)–(13) is locally soluble. But, taking into account the estimates above we conclude that the solution to (12)–(13) is defined on $[0,T]$. Particularly, $\varphi_1^m,\ldots,\varphi_m^m,\psi_1^m,\ldots,\psi_m^m\in C^1([0,T])$, so

$$u^m,v^m\in C^1(\overline{Q}_{0,T}).\tag{16}$$

2. Multiplying each μ^{th}-equation (12)–(13) ($\mu\in\{1,2,\ldots,m\}$, m is fixed) by $\varphi_\mu^m(t)$ and $\psi_\mu^m(t)$, respectively, summarizing all such formed equations of the system and then integrating it over $[0,\tau]$, $\tau\le T$, we get

$$J_1+J_2+\ldots+J_{18}=J_{19}+J_{20},\tag{17}$$

where

$$J_1 = \int_{Q_{0,\tau}} (\mathcal{P}u^m)_t u^m dxdt, \quad J_2 = \int_{Q_{0,\tau}} \sum_{i,j=1}^n a_{ij} u^m_{x_i} u^m_{x_j} dxdt, \quad J_3 = \int_{Q_{0,\tau}} \sum_{i=1}^n z_i u^m_{x_i} u^m dxdt,$$

$$J_4 = \int_{Q_{0,\tau}} d|u^m|^2 dxdt, \quad J_5 = \int_{Q_{0,\tau}} g|u^m|^{q(x)} dxdt, \quad J_6 = \int_{Q_{0,\tau}} \sum_{i,j=1}^n b_{ij} v^m_{x_i x_j} u^m dxdt,$$

$$J_7 = \int_{Q_{0,\tau}} \sum_{i=1}^n l_i v^m_{x_i} u^m dxdt, \quad J_8 = \int_{Q_{0,\tau}} h v^m u^m dxdt, \quad J_9 = \int_{Q_{0,\tau}} v^m_t v^m dxdt,$$

$$J_{10} = \int_{Q_{0,\tau}} \sum_{i,j,k,l=1}^n A_{ijkl} v^m_{x_i x_j} v^m_{x_k x_l} dxdt, \quad J_{11} = \int_{Q_{0,\tau}} \sum_{i,j,k=1}^n L_{ijk} v^m_{x_i x_j} v^m_{x_k} dxdt,$$

$$J_{12} = \int_{Q_{0,\tau}} \sum_{i,j=1}^n B_{ij} v^m_{x_i x_j} v^m dxdt, \quad J_{13} = \int_{Q_{0,\tau}} \sum_{i=1}^n Z_i v^m_{x_i} v^m dxdt, \quad J_{14} = \int_{Q_{0,\tau}} D|v^m|^2 dxdt,$$

$$J_{15} = \int_{Q_{0,\tau}} G|v^m|^{s(x)} dxdt, \quad J_{16} = -\int_{Q_{0,\tau}} \sum_{i,j=1}^n E_{ij} v^m_{x_i} u^m_{x_j} dxdt, \quad J_{17} = \int_{Q_{0,\tau}} \sum_{i=1}^n H_i u^m_{x_i} v^m dxdt,$$

$$J_{18} = \int_{Q_{0,\tau}} I u^m v^m dxdt, \quad J_{19} = \int_{Q_{0,\tau}} f u^m dxdt, \quad J_{20} = \int_{Q_{0,\tau}} F v^m dxdt.$$

Further, we are going to make an estimates of $J_1 - J_{20}$. Since (16) holds, we have

$$J_1 = \int_{\Omega_\tau} \frac{1}{r(x)} |u^m|^{r(x)} dx - \int_{\Omega_0} \frac{1}{r(x)} |u^m_0|^{r(x)} dx + \frac{1}{2}\int_{\Omega_\tau} |u^m|^2 dx - \frac{1}{2}\int_{\Omega_0} |u^m_0|^2 dx.$$

Using condition **(a)**–**(F)** and Young's estimate $ab \le \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$, we get

$$J_2 \ge a_0 \int_{Q_{0,\tau}} \sum_{i=1}^n |u^m_{x_i}|^2 dxdt, \quad |J_3| \le \frac{z^0 \varepsilon_1}{2} \int_{Q_{0,\tau}} \sum_{i=1}^n |u^m_{x_i}|^2 dxdt + \frac{z^0 n}{2\varepsilon_1} \int_{Q_{0,\tau}} |u^m|^2 dxdt,$$

$$|J_4| \le d^0 \int_{Q_{0,\tau}} |u^m|^2 dxdt, \quad J_5 \ge g_0 \int_{Q_{0,\tau}} |u^m|^{q(x)} dxdt,$$

$$|J_6| \le \frac{b^0 \varepsilon_2}{2} \int_{Q_{0,\tau}} \sum_{i,j=1}^n |v^m_{x_i x_j}|^2 dxdt + \frac{b^0 n^2}{2\varepsilon_2} \int_{Q_{0,\tau}} |u^m|^2 dxdt,$$

$$|J_7| \le \frac{l^0 \varepsilon_3}{2} \int_{Q_{0,\tau}} \sum_{i,j=1}^n |v^m_{x_i}|^2 dxdt + \frac{l^0 n}{2\varepsilon_3} \int_{Q_{0,\tau}} |u^m|^2 dxdt,$$

$$|J_8| \le \frac{h^0 \varepsilon_4}{2} \int_{Q_{0,\tau}} |v^m|^2 dxdt + \frac{h^0}{2\varepsilon_4} \int_{Q_{0,\tau}} |u^m|^2 dxdt,$$

$$J_9 = \frac{1}{2}\int_{\Omega_\tau} |v^m|^2 dx - \frac{1}{2}\int_{\Omega_0} |v^m_0|^2 dx, \quad J_{10} \ge A_0 \int_{Q_{0,\tau}} \sum_{i,j=1}^n |u^m_{x_i x_j}|^2 dxdt,$$

$$|J_{11}| \le \frac{L^0 \varepsilon_5 n}{2} \int_{Q_{0,\tau}} \sum_{i,j=1}^n |v^m_{x_i x_j}|^2 dxdt + \frac{L^0 n^2}{2\varepsilon_5} \int_{Q_{0,\tau}} \sum_{i=1}^n |v^m_{x_i}|^2 dxdt,$$

$$|J_{12}| \le \frac{B^0 \varepsilon_6}{2} \int_{Q_{0,\tau}} \sum_{i,j=1}^n |v^m_{x_i x_j}|^2 dxdt + \frac{B^0 n^2}{2\varepsilon_6} \int_{Q_{0,\tau}} |v^m|^2 dxdt,$$

$$|J_{13}| \le \frac{Z^0\varepsilon_7}{2}\int_{Q_{0,\tau}}\sum_{i=1}^n |v^m_{x_i}|^2dxdt + \frac{Z^0 n}{2\varepsilon_7}\int_{Q_{0,\tau}}|v^m|^2dxdt,$$

$$|J_{14}| \le D^0\int_{Q_{0,\tau}}|u^m|^2dxdt, \quad J_{15} \ge G_0\int_{Q_{0,\tau}}|v^m|^{s(x)}dxdt,$$

$$|J_{16}| \le \frac{E^0\varepsilon_8 n}{2}\int_{Q_{0,\tau}}\sum_{i=1}^n |u^m_{x_i}|^2dxdt + \frac{E^0 n}{2\varepsilon_8}\int_{Q_{0,\tau}}\sum_{i=1}^n|v^m_{x_i}|^2dxdt,$$

$$|J_{17}| \le \frac{H^0\varepsilon_9}{2}\int_{Q_{0,\tau}}\sum_{i=1}^n |u^m_{x_i}|^2dxdt + \frac{H^0 n}{2\varepsilon_9}\int_{Q_{0,\tau}}|v^m|^2dxdt,$$

$$|J_{18}| \le \frac{I^0\varepsilon_{10}}{2}\int_{Q_{0,\tau}}|u^m|^2dxdt + \frac{I^0}{2\varepsilon_{10}}\int_{Q_{0,\tau}}|v^m|^2dxdt,$$

$$|J_{19}| + |J_{20}| \le \frac{1}{2}\int_{Q_{0,\tau}}\{|f|^2+|F|^2\}dxdt + \frac{1}{2}\int_{Q_{0,\tau}}\{|u^m|^2+|v^m|^2\}dxdt.$$

Using the priory estimates, (17) we obtain the following

$$\int_{\Omega_\tau}\Big\{\frac{1}{r(x)}|u^m|^{r(x)} + \frac{1}{2}(|u^m|^2+|v^m|^2)\Big\}dx + \int_{Q_{0,\tau}}\Big\{\Big(a_0 - \frac{z^0\varepsilon_1}{2} - \frac{E^0\varepsilon_8 n}{2} - \frac{H^0\varepsilon_9}{2}\Big)\sum_{i=1}^n|u^m_{x_i}|^2 +$$
$$+\Big(A_0 - \frac{b^0\varepsilon_2}{2} - \frac{B^0\varepsilon_6}{2} - \frac{L^0\varepsilon_5 n}{2}\Big)\sum_{i,j=1}^n |v^m_{x_ix_j}|^2 + g_0|u^m|^{q(x)} + G_0|v^m|^{s(x)}\Big\}dxdt \le C_1\mathcal{F}_1(\tau) +$$
$$+C_2(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_6,\varepsilon_7,\varepsilon_9,\varepsilon_{10})\int_{Q_{0,\tau}}\{|u^m|^2+|v^m|^2\}dxdt +$$
$$+\int_{Q_{0,\tau}}\Big(\frac{l^0\varepsilon_3}{2} + \frac{L^0n^2}{2\varepsilon_5} + \frac{Z^0\varepsilon_7}{2} + \frac{E^0 n}{2\varepsilon_8}\Big)\sum_{i=1}^n|v^m_{x_i}|^2dxdt, \tag{18}$$

where $\mathcal{F}_1(\tau) = \int_{\Omega_0}\{|u^m_0|^{r(x)} + |u^m_0|^2 + |v^m_0|^2\}dx + \int_{Q_{0,\tau}}\{|f|^2+|F|^2\}dxdt$.

Next, applying Friedrichs' inequality (10) to the last term of (18) we obtain

$$\int_{\Omega_\tau}\Big\{\frac{1}{r(x)}|u^m|^{r(x)} + \frac{1}{2}(|u^m|^2+|v^m|^2)\Big\}dx + \int_{Q_{0,\tau}}\Big\{\Big(a_0 - \frac{z^0\varepsilon_1}{2} - \frac{E^0\varepsilon_8 n}{2} - \frac{H^0\varepsilon_9}{2}\Big)\sum_{i=1}^n|u^m_{x_i}|^2 +$$
$$+\Big(A_0 - \frac{b^0\varepsilon_2}{2} - \frac{B^0\varepsilon_6}{2} - \frac{L^0\varepsilon_5 n}{2} - M_\Omega\Big(\frac{l^0\varepsilon_3}{2} + \frac{L^0n^2}{2\varepsilon_5} + \frac{Z^0\varepsilon_7}{2} + \frac{E^0 n}{2\varepsilon_8}\Big)\Big)\sum_{i,j=1}^n|v^m_{x_ix_j}|^2 + g_0|u^m|^{q(x)} +$$
$$+G_0|v^m|^{s(x)}\Big\}dxdt \le C_3\mathcal{F}_1(\tau) + C_4(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_5,\varepsilon_7,\varepsilon_8,\varepsilon_9,\varepsilon_{10})\int_{Q_{0,\tau}}\{|u^m|^2+|v^m|^2\}dxdt.$$

Selecting $\varepsilon_5 = \eta_2$, $\varepsilon_8 = \eta_1$ and $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_5,\varepsilon_7,\varepsilon_8,\varepsilon_9,\varepsilon_{10} > 0$ small enough, we get

$$\int_{\Omega_\tau}\Big\{\frac{1}{r(x)}|u^m|^{r(x)} + \frac{1}{2}(|u^m|^2+|v^m|^2)\Big\}dx + \int_{Q_{0,\tau}}\Big\{\Big(a_0 - \frac{E^0 n}{2}\eta_1 - \eta_3\Big)\sum_{i=1}^n|u^m_{x_i}|^2 +$$
$$+\Big(A_0 - M_\Omega\frac{E^0 n}{2\eta_1} - \frac{L^0\eta_2 n}{2} - M_\Omega\frac{L^0n^2}{2\eta_2} - \eta_4\Big)\Big)\sum_{i,j=1}^n|v^m_{x_ix_j}|^2 + g_0|u^m|^{q(x)} +$$
$$+G_0|v^m|^{s(x)}\Big\}dxdt \le C_3\mathcal{F}_1(\tau) + C_5(\eta_3,\eta_4)\int_{Q_{0,\tau}}\{|u^m|^2+|v^m|^2\}dxdt,$$

where $\eta_3, \eta_4 > 0$. Since the **(T)** is carried out, then from the Proposition we obtain

$$\int_{\Omega_\tau} \{|u^m|^{r(x)} + |u^m|^2 + |v^m|^2\}dx + \int_{Q_{0,\tau}} \Big\{\sum_{i=1}^n |u^m_{x_i}|^2 + \sum_{i,j=1}^n |v^m_{x_ix_j}|^2 + \\ + |u^m|^{q(x)} + |v^m|^{s(x)}\Big\}dxdt \le C_3\mathcal{F}_1(\tau) + C_6 \int_{Q_{0,\tau}} \{|u^m|^2 + |v^m|^2\}dxdt.$$

Using Gronwall's lemma we reach the estimate

$$\int_{\Omega_\tau} \{|u^m|^{r(x)} + |u^m|^2 + |v^m|^2\}dx + \int_{Q_{0,\tau}} \Big\{\sum_{i=1}^n |u^m_{x_i}|^2 + \sum_{i,j=1}^n |v^m_{x_ix_j}|^2 + \\ + |u^m|^{q(x)} + |v^m|^{s(x)}\Big\}dxdt \le C_7, \tag{19}$$

where $C_7 > 0$ is a constant independent on m, τ.

From Friedrichs' inequality (10) and (19) we obtain

$$\int_{\Omega_\tau} \{|u^m|^{r(x)} + |u^m|^2 + |v^m|^2\}dx + \int_{Q_{0,\tau}} \Big\{\sum_{i=1}^n |u^m_{x_i}|^2 + |u^m|^2 + \sum_{i,j=1}^n |v^m_{x_ix_j}|^2 + \sum_{i=1}^n |v^m_{x_i}|^2 + \\ + |v^m|^2 + |u^m|^{q(x)} + |v^m|^{s(x)}\Big\}dxdt \le C_8. \tag{20}$$

Hence, the functions $\varphi_1^m, \dots, \varphi_m^m$, $\psi_1^m, \dots, \psi_m^m$, and therefore u^m, v^m can be proceed to $[0,T]$ for all $m \in \mathbb{N}$.

Moreover, we have

$$\int_{\Omega_\tau} ||u^m|^{r(x)-2}u^m|^{r'(x)}dx = \int_{\Omega_\tau} |u^m|^{r(x)}dx \le C_8, \tag{21}$$

$$\int_{Q_{0,\tau}} ||u^m|^{q(x)-2}u^m|^{q'(x)}dxdt = \int_{Q_{0,\tau}} |u^m|^{q(x)}dxdt \le C_8, \tag{22}$$

$$\int_{Q_{0,\tau}} ||v^m|^{s(x)-2}v^m|^{s'(x)}dxdt = \int_{Q_{0,\tau}} |v^m|^{s(x)}dxdt \le C_8. \tag{23}$$

3. From the previous estimates we obtain the existence of subsequences $\{u^{m_j}\}_{j\in\mathbb{N}} \subset \{u^m\}_{m\in\mathbb{N}}$, $\{v^{m_j}\}_{j\in\mathbb{N}} \subset \{v^m\}_{m\in\mathbb{N}}$ such that

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{weakly star in } L^\infty(0,T; L^{r(x)}(\Omega) \cap L^2(\Omega)),$$

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \ \text{ weakly in } U_1(Q_{0,T}) \cap L^{r(x)}(Q_{0,T}), \quad |u^{m_j}|^{q(x)-2}u^{m_j} \underset{j\to\infty}{\longrightarrow} \chi_0 \ \text{ weakly in } L^{q'(x)}(Q_{0,T}),$$

$$\mathcal{R}u^{m_j} \underset{j\to\infty}{\longrightarrow} \widetilde{\chi}_0 \quad \text{weakly star in } L^\infty(0,T; L^{r'(x)}(\Omega)) \text{ and weakly in } L^{r'(x)}(Q_{0,T}),$$

$$v^{m_j} \underset{j\to\infty}{\longrightarrow} v \quad \text{weakly star in } L^\infty(0,T; L^2(\Omega)), \quad v^{m_j} \underset{j\to\infty}{\longrightarrow} v \text{ weakly in } U_2(Q_{0,T}),$$

$$|v^{m_j}|^{s(x)-2}v^{m_j} \underset{j\to\infty}{\longrightarrow} \chi_1 \quad \text{weakly in } L^{s'(x)}(Q_{0,T}).$$

4. Now we can tend $\mu \to \infty$ in (13) after applying the monotonicity method. Nonetheless, it is not enough to do the same in (12). It is required some additional estimates. Let us multiply it by $\frac{d}{dt}\varphi_j^m(t)$, summarize it on μ from 1 to m and integrate the result over t from 0 to $\tau \in (0, T]$. We obtain the identity

$$\widetilde{J}_1 + \widetilde{J}_2 + \ldots + \widetilde{J}_8 = \widetilde{J}_9, \tag{24}$$

where

$$\widetilde{J}_1 = \int_{Q_{0,\tau}} (\mathcal{P}u^m)_t u_t^m dxdt, \ \widetilde{J}_2 = \int_{Q_{0,\tau}} \sum_{i,j=1}^n a_{ij} u_{x_i}^m u_{x_j t}^m dxdt, \ \widetilde{J}_3 = \int_{Q_{0,\tau}} \sum_{i=1}^n z_i u_{x_i}^m u_t^m dxdt,$$

$$\widetilde{J}_4 = \int_{Q_{0,\tau}} du^m u_t^m dxdt, \ \widetilde{J}_5 = \int_{Q_{0,\tau}} g|u^m|^{q(x)-2} u^m u_t^m dxdt, \ \widetilde{J}_6 = \int_{Q_{0,\tau}} \sum_{i,j=1}^n b_{ij} v_{x_i x_j}^m u_t^m dxdt,$$

$$\widetilde{J}_7 = \int_{Q_{0,\tau}} \sum_{i=1}^n l_i v_{x_i}^m u_t^m dxdt, \ \widetilde{J}_8 = \int_{Q_{0,\tau}} hv^m u_t^m dxdt, \ \widetilde{J}_9 = \int_{Q_{0,\tau}} f u_t^m dxdt.$$

Let us estimate each term form (24). We have

$$\widetilde{J}_1 = \int_{Q_{0,\tau}} (|u^m|^{r(x)-2} + 1)|u_t^m|^2 dxdt = \int_{Q_{0,\tau}} |u^m|^{r(x)-2}|u_t^m|^2 dxdt + \int_{Q_{0,\tau}} |u_t^m|^2 dxdt;$$

$$\widetilde{J}_2 = \frac{1}{2}\int_{\Omega_\tau} \sum_{i,j=1}^n a_{ij} u_{x_i}^m u_{x_j}^m dx - \frac{1}{2}\int_{\Omega_0} \sum_{i,j=1}^n a_{ij} u_{0x_i}^m u_{0x_j}^m dx - \frac{1}{2}\int_{Q_{0,\tau}} \sum_{i,j=1}^n (a_{ij})_t u_{x_i}^m u_{x_j}^m dxdt \geq$$

$$\geq \frac{a_0}{2}\int_{\Omega_\tau} \sum_{i=1}^n |u_{x_i}^m|^2 dx - \frac{a^0}{2}\int_{\Omega_0} \sum_{i=1}^n |u_{0x_i}^m|^2 dx - C_7 \int_{Q_{0,\tau}} \sum_{i=1}^n |u_{x_i}^m|^2 dxdt;$$

$$|\widetilde{J}_3| \leq \frac{z^0}{2\varepsilon_1}\int_{Q_{0,\tau}} \sum_{i=1}^n |u_{x_i}^m|^2 dxdt + \frac{z^0 n\varepsilon_1}{2}\int_{Q_{0,\tau}} |u_t^m|^2 dxdt;$$

$$|\widetilde{J}_4| \leq \frac{d^0}{2\varepsilon_2}\int_{Q_{0,\tau}} |u^m|^2 dxdt + \frac{d^0\varepsilon_2}{2}\int_{Q_{0,\tau}} |u_t^m|^2 dxdt;$$

$$\widetilde{J}_5 = \int_{\Omega_\tau} \frac{g}{q(x)}|u^m|^{q(x)} dx - \int_{\Omega_0} \frac{g}{q(x)}|u_0^m|^{q(x)} dx - \int_{Q_{0,\tau}} \frac{g_t}{q(x)}|u^m|^{q(x)} dxdt \geq$$

$$\geq \frac{g_0}{q^0}\int_{\Omega_\tau} |u^m|^{q(x)} dx - \frac{g^0}{q_0}\int_{\Omega_0} |u_0^m|^{q(x)} dx - C_8 \int_{Q_{0,\tau}} |u^m|^{q(x)} dxdt;$$

$$|\widetilde{J}_6| \leq \frac{b^0}{2\varepsilon_3}\int_{Q_{0,\tau}} \sum_{i,j=1}^n |v_{x_i x_j}^m|^2 dxdt + \frac{b^0 n^2\varepsilon_3}{2}\int_{Q_{0,\tau}} |u_t^m|^2 dxdt;$$

$$|\widetilde{J}_7| \leq \frac{l^0}{2\varepsilon_4}\int_{Q_{0,\tau}} \sum_{i=1}^n |v_{x_i}^m|^2 dxdt + \frac{l^0 n\varepsilon_4}{2}\int_{Q_{0,\tau}} |u_t^m|^2 dxdt;$$

$$|\widetilde{J}_8| \leq \frac{h^0}{2\varepsilon_5}\int_{Q_{0,\tau}} |v^m|^2 dxdt + \frac{h^0\varepsilon_5}{2}\int_{Q_{0,\tau}} |u_t^m|^2 dxdt; \ |\widetilde{J}_9| = \frac{1}{2\varepsilon_6}\int_{Q_{0,\tau}} |f|^2 dxdt + \frac{\varepsilon_6}{2}\int_{Q_{0,\tau}} |u_t|^2 dxdt.$$

From (24) and reached priory estimates we get

$$\int_{Q_{0,\tau}} |u^m|^{r(x)-2}|u_t^m|^2 dxdt + \Big(1 - \frac{b^0 n^2\varepsilon_3}{2} - \frac{z^0 n\varepsilon_1}{2} - \frac{d^0\varepsilon_2}{2} - \frac{l^0 n\varepsilon_4}{2} - \frac{h^0\varepsilon_5}{2} - \frac{\varepsilon_6}{2}\Big)\times$$

$$\times \int_{Q_{0,\tau}} |u_t^m|^2 dxdt + \frac{g_0}{q^0}\int_{\Omega_\tau} |u^m|^{q(x)} dx + \frac{a_0}{2}\int_{\Omega_\tau} \sum_{i=1}^n |u_{x_i}^m|^2 dx \leq$$

$$\leq \int_{\Omega_0}\Big(\frac{a^0}{2}\sum_{i=1}^n |u^m_{0x_i}|^2 + \frac{g^0}{q_0}|u^m_0|^{q(x)}\Big)dx + \frac{1}{2\varepsilon_6}\int_{Q_{0,\tau}} |f|^2 dxdt + C_9(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)\times$$
$$\times\int_{Q_{0,\tau}}\Big(\sum_{i,j=1}^n |v^m_{x_ix_j}|^2 + \sum_{i=1}^n |v^m_{x_i}|^2 + \sum_{i=1}^n |u^m_{x_i}|^2 + |u^m|^2 + |u^m|^{q(x)} + |v^m|^2\Big)dx. \tag{25}$$

But now let $\varepsilon_i > 0$ ($i \in \{1,2,\dots,5\}$) are small enough and apply (20) to (25) we get

$$\int_{Q_{0,\tau}} (|(|u^m|^{\frac{r(x)-2}{2}}u^m)_t|^2 + |u^m_t|^2)dxdt + \int_{\Omega_\tau}\Big[\sum_{i=1}^n |u^m_{x_i}|^2 + |u^m|^{q(x)}\Big]dx \leq C_{10}, \tag{26}$$

where $C_{10} > 0$ is a constant independent of m and τ.

Hence, passing to the new subsequence $\{u^{m_j}\}_{j\in\mathbb{N}}$ ((let us leave the same notation) we have

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{weakly star in } L^\infty(0,T;H^1_0(\Omega)\cap L^{q(x)}(\Omega)),$$
$$(|u^{m_j}|^{\frac{r(x)-2}{2}}u^{m_j})_t \underset{j\to\infty}{\longrightarrow} \widetilde{\chi}_1 \quad \text{weakly in } L^2(Q_{0,T}),\quad u^{m_j}_t \underset{j\to\infty}{\longrightarrow} u_t \quad \text{weakly in } L^2(Q_{0,T}). \tag{27}$$

These convergences, particulary, mean that

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{weakly in } H^1(Q_{0,T}).$$

Thus, using the Embedding Theorem (Rellich-Kondrachov) and Lemma 1.18 ([13, c. 39]), and maybe passing to a new subsequence we reach

$$u^{m_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{in } L^2(Q_{0,T}) \text{ and a. e. in } Q_{0,T}. \tag{28}$$

Thus, $\chi_0 = |u|^{q(x)-2}u$, $\widetilde{\chi}_0 = \mathcal{R}u$, $\widetilde{\chi}_1 = (|u|^{\frac{r(x)-2}{2}}u)_t$.

From (26) and the inequality $|y(\tau)|^2 \leq C_{10}(|y(0)|^2 + \int_0^\tau |y'(t)|^2dt)$, where C_{10} is independent of y, we have

$$\int_{\Omega_\tau} |u^m|^{r(x)}dx = \int_{\Omega_\tau} ||u^m|^{\frac{r(x)-2}{2}}u^m|^2dx \leq C_{10}\Big(\int_{\Omega_0} ||u^m_0|^{\frac{r(x)-2}{2}}u^m_0|^2dx +$$
$$+\int_{Q_{0,\tau}} |(|u^m|^{\frac{r(x)-2}{2}}u^m)_t|^2dxdt\Big) \leq C_{10}\Big(\int_{\Omega_0} |u^m_0|^{r(x)}dx + C_{10}\Big) \leq C_{11}.$$

Then

$$\|u^m; L^\infty(0,T;L^{r(x)}(\Omega))\| \leq C_{11}, \tag{29}$$

where C_{11} is a constant independent of m.

Now let us derive the estimates of the expressions below

$$(\mathcal{R}u^m)_t = |u^m|^{r(x)-2}u^m_t = |u^m|^{\frac{r(x)-2}{2}}|u^m|^{\frac{r(x)-2}{2}}u^m_t. \tag{30}$$

From the Young inequality with the exponent $\frac{2}{r'(x)} > 1$, we get

$$|(\mathcal{R}u^m)_t|^{r'(x)} = \Big\||u^m|^{r(x)-2}\Big|^{\frac{r'(x)}{2}}\cdot\Big\||u^m|^{\frac{r(x)-2}{2}}u^m_t\Big|^{r'(x)} \leq C_{12}(|u^m|^{(r(x)-2)\frac{r'(x)}{2}(\frac{2}{r'(x)})'} + |u^m|^{r(x)-2}|u^m_t|^2).$$

Since

$$(r-2)\frac{r'}{2}(\frac{2}{r'})' = (r-2)\frac{r'}{2}\frac{\frac{2}{r'}}{\frac{2}{r'}-1} = (r-2)\frac{r'}{2}\frac{2}{2-r'} = (r-2)\frac{r'}{2-r'} =$$

$$= (r-2)\frac{\frac{r}{r-1}}{2-\frac{r}{r-1}} = (r-2)\frac{r}{2r-2-r} = (r-2)\frac{r}{r-2} = r,$$

then $|(\mathcal{R}u^m)_t|^{r'(x)} \le C_{12}(|u^m|^{r(x)} + |u^m|^{r(x)-2}|u_t^m|^2)$. Thus,

$$\|(\mathcal{R}u^m)_t; L^{r'(x)}(Q_{0,T})\| \le C_{13}, \tag{31}$$

where C_{13} is independent of m.

5. From the Monotonicity Method ([2]) we derive $|v|^{s(x)-2}v = \chi_1$.

6. For all $\omega \in U_1(Q_{0,T}) \cap U_2(Q_{0,T}) \cap L^{r(x)}(Q_{0,T})$ we consider a sequence of functions $\{z_k\}_{k\in\mathbb{N}}$ such that $z_k \in \mathcal{L}_m = \{\sum_{k=1}^m d_k^m(t)\omega^k(x) \colon d_1,\dots,d_m \in C^1([0,T])\}$, $z_k \to \omega$ in $\mathcal{U}$ as $k \to \infty$. Then (12)–(13) imply

$$[(\mathcal{R}u^{m_j})_t + u_t^{m_j}, z_k]_Q + \sum_{i,j=1}^n [a_{ij}u_{x_i}^{m_j}, (z_k)_{x_j}]_Q + \sum_{i=1}^n [z_i u_{x_i}^{m_j}, z_k]_Q + [du^{m_j}, z_k]_Q +$$

$$+[g|u^{m_j}|^{q(x)-2}u^{m_j}, z_k]_Q + \sum_{i,j=1}^n [b_{ij}v_{x_ix_j}^{m_j}, z_k]_Q + \sum_{i=1}^n [l_i v_{x_i}^{m_j}, z_k]_Q + [hv^{m_j}, z_k]_Q = [f, z_k]_Q,$$

$$[v_t^{m_j}, z_k]_Q + \sum_{i,j,k,l=1}^n [A_{ijkl}v_{x_ix_j}^{m_j}, (z_k)_{x_kx_l}]_Q + \sum_{i,j,k=1}^n [L_{ijk}v_{x_ix_j}^{m_j}, (z_k)_{x_k}]_Q + \sum_{i,j=1}^n [B_{ij}v_{x_ix_j}^{m_j}, z_k]_Q +$$

$$+\sum_{i=1}^n [Z_i v_{x_i}^{m_j}, z_k]_Q + [Du^m, z_k]_Q + [G|v^{m_j}|^{s(x)-2}v^{m_j}, z_k]_Q - \sum_{i,j=1}^n [E_{ij}u_{x_i}^m, (z_k)_{x_j}]_Q +$$

$$+\sum_{i=1}^n [Hu_{x_i}^m, z_k]_Q + [Iu^m, z_k]_Q = [F, z_k]_Q.$$

Tending $m_j \to \infty$, we get

$$[(\mathcal{R}u)_t + u_t, z_k]_Q + \sum_{i,j=1}^n [a_{ij}u_{x_i}, (z_k)_{x_j}]_Q + \sum_{i=1}^n [z_i u_{x_i}, z_k]_Q + [du, z_k]_Q + [g|u|^{q(x)-2}u, z_k]_Q +$$

$$+\sum_{i,j=1}^n [b_{ij}v_{x_ix_j}, z_k]_Q + \sum_{i=1}^n [l_i v_{x_i}, z_k]_Q + [hv, z_k]_Q = [f, z_k]_Q,$$

$$[v_t, z_k]_Q + \sum_{i,j,k,l=1}^n [A_{ijkl}v_{x_ix_j}, (z_k)_{x_kx_l}]_Q + \sum_{i,j,k=1}^n [L_{ijk}v_{x_ix_j}, (z_k)_{x_k}]_Q + \sum_{i,j=1}^n [B_{ij}v_{x_ix_j}, z_k]_Q +$$

$$+\sum_{i=1}^n [Z_i v_{x_i}, z_k]_Q + [Du, z_k]_Q + [G|v|^{s(x)-2}v, z_k]_Q - \sum_{i,j=1}^n [E_{ij}u_{x_i}, (z_k)_{x_j}]_Q +$$

$$+\sum_{i=1}^n [Hu_{x_i}, z_k]_Q + [Iu, z_k]_Q = [F, z_k]_Q,$$

Hence, letting $k \to \infty$ we obtain (8).

Let us mention that the uniqueness of a solution of problem (4)–(7) was not an object of our investigation. It worth mentioning that the uniqueness of a solution of a mixed problem for (5) is separately discovered in [2]. The uniqueness of a solution for the model doubly nonlinear equation of a type (4) is proved in [14].

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Received 3.12.2011
Revised 7.06.2012