INITIAL-BOUNDARY PROBLEMS FOR SYSTEMS OF A HIGH ORDER
DOUBLY NONLINEAR PARABOLIC EQUATIONS WITH VARIABLE
EXponent OF NONLINEarity

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Consider a mixed problem for a class of system of a high order doubly nonlinear parabolic
equations with variable exponent of nonlinearity. This problem is considered in generalized
Lebesgue-Sobolev spaces. As a result, we reached a condition of the existence of a solution. We
use here Galerkin’s procedure.

Thin liquid films are important in biophysics, physics, and engineering, as well as in
natural settings. They can be composed of common liquids such as water, oil, or complex
mixtures of components. Let us consider length scales in the x direction.

The fluid motion is described by the equations in two dimensions of the form ( [1, p. 936])

\[
\begin{align*}
\rho(u_t + uu_x + wu_z) &= -p_x + \mu u_{xx} - \varphi_x, \\
\rho(w_t + uw_x + ww_z) &= -p_z + \mu w_{xx} - \varphi_z, \\
u_x + w_z &= 0.
\end{align*}
\]

If we switch to a new dimensionless parameters and applied some additional assumptions,
this problem is reduced to an equation with respect to a function \( h \) of the form

\[ h_t + h_{xxxx} + h_{xx} = f(x,t). \]  (1)

In [3, p. 1034] it is shown that the movement of the considered liquid can be described by a
system of equations, much more precisely than by (1). A model example of such a system is

\[
\begin{align*}
n_t - n_{xx} + h_{xx} &= F(x,t), \\
h_t + h_{xxxx} + h_{xx} + n_{xx} &= f(x,t),
\end{align*}
\]  (2)

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where dimensionless parameter \( n \) is the concentration of a liquid.

Particularly, A. N. Tikhonov and A. A. Samarskii (\cite{4, p. 184}) state that the diffusion process in a porous medium, which is described by the linear equation \( n_t - L n_{xx} = 0 \), can be better described by the nonlinear equation \((\varphi(n))_t - L n_{xx} = 0\), where \( \varphi \) is some nonlinear function, for instance \( \varphi(n) = n^\alpha n + n \), \( \alpha > 0 \).

Thus instead of (2) we study the model system

\[
\begin{align*}
(n^\alpha n + n)_t - n_{xx} + h_{xx} &= F(x,t), \\
h_t + h_{xxxx} + h_{xx} + n_{xx} &= f(x,t),
\end{align*}
\]

which drops into a class of systems of equations we consider.

1. **Problem statement.** Let \( T > 0 \) be an arbitrary number, \( \Omega \) be a bounded domain, \( Q_{t_1,t_2} = \Omega \times (t_1,t_2) \), \( 0 \leq t_1 < t_2 \leq T \), \( \Omega_\tau = \{(x,t) : x \in \Omega, t = \tau\} \), \( \tau \in [0,T], \)

\[
L_+^\infty(\Omega) = \left\{ v \in L^\infty(\Omega) : \text{ess inf}_{x \in \Omega} v(x) > 1 \right\}.
\]

We define \( s_0 \equiv \text{ess inf}_{x \in \Omega} s(x) \), \( s^0 \equiv \text{ess sup}_{x \in \Omega} s(x) \) for all \( s \in L_+^\infty(\Omega) \), and by \( s' \) we denote the element from \( L_+^\infty(\Omega) \) satisfying \( \frac{1}{s(x)} + \frac{1}{s'(x)} = 1 \) for almost all \( x \in \Omega \). The symbols \( L^{s(x)}(\Omega) \) and \( L^{s(x)}(Q_{0,T}) \) stand for the generalized Lebesgue spaces (\cite{5–7}).

Let \( r, s, q \in L_+^\infty(\Omega) \), and

\[
\begin{align*}
V_1 &= H^1_0(\Omega) \cap L^{q(x)}(\Omega), \\
V_2 &= H^2_0(\Omega) \cap L^{s(x)}(\Omega), \\
U_1(Q_{0,T}) &= L^2(0,T; H^2_0(\Omega)) \cap L^{q(x)}(Q_{0,T}), \\
U_2(Q_{0,T}) &= L^2(0,T; H^2_0(\Omega)) \cap L^{s(x)}(Q_{0,T}), \\
U_3(Q_{0,T}) &= L^2(0,T; H^2_0(\Omega)) \cap L^{r(x)}(Q_{0,T}) \cap L^{q(x)}(Q_{0,T}) \cap L^{s(x)}(Q_{0,T}).
\end{align*}
\]

We consider the following problem

\[
(Pu)_t - \sum_{i,j=1}^n (a_{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^n z_i(x,t)u_{x_i} + d(x,t)u + g(x,t)|u|^{q(x)-2}u + \\
+ \sum_{i,j=1}^n b_{ij}(x,t)v_{x_i}v_{x_j} + \sum_{i=1}^n l_i(x,t)v_{x_i} + h(x,t)v = f(x,t), \\
v_t + \sum_{i,j,k,l=1}^n (A_{ijkl}(x,t)v_{x_i}x_j)_{x_k} + \sum_{i,j,k,l=1}^n (L_{ijkl}(x,t)v_{x_i}x_j)_{x_k} + \sum_{i,j=1}^n B_{ij}(x,t)v_{x_i}v_{x_j} + \\
+ \sum_{i=1}^n Z_i(x,t)v_{x_i} + D(x,t)v + G(x,t)|v|^{s(x)-2}v + \sum_{i,j=1}^n (E_{ij}(x,t)u_{x_i})_{x_j} + \\
+ \sum_{i=1}^n H_i(x,t)u_{x_i} + I(x,t)u = F(x,t),
\]

\[
|u|_{\partial \Omega \times [0,T]} = v|_{\partial \Omega \times [0,T]} = \frac{\partial v}{\partial \nu}|_{\partial \Omega \times [0,T]} = 0, \\
u|_{t=0} = u_0, \quad v|_{t=0} = v_0,
\]

where \( Pu = Ru + u \), \( Ru = \frac{1}{r(x)-1}|u|^{r(x)-2}u \). Assume the measurable coefficients of (4), (5) satisfy the following conditions:
(a): \( a_{ij}, (a_{ij})_t \in L^\infty(Q_{0,T}) \), \( a_0|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \leq a^0|\xi|^2 \) for every \( \xi \in \mathbb{R}^n \) and a. e. \( x \in \Omega \), where \( a_0, a^0 > 0 \);
(b): \( |b_{ij}(x,t)| \leq b^0 \) for every \( i, j \in \{1, 2, \ldots, n\} \) and a. e. \( (x,t) \in Q_{0,T} \), where \( b^0 > 0 \);
(d): \( |d(x,t)| \leq d^0 \) for a. e. \( (x,t) \in Q_{0,T} \), where \( d^0 > 0 \);
(g): \( g, g_t \in L^\infty(Q_{0,T}) \), \( 0 < g_0 \leq g(x,t) \leq g^0 \) for a. e. \( (x,t) \in Q_{0,T} \);
(h): \( |h(x,t)| < h^0 \) for a. e. \( (x,t) \in Q_{0,T} \), where \( h^0 > 0 \);
(l): \( |l_i(x,t)| < l^0 \) for every \( i \in \{1, 2, \ldots, n\} \) and a. e. \( (x,t) \in Q_{0,T} \), where \( l^0 > 0 \);
(z): \( |z_i(x,t)| \leq z^0 \) for every \( i \in \{1, 2, \ldots, n\} \) and a. e. \( (x,t) \in Q_{0,T} \), where \( z^0 > 0 \);
(A): \( A_0 \sum_{i,j=1}^n |\eta_{ij}|^2 \leq \sum_{i,j,k,l=1}^n A_{ijkl}(x,t)\eta_{ij}\eta_{kl} \leq A^0 \sum_{i,j=1}^n |\eta_{ij}|^2 \) for every \( \eta \in \mathbb{R}^n \times \mathbb{R}^n \) and a. e. \( (x,t) \in Q_{0,T} \), where \( A_0, A^0 > 0 \);
(B): \( |B_{ij}(x,t)| \leq B^0 \) for every \( i, j \in \{1, 2, \ldots, n\} \) and a. e. \( (x,t) \in Q_{0,T} \), where \( B^0 > 0 \);
(D): \( |D(x,t)| \leq D^0 \) for a. e. \( (x,t) \in Q_{0,T} \), where \( D^0 > 0 \);
(E): \( |E_{ij}(x,t)| \leq E^0 \) for every \( i, j \in \{1, 2, \ldots, n\} \) and a. e. \( (x,t) \in Q_{0,T} \), where \( E^0 > 0 \);
(G): \( G(x,t) \geq G_0 > 0 \) for a. e. \( (x,t) \in Q_{0,T} \);
(H): \( |H_i(x,t)| \leq H^0 \) for every \( i \in \{1, 2, \ldots, n\} \) and a. e. \( (x,t) \in Q_{0,T} \), where \( H^0 > 0 \);
(I): \( |I(x,t)| \leq I^0 \) for a. e. \( (x,t) \in Q_{0,T} \), where \( I^0 > 0 \);
(L): \( |L_{ijk}(x,t)| \leq L^0 \) for every \( i, j, k \in \{1, 2, \ldots, n\} \) and a. e. \( (x,t) \in Q_{0,T} \), where \( L^0 > 0 \);
(Z): \( |Z_i(x,t)| \leq Z^0 \) for every \( i \in \{1, 2, \ldots, n\} \) and a. e. \( (x,t) \in Q_{0,T} \), where \( Z^0 > 0 \);
(F): \( f, f_t, F, F_t \in L^2(Q_{0,T}) \).

To simplify the estimations we define

\[
[u, v]_Q = \int_{Q_{0,T}} u(x,t)v(x,t)dxdt, \quad (y, z)_\Omega = \int_\Omega y(x)z(x)dx.
\]

**Definition 1.** A pair of functions \((u, v)\) such that
\((u, v) \in U_1(Q_{0,T}) \times U_2(Q_{0,T}), u \in L^*Q_{0,T}, u_t, (Ru)_t \in L^2(Q_{0,T}), v_t \in L^2(0, T; H^{-2}(\Omega))\)
is called a **weak solution of problem** (4)–(7), if it satisfies (7) and the following equalities

\[
[(Ru)_t + u_t, w]_Q + \sum_{i,j=1}^n [a_{ij}u_{x_i}, w_{x_j}]_Q + \sum_{i=1}^n [z_iu_{x_i}, w]_Q + [du, w]_Q + [g|u|^q(x)-2u, w]_Q + \sum_{i,j=1}^n [b_{ij}v_{x_i,x_j}, w]_Q + \sum_{i=1}^n [l_i v_{x_i}, w]_Q + [hv, w]_Q = [f, w]_Q,
\]

\[
[v_t, w]_Q + \sum_{i,j,k,l=1}^n [A_{ijkl}v_{x_i,x_j,x_k}, w_{x_l}]_Q + \sum_{i,j,k=1}^n [L_{ijk}v_{x_i,x_j}, w_{x_k}]_Q + \sum_{i,j=1}^n [B_{ij}v_{x_i,x_j}, w]_Q + \sum_{i=1}^n [Z_i v_{x_i}, w]_Q + [Du, w]_Q + [G|v|^{s(x)}-2, v]_Q - \sum_{i,j=1}^n [E_{ij}u_{x_i}, w_{x_j}]_Q + \sum_{i=1}^n [Hu_{x_i}, w]_Q + [Iu, w]_Q = [F, w]_Q,
\]

for every \( w \in U_3(Q_{0,T}) \).
Assume, the next condition fulfills

\((T)\): \(4a_0 \tilde{A}^0 < M_\Omega (E^0 n)^2, \ A_0^2 > M_\Omega (L^0 n)^2 n, \ \tilde{A}^0 = \frac{A_0^2 - M_\Omega (L^0)^2 n^3}{2 A_0},\)

where \(M_\Omega\) is taken from Friedrich’s inequality (Lemma 1.36 [13, p. 50]),

\[
\int |v|^2 dx \leq M_\Omega \int |\nabla v|^2 dx \quad \text{for every} \ v \in H^1_0(\Omega),
\]

\(n\) is a space dimension parameter, \(a_0, A_0, E^0, L^0\) from (a), (A), (E), (L), respectively.

**Proposition 1.** If condition \((T)\) is carried out, then there exist \(\eta_1, \eta_2 > 0\), such that we have

\[
a_0 - \frac{E^0 n}{2} \eta_1 > 0, \quad A_0 - \frac{L^0 n}{2} \eta_2 - \frac{M_\Omega L^0 n^2}{2} \frac{1}{\eta_2} = \frac{M_\Omega E^0 n}{2} \eta_1 > 0,
\]

where \(a_0, A_0, E^0, L^0, n, M_\Omega\) are as stated above.

**Proof.** Define \(\tilde{A}(\eta_2) = A_0 - \frac{L^0 n}{2} \eta_2 - \frac{M_\Omega L^0 n^2}{2} \frac{1}{\eta_2}\). Let us find \(\eta_2 = \eta_2^0 > 0\), such that \(\tilde{A}(\eta_2^0) > 0\). Since

\[
\tilde{A}(\eta_2) = A_0 - \frac{L^0 n}{2} \eta_2 - \frac{M_\Omega L^0 n^2}{2} \frac{1}{\eta_2} = \frac{2 A_0 \eta_2 - L^0 n \eta_2^2 - M_\Omega L^0 n^2}{2 \eta_2} = \frac{-L^0 n \eta_2^2 + 2 A_0 \eta_2 - M_\Omega L^0 n^2}{2 \eta_2},
\]

we are going to find out the abscissa of the culmination point of the parabola registered in the numerator of the expression above (such a point is \(\eta_2 = \eta_2^0\)) \(\eta_2^0 = \frac{A_0}{E^0 n}\). Obviously, \(\eta_2^0 > 0\).

Thus, taken into consideration condition \((T)\) we have

\[
\tilde{A}^0 = A_0 - \frac{L^0 n}{2} \frac{A_0}{L^0 n} - \frac{M_\Omega L^0 n^2}{2} \frac{L^0 n}{A_0} = \frac{A_0}{2} - \frac{M_\Omega (L^0)^2 n^3}{2 A_0} = \frac{A_0^2 - M_\Omega (L^0)^2 n^3}{2 A_0} > 0.
\]

Now we get

\[
\frac{M_\Omega E^0 n}{2 \tilde{A}^0} < \eta_1 < \frac{2a_0}{E^0 n}.
\]

Hence, from \((T)\) we obtain \(\frac{M_\Omega E^0 n}{2 \tilde{A}^0} < \frac{2a_0}{E^0 n}\). So such \(\eta_1 > 0\) exists and therefore (11) is to be fulfilled.

**Theorem 1.** If \(u_0 \in V_1 \cap L^p(\Omega), \ v_0 \in V_2, \) conditions \((a)-(F), (T)\) hold and \(r_0 > 2\), then problem (4)–(7) has a weak solution.

We use the Galerkin procedure to proof the theorem. The proof is located in a separate section just only for convenience of our investigation process.

Systems of nonlinear parabolic equations with variable exponent of nonlinearity is widely studied, particularly (see [2]). As mentioned above, the system of equations of form (3) has been occurred in [3]. The solvability of some doubly nonlinear parabolical equations by elliptic regularization method is shown in [8, 9]. A model example of the equations is

\[
|u|^{r(x)} - u_t - (|u|^{p-2} u_x)_x = f.
\]

Moreover, in [9] one can find a proof of solution existence to a homogenous mixed problem for the model equation \((|u|^{r(x)} - u + u)_t - (|u|^{p-2} u_x)_x = f\). Other nonlinear equations in Lebesgue and Sobolev spaces were investigated in [10–12]. Conditions of solution existence and uniqueness of correspondent parabolic variational inequalities are mentioned in [6, 7].
2. Proof of the main result.
1. Let \( \omega^1, \omega^2, \ldots, \omega^m, \ldots \) be a basis in \( H^2_0(\Omega) \). A solution to (4)–(7) we start to search in the form

\[
    u^m(x, t) = \sum_{\mu=1}^{m} \varphi^m_\mu(t) \omega^\mu(x), \quad v^m(x, t) = \sum_{\mu=1}^{m} \psi^m_\mu(t) \omega^\mu(x), \quad (x, t) \in Q_{0,T},
\]

where functions \( \varphi^m_\mu, \psi^m_\mu (\mu \in \{1, 2, \ldots, m\}) \) are solutions to the following Cauchy problem

\[
    \begin{align*}
    ((P u^m(t))_t, \omega^\mu)_\Omega & + \sum_{i,j=1}^{n} (a_{ij} u^m_{x_i}(t), \omega^\mu_{x_j})_\Omega + \sum_{i=1}^{n} (z_i u^m_x(t), \omega^\mu)_\Omega + (du^m(t), \omega^\mu)_\Omega + \\
    +(g|u^m(t)|^{q(x)-2} u^m(t), \omega^\mu)_\Omega + \sum_{i,j=1}^{n} (b_{ij} v^m_{x_i x_j}(t), \omega^\mu)_\Omega + \sum_{i=1}^{n} (l_i v^m_x(t), \omega^\mu)_\Omega + \\
    +(h v^m(t), \omega^\mu)_\Omega = (f(t), \omega^\mu)_\Omega, \quad (12) \\
    (v^m_t(t), \omega^\mu)_\Omega & + \sum_{i,j,k,l=1}^{n} (A_{ijkl} v^m_{x_i x_j x_k x_l}(t), \omega^\mu_{x_l})_\Omega + \sum_{i,j,k=1}^{n} (L_{ijkl} v^m_{x_i x_j x_k}(t), \omega^\mu_{x_l})_\Omega + \\
    + \sum_{i,j=1}^{n} (B_{ij} v^m_{x_i x_j}(t), \omega^\mu)_\Omega + \sum_{i=1}^{n} (Z_i v^m_x(t), \omega^\mu)_\Omega + (D v^m(t), \omega^\mu)_\Omega + \\
    +(G|v^m(t)|^{s(x)-2} v^m(t), \omega^\mu)_\Omega - \sum_{i=1}^{n} (E_{ij} u^m_{x_i x_j}(t), \omega^\mu)_\Omega + \sum_{i=1}^{n} (H_i u^m_{x_i}(t), \omega^\mu)_\Omega + \\
    +(I u^m(t), \omega)_\Omega = (F(t), \omega^\mu)_\Omega, \quad t \in (0, T), \quad (13) \\
    \varphi_j^m(0) = \alpha_j^m, \quad \psi_j^m(0) = \beta_j^m, \quad j \in \{1, 2, \ldots, m\},
\end{align*}
\]

where \( \alpha_1^m, \ldots, \alpha_m^m, \beta_1^m, \ldots, \beta_m^m \in \mathbb{R} \) are such numbers that

\[
    u_0^m = \sum_{\mu=1}^{m} \alpha^m_\mu \omega^\mu \quad \text{strongly in } L^r(x)(\Omega), \quad v_0^m = \sum_{\mu=1}^{m} \beta^m_\mu \omega^\mu \quad \text{strongly in } V_2.
\]

Additionally, note that

\[
    u^m \bigg|_{t=0} = v_0^m, \quad v^m \bigg|_{t=0} = v_0^m. \quad (15)
\]

The system of ordinary differential equations (12)–(13) is locally soluble. But, taking into account the estimates above we conclude that the solution to (12)–(13) is defined on \([0, T]\).

Particularly, \( \varphi^m_1, \ldots, \varphi^m_m, \psi^m_1, \ldots, \psi^m_m \in C^1([0, T]) \), so

\[
    u^m, \quad v^m \in C^1(\overline{Q}_{0,T}). \quad (16)
\]

2. Multiplying each \( \mu \)-th equation (12)–(13) \( (\mu \in \{1, 2, \ldots, m\}) \), \( m \) is fixed) by \( \varphi^m_\mu(t) \) and \( \psi^m_\mu(t) \), respectively, summarizing all such formed equations of the system and then integrating it over \([0, \tau], \tau \leq T\), we get

\[
    J_1 + J_2 + \ldots + J_{18} = J_{19} + J_{20}, \quad (17)
\]
where

\[
J_1 = \int_{Q_0} \left(Pu^m\right)_t u^m dx dt, \quad J_2 = \int_{Q_0} \sum_{i,j=1}^n a_{ij} u^m_i u^m_j dx dt, \quad J_3 = \int_{Q_0} \sum_{i=1}^n z_i u^m_i dx dt,
\]

\[
J_4 = \int_{Q_0} d|u^m|^2 dx dt, \quad J_5 = \int_{Q_0} g|u^m|^q(x) dx dt, \quad J_6 = \int_{Q_0} \sum_{i,j=1}^n b_{ij} v^m_{x_i x_j} u^m dx dt,
\]

\[
J_7 = \int_{Q_0} \sum_{i=1}^n l_i v^m_{x_i} u^m dx dt, \quad J_8 = \int_{Q_0} h v^m u^m dx dt, \quad J_9 = \int_{Q_0} v^m_i v^m dx dt,
\]

\[
J_{10} = \int_{Q_0} \sum_{i,j,k,l=1}^n A_{ijkl} v^m_{x_i x_j x_k x_l} dx dt, \quad J_{11} = \int_{Q_0} \sum_{i,j,k,l=1}^n L_{ijkl} v^m_{x_i x_j x_k x_l} dx dt,
\]

\[
J_{12} = \int_{Q_0} \sum_{i,j=1}^n B_{ij} v^m_{x_i x_j} u^m dx dt, \quad J_{13} = \int_{Q_0} \sum_{i=1}^n Z_i v^m_i u^m dx dt, \quad J_{14} = \int_{Q_0} D|v^m|^2 dx dt,
\]

\[
J_{15} = \int_{Q_0} G|v^m|^\alpha(x) dx dt, \quad J_{16} = -\int_{Q_0} \sum_{i,j=1}^n E_{ij} v^m_{x_i} u^m_{x_j} dx dt, \quad J_{17} = \int_{Q_0} \sum_{i=1}^n H_i u^m_i v^m dx dt,
\]

\[
J_{18} = \int_{Q_0} I u^m v^m dx dt, \quad J_{19} = \int_{Q_0} f u^m dx dt, \quad J_{20} = \int_{Q_0} F v^m dx dt.
\]

Further, we are going to make an estimates of \(J_1 - J_{20}\). Since (16) holds, we have

\[
J_1 = \int_{\Omega} \left(\frac{1}{r(x)}|u^m|^r(x) dx - \frac{1}{r(x)}|u^0|^r(x) dx + \frac{1}{2} \int_{\Omega_r} |u^m|^2 dx + \frac{1}{2} \int_{\Omega_r} |u^0|^2 dx\right).
\]

Using condition (a)–(F) and Young’s estimate \(ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}\), we get

\[
J_2 \geq a_0 \int_{Q_0} \sum_{i=1}^n |u^m_{x_i}|^2 dx dt, \quad |J_3| \leq \frac{\varepsilon_1}{2} \int_{Q_0} \sum_{i=1}^n |u^m_{x_i}|^2 dx dt + \frac{\varepsilon_0 n}{2\varepsilon_1} \int_{Q_0} |u^m|^2 dx dt,
\]

\[
|J_4| \leq \frac{\varepsilon_2}{2} \int_{Q_0} |u^m|^2 dx dt, \quad J_5 \geq g_0 \int_{Q_0} |u^m|^q(x) dx dt,
\]

\[
|J_6| \leq \frac{\varepsilon_2 n}{2} \int_{Q_0} \sum_{i,j=1}^n |v^m_{x_i x_j}|^2 dx dt + \frac{\varepsilon_2 n}{2} \int_{Q_0} |u^m|^2 dx dt,
\]

\[
|J_7| \leq \frac{\varepsilon_3}{2} \int_{Q_0} \sum_{i,j=1}^n |v^m_{x_i}|^2 dx dt + \frac{\varepsilon_3 n}{2} \int_{Q_0} |u^m|^2 dx dt,
\]

\[
|J_8| \leq \frac{\varepsilon_4}{2} \int_{Q_0} |v^m|^2 dx dt + \frac{\varepsilon_4 n}{2} \int_{Q_0} |u^m|^2 dx dt,
\]

\[
J_9 = \frac{1}{2} \int_{\Omega_r} |v^m|^2 dx - \frac{1}{2} \int_{\Omega_r} |v^0|^2 dx, \quad J_{10} \geq A_0 \int_{Q_0} \sum_{i,j=1}^n |v^m_{x_i x_j}|^2 dx dt,
\]

\[
|J_{11}| \leq \frac{\varepsilon_5 n}{2} \int_{Q_0} \sum_{i,j=1}^n |v^m_{x_i x_j}|^2 dx dt + \frac{\varepsilon_5 n}{2} \int_{Q_0} \sum_{i=1}^n |v^m_{x_i}|^2 dx dt,
\]

\[
|J_{12}| \leq \frac{\varepsilon_6}{2} \int_{Q_0} \sum_{i,j=1}^n |v^m_{x_i x_j}|^2 dx dt + \frac{\varepsilon_6 n}{2} \int_{Q_0} |v^m|^2 dx dt,
\]
\[ |J_{13}| \leq \frac{Z^0 \varepsilon}{2} \int_{Q_{0, r}} \sum_{i=1}^{n} |u_{x_i}^m|^2 dx dt + \frac{Z^0 n}{2\varepsilon} \int_{Q_{0, r}} |v^m|^2 dx dt, \]

\[ |J_{14}| \leq D^0 \int_{Q_{0, r}} |u^m|^2 dx dt, \quad J_{15} \geq G_0 \int_{Q_{0, r}} |v^m|^s(x) dx dt, \]

\[ |J_{16}| \leq \frac{E^0 \varepsilon n}{2} \int_{Q_{0, r}} \sum_{i=1}^{n} |u_{x_i}^m|^2 dx dt + \frac{E^0 n}{2\varepsilon} \int_{Q_{0, r}} \sum_{i=1}^{n} |v_{x_i}^m|^2 dx dt, \]

\[ |J_{17}| \leq \frac{H^0 \varepsilon}{2} \int_{Q_{0, r}} \sum_{i=1}^{n} |u_{x_i}^m|^2 dx dt + \frac{H^0 n}{2\varepsilon} \int_{Q_{0, r}} |v^m|^2 dx dt, \]

\[ |J_{18}| \leq \frac{I^0 \varepsilon_1}{2} \int_{Q_{0, r}} |u^m|^2 dx dt + \frac{I^0}{2\varepsilon_1} \int_{Q_{0, r}} |v^m|^2 dx dt, \]

\[ |J_{19}| + |J_{20}| \leq \frac{1}{2} \int_{Q_{0, r}} \{|f|^2 + |F|^2\} dx dt + \frac{1}{2} \int_{Q_{0, r}} \{|u^m|^2 + |v^m|^2\} dx dt. \]

Using the priory estimates, (17) we obtain the following

\[ \int_{\Omega} \left\{ \frac{1}{r(x)} |u^m|^r(x) + \frac{1}{2} (|u^m|^2 + |v^m|^2) \right\} dx + \int_{Q_{0, r}} \left\{ \left( a_0 - \frac{z^0 \varepsilon_1}{2} - \frac{E^0 \varepsilon n}{2} - \frac{H^0 \varepsilon}{2} \right) \sum_{i=1}^{n} |u_{x_i}^m|^2 + \right. \]

\[ + \left( A_0 - \frac{b^0 \varepsilon_2}{2} - \frac{B^0 \varepsilon_6}{2} - \frac{L^0 \varepsilon_5 n}{2} \right) \sum_{i,j=1}^{n} |u_{x_i x_j}^m|^2 + g_0 |u^m|^q(x) + G_0 |v^m|^s(x) \right\} dx dt \leq C_1 F_1(\tau) + \]

\[ + C_2 (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_6, \varepsilon_7, \varepsilon_9, \varepsilon_{10}) \int_{Q_{0, r}} \{|u^m|^2 + |v^m|^2\} dx dt + \]

\[ \int_{Q_{0, r}} \left( \frac{I^0 \varepsilon_3}{2} + \frac{L^0 n^2}{2\varepsilon_5} + \frac{Z^0 \varepsilon_7}{2} + \frac{E^0 n}{2\varepsilon_8} \right) \sum_{i=1}^{n} |v_{x_i}^m|^2 dx dt, \tag{18} \]

where \( F_1(\tau) = \int_{\Omega_0} \{|u^0_m|^r(x) + |u^0_m|^2 + |v^0_m|^2\} dx + \int_{Q_{0, r}} \{|f|^2 + |F|^2\} dx dt. \)

Next, applying Friedrichs’ inequality (10) to the last term of (18) we obtain

\[ \int_{\Omega} \left\{ \frac{1}{r(x)} |u^m|^r(x) + \frac{1}{2} (|u^m|^2 + |v^m|^2) \right\} dx + \int_{Q_{0, r}} \left\{ \left( a_0 - \frac{z^0 \varepsilon_1}{2} - \frac{E^0 \varepsilon n}{2} - \frac{H^0 \varepsilon}{2} \right) \sum_{i=1}^{n} |u_{x_i}^m|^2 + \right. \]

\[ + \left( A_0 - \frac{b^0 \varepsilon_2}{2} - \frac{B^0 \varepsilon_6}{2} - \frac{L^0 \varepsilon_5 n}{2} - M_\Omega \left( \frac{I^0 \varepsilon_3}{2} + \frac{L^0 n^2}{2\varepsilon_5} + \frac{Z^0 \varepsilon_7}{2} + \frac{E^0 n}{2\varepsilon_8} \right) \right) \sum_{i,j=1}^{n} |u_{x_i x_j}^m|^2 + g_0 |u^m|^q(x) + \]

\[ + G_0 |v^m|^s(x) \right\} dx dt \leq C_3 F_1(\tau) + C_4 (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10}) \int_{Q_{0, r}} \{|u^m|^2 + |v^m|^2\} dx dt. \]

Selecting \( \varepsilon_5 = \eta_2, \varepsilon_8 = \eta_1 \) and \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_5, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10} > 0 \) small enough, we get

\[ \int_{\Omega} \left\{ \frac{1}{r(x)} |u^m|^r(x) + \frac{1}{2} (|u^m|^2 + |v^m|^2) \right\} dx + \int_{Q_{0, r}} \left\{ \left( a_0 - \frac{E^0 n}{2} \eta_1 - \eta_3 \right) \sum_{i=1}^{n} |u_{x_i}^m|^2 + \right. \]

\[ + \left( A_0 - M_\Omega \frac{E^0 n}{2\eta_1} - \frac{L^0 \eta_2 n}{2} - M_\Omega \frac{L^0 n^2}{2\eta_2} - \eta_1 \right) \sum_{i,j=1}^{n} |u_{x_i x_j}^m|^2 + g_0 |u^m|^q(x) + \]

\[ + G_0 |v^m|^s(x) \right\} dx dt \leq C_3 F_1(\tau) + C_5 (\eta_3, \eta_4) \int_{Q_{0, r}} \{|u^m|^2 + |v^m|^2\} dx dt, \]
where $\eta_3, \eta_4 > 0$. Since the (T) is carried out, then from the Proposition we obtain
\[
\int_{\Omega_r} \left\{ |u^m|^r(x) + |u^m|^2 + |v^m|^2 \right\} dx + \int_{Q_0,\tau} \left\{ \sum_{i=1}^{n} |u_{x_i}^m|^2 + \sum_{i,j=1}^{n} |v_{x_i,x_j}^m|^2 \right. \\
+ \left. |v^m|^q(x) + |v^m|^s(x) \right\} dxdt \leq C_3 F_1(\tau) + C_6 \int_{Q_0,\tau} \left\{ |u^m|^2 + |v^m|^2 \right\} dxdt.
\]
Using Gronwall’s lemma we reach the estimate
\[
\int_{\Omega_r} \left\{ |u^m|^r(x) + |u^m|^2 + |v^m|^2 \right\} dx + \int_{Q_0,\tau} \left\{ \sum_{i=1}^{n} |u_{x_i}^m|^2 + \sum_{i,j=1}^{n} |v_{x_i,x_j}^m|^2 \right. \\
+ \left. |u^m|^q(x) + |v^m|^s(x) \right\} dxdt \leq C_7, \tag{19}
\]
where $C_7 > 0$ is a constant independent on $m$, $\tau$.

From Friedrichs’ inequality (10) and (19) we obtain
\[
\int_{\Omega_r} \left\{ |u^m|^r(x) + |u^m|^2 + |v^m|^2 \right\} dx + \int_{Q_0,\tau} \left\{ \sum_{i=1}^{n} |u_{x_i}^m|^2 + \sum_{i,j=1}^{n} |v_{x_i,x_j}^m|^2 + \sum_{i,j=1}^{n} |v_{x_i,x_j}^m|^2 \right. \\
+ \left. |u^m|^q(x) + |v^m|^s(x) \right\} dxdt \leq C_8. \tag{20}
\]
Hence, the functions $\varphi_1^m, \ldots, \varphi_n^m, \psi_1^m, \ldots, \psi_n^m$, and therefore $u^m, v^m$ can be proceed to $[0, T]$ for all $m \in \mathbb{N}$.

Moreover, we have
\[
\int_{\Omega_r} |u^m|^r(x) - 2u^m|^{r}(x) dx = \int_{\Omega_r} |u^m|^r(x) dx \leq C_8, \tag{21}
\]
\[
\int_{Q_0,\tau} |u^m|^q(x) - 2u^m|^{q}(x) dxdt = \int_{Q_0,\tau} |u^m|^q(x) dxdt \leq C_8, \tag{22}
\]
\[
\int_{Q_0,\tau} |v^m|^s(x) - 2v^m|^{s}(x) dxdt = \int_{Q_0,\tau} |v^m|^s(x) dxdt \leq C_8 \tag{23}
\]

3. From the previous estimates we obtain the existence of subsequences $\{u^{m_j}\}_{j \in \mathbb{N}} \subset \{u^m\}_{m \in \mathbb{N}}$, $\{v^{m_j}\}_{j \in \mathbb{N}} \subset \{v^m\}_{m \in \mathbb{N}}$ such that
\[
u^{m_j} \rightharpoonup u \quad \text{weakly star in } L^{\infty}(0, T; L^r(\Omega) \cap L^2(\Omega)),
\]
\[
u^{m_j} \rightharpoonup u \quad \text{weakly in } U_1(Q_0,T) \cap L^r(\Omega) \cap L^2(\Omega), \quad |u^{m_j}|^q(x) - 2u^{m_j} \rightharpoonup \chi_0 \quad \text{weakly in } L^q(\Omega),
\]
\[
\mathcal{R}u^{m_j} \rightharpoonup \chi_0 \quad \text{weakly star in } L^\infty(0, T; L^r(\Omega)) \text{ and weakly in } L^r(\Omega),
\]
\[
u^{m_j} \rightharpoonup v \quad \text{weakly in } L^{\infty}(0, T; L^2(\Omega)), \quad |v^{m_j}|^s(x) - 2v^{m_j} \rightharpoonup \chi_1 \quad \text{weakly in } L^s(\Omega).
\]

4. Now we can tend $\mu \to \infty$ in (13) after applying the monotonicity method. Nonetheless, it is not enough to do the same in (12). It is required some additional estimates. Let us
multiply it by \( \frac{d}{dt} \varphi_j^m(t) \), summarize it on \( \mu \) from 1 to \( m \) and integrate the result over \( t \) from 0 to \( \tau \in (0, T] \). We obtain the identity

\[
\tilde{J}_1 + \tilde{J}_2 + \ldots + \tilde{J}_8 = \tilde{J}_9, \tag{24}
\]

where

\[
\tilde{J}_1 = \int_{Q_0, \tau} (\mathcal{F}u^m) \xi u^m dx dt, \quad \tilde{J}_2 = \int_{Q_0, \tau} \sum_{i,j=1}^n a_{ij} u^m_{|x_i} u^m_{x_j} dx dt, \quad \tilde{J}_3 = \int_{Q_0, \tau} \sum_{i=1}^n z_i u^m_{x_i} u^m dx dt,
\]

\[
\tilde{J}_4 = \int_{Q_0, \tau} du^m u^m dx dt, \quad \tilde{J}_5 = \int_{Q_0, \tau} g|u^m|^{q(x)}-2 u^m u^m dx dt, \quad \tilde{J}_6 = \int_{Q_0, \tau} \sum_{i,j=1}^n b_{ij} u^m_{x_i x_j} u^m dx dt,
\]

\[
\tilde{J}_7 = \int_{Q_0, \tau} \sum_{i=1}^n l_i u^m_{x_i} u^m dx dt, \quad \tilde{J}_8 = \int_{Q_0, \tau} h u^m u^m dx dt, \quad \tilde{J}_9 = \int_{Q_0, \tau} f u^m dx dt.
\]

Let us estimate each term form (24). We have

\[
\tilde{J}_1 = \int_{Q_0, \tau} (|u^m|^{q(x)}-2 + 1)|u^m| dx dt = \int_{Q_0, \tau} |u^m|^{q(x)}-2 |u^m| dx dt + \int_{Q_0, \tau} |u^m|^{2} dx dt;
\]

\[
\tilde{J}_2 = \frac{1}{2} \int_{Q_0, \tau} \sum_{i,j=1}^n a_{ij} u^m_{|x_i} u^m_{x_j} dx = \frac{1}{2} \int_{Q_0, \tau} \sum_{i,j=1}^n a_{ij} u^m_{|x_i} u^m_{x_j} dx - \frac{1}{2} \int_{Q_0, \tau} \sum_{i,j=1}^n (a_{ij})_{u^m_{|x_i} u^m_{x_j}} dx \geq \frac{a_0}{2} \int_{Q_0, \tau} \sum_{i=1}^n |u^m_{x_i}| dx - \frac{a_0}{2} \int_{Q_0, \tau} \sum_{i=1}^n |u^m_{x_i}| dx - C_7 \int_{Q_0, \tau} \sum_{i=1}^n |u^m_{x_i}| dx;
\]

\[
|\tilde{J}_3| \leq \frac{z^0}{2 \varepsilon_1} \int_{Q_0, \tau} \sum_{i=1}^n |u^m_{x_i}| dx + \frac{z^0 n \varepsilon_1}{2} \int_{Q_0, \tau} |u^m_{x_i}| dx;
\]

\[
|\tilde{J}_4| \leq \frac{a_0^0}{2 \varepsilon_2} \int_{Q_0, \tau} |u^m|^{2} dx dt + \frac{a_0^0 \varepsilon_2}{2} \int_{Q_0, \tau} |u^m|^{2} dx dt;
\]

\[
|\tilde{J}_5| = \int_{Q_0, \tau} g \frac{|u^m| |q(x)| dx}{q(x)} - \int_{Q_0, \tau} \frac{g}{q(x)} |u^m|^{q(x)} dx - \int_{Q_0, \tau} \frac{g_t}{q(x)} |u^m|^{q(x)} dx \geq \frac{g_0}{q_0} \int_{Q_0, \tau} |u^m|^{q(x)} dx - \frac{g_0}{q_0} \int_{Q_0, \tau} |u^m|^{q(x)} dx - C_8 \int_{Q_0, \tau} |u^m|^{q(x)} dx;
\]

\[
|\tilde{J}_6| \leq \frac{b_0^0}{2 \varepsilon_3} \int_{Q_0, \tau} \sum_{i,j=1}^n |v^m_{x_i x_j}| dx dt + \frac{b_0^0 n \varepsilon_3}{2} \int_{Q_0, \tau} |u^m_{x_i}| dx dt;
\]

\[
|\tilde{J}_7| \leq \frac{b_0^0}{2 \varepsilon_4} \int_{Q_0, \tau} \sum_{i,j=1}^n |v^m_{x_i x_j}| dx dt + \frac{b_0^0 n \varepsilon_4}{2} \int_{Q_0, \tau} |u^m_{x_i}| dx dt;
\]

\[
|\tilde{J}_8| \leq \frac{h_0^0}{2 \varepsilon_5} \int_{Q_0, \tau} |v^m|^{2} dx dt + \frac{h_0^0 \varepsilon_5}{2} \int_{Q_0, \tau} |u^m|^{2} dx dt; \quad |\tilde{J}_9| = \frac{1}{2 \varepsilon_6} \int_{Q_0, \tau} |f|^2 dx dt + \frac{\varepsilon_6}{2} \int_{Q_0, \tau} |u^m|^2 dx dt.
\]

From (24) and reached priory estimates we get

\[
\int_{Q_0, \tau} |u^m|^{q(x)}-2 |u^m| dx dt + \left( 1 - \frac{b_0^0 n \varepsilon_3}{2} - \frac{z^0 n \varepsilon_1}{2} - \frac{a_0^0 \varepsilon_2}{2} - \frac{b_0^0 n \varepsilon_4}{2} - \frac{h_0^0 \varepsilon_5}{2} - \frac{\varepsilon_6}{2} \right) \times
\]

\[
\times \int_{Q_0, \tau} |u^m|^2 dx dt + \frac{g_0}{q_0} \int_{Q_0, \tau} |u^m|^{q(x)} dx + \frac{a_0}{2} \int_{Q_0, \tau} \sum_{i=1}^n |u^m_{x_i}|^2 dx \leq
\]
Thus, \(\chi|\cdot|C\) and maybe passing to a new subsequence we reach
\[
\int_{\Omega_t} \left( \sum_{i=1}^{n} |u_{x_i}^{m_j}|^2 + \sum_{i=1}^{n} |v_{x_i}^{m_j}|^2 + \sum_{i=1}^{n} |u_i^{m_j}|^2 + |u^{m_j}|^2 + |u^{m_j}|^2 \right) dx. \tag{25}
\]
But now let \(\varepsilon_i > 0\) \((i \in \{1, 2, \ldots, 5\})\) are small enough and apply (20) to (25) we get
\[
\int_{\Omega_t} \left( \sum_{i=1}^{n} |u_{x_i}^{m_j}|^2 + |u_i^{m_j}|^2 \right) dx + \int_{\Omega_t} \left[ \sum_{i=1}^{n} |u_{x_i}^{m_j}|^2 + |u^{m_j}|^2 \right] dx \leq C_{10}. \tag{26}
\]
where \(C_{10} > 0\) is a constant independent of \(m\) and \(\tau\).

Hence, passing to the new subsequence \(\{u^{m_j}\}_{j \in \mathbb{N}}\) ((let us leave the same notation) we have
\[
u^{m_j} \rightarrow u \quad \text{weakly star in } L^\infty(0,T;H_0^1(\Omega) \cap L^{q(\cdot)}(\Omega)),
\]

\[
(|u^{m_j}|^{r(\cdot)-2}u^{m_j})_t \rightarrow \tilde{\chi}_1 \quad \text{weakly in } L^2(\Omega_t), \quad u^{m_j}_t \rightarrow u_t \quad \text{weakly in } L^2(\Omega_t). \tag{27}
\]
These convergences, particular, mean that
\[
u^{m_j} \rightarrow u \quad \text{weakly in } H^1(\Omega_t).
\]
Thus, using the Embedding Theorem (Rellich-Kondrachov) and Lemma 1.18 ([13, c. 39]), and maybe passing to a new subsequence we reach
\[
\nu^{m_j} \rightarrow u \quad \text{in } L^2(\Omega_t) \text{ and a. e. in } Q_{0,T}. \tag{28}
\]
Thus, \(\chi_0 = |u|^{q(\cdot)-2}u, \tilde{\chi}_0 = \mathcal{R}u, \tilde{\chi}_1 = (|u|^{r(\cdot)-2}u)_t\).

From (26) and the inequality \(|y(\tau)|^2 \leq C_{10}(|y(0)|^2 + \int_0^\tau |y'(t)|^2 dt\), where \(C_{10}\) is independent of \(y\), we have
\[
\int_{\Omega_T} |u^{m_j}|^{r(\cdot)} \, dx = \int_{\Omega_T} \left| \left| u^{m_j} \right|^{r(\cdot)-2} u^{m_j} \right|^2 \, dx \leq C_{10} \left( \int_{\Omega_0} \left| u^{m_j}_0 \right|^{r(\cdot)-2} u^{m_j}_0 \, dx + \int_{\Omega_T} \left( \left| \left| u^{m_j} \right|^{r(\cdot)-2} u^{m_j} \right|^2 \right) \, dx \right) \leq C_{10} \left( \int_{\Omega_0} u^{m_j} \, dx + C_{10} \right) \leq C_{11}.
\]
Then
\[
\|u^m; L^\infty(0,T;L^{r(\cdot)}(\Omega))\| \leq C_{11}, \tag{29}
\]
where \(C_{11}\) is a constant independent of \(m\).

Now let us derive the estimates of the expressions below
\[
(\mathcal{R}u^m)_t = |u^m|^{r(\cdot)-2}u^m = |u^m|^{r(\cdot)-2}u^m_t + u^m_t \tag{30}
\]
From the Young inequality with the exponent \(\frac{2}{r(\cdot)} > 1\), we get
\[
| (\mathcal{R}u^m)_t |^{r'(\cdot)} = \left| u^m |^{r(\cdot)-2} u^m_t \right|^{r'(\cdot)} \leq C_{12} \left( \left| u^m |^{r(\cdot)-2} u^m_t \right|^{2} + \left| u^m |^{r(\cdot)-2} u^m_t \right|^{2} \right).
\]
Since
\[(r - 2)\frac{r'}{2} (\frac{2}{r'}) = (r - 2) \frac{r'}{2} \frac{2}{r'} - 1 = (r - 2) \frac{r'}{2} - r\frac{r'}{2} - r = (r - 2) \frac{2 - r}{r - 1},\]
then \(|(u^R)|^{r(x)} \leq C_{12}(|u^m|^{r(x)} + |u^m|^{r(x)} - 2|u^m|^2)|. Thus,
\[
\|(R^m)_t; L^{r(x)}(Q_{0,T})\| \leq C_{13},
\]
where \(C_{13}\) is independent of \(m\).

5. From the Monotonicity Method (\([2]\)) we derive \(|v|^s(x) - 2v = \chi_1\).

6. For all \(x_{1} \in U_{1}(Q_{0,T}) \cap U_{2}(Q_{0,T}) \cap L^{r(x)}(Q_{0,T})\) we consider a sequence of functions \(\{z_k\}_{k \in \mathbb{N}}\) such that \(z_k \in L_m = \{\sum_{i=1}^{m} d_k(t) \omega_k(x): d_1, \ldots, d_m \in C^1([0, T])\}, z_k \to \omega \) in \(U\) as \(k \to \infty\). Then (12)–(13) imply
\[
\left[(R^m)_t + u^m, z_k\right] + \sum_{i,j=1}^{n} [a_{ij} u^m_{x_{i}}, (z_k)_{x_{j}}] + \sum_{i=1}^{n} [z_i u^m_{x_{i}}, z_k] + [d u^m, z_k] +
\]
\[
+ |g| u^m|^{q(x)} - 2 u^m, z_k| + \sum_{i,j=1}^{n} [b_{ij} v^m_{x_{i} x_{j}}, z_k] + \sum_{i=1}^{n} [l v^m_{x_{i} x_{j}}, z_k] + [h v^m, z_k] = [f, z_k],
\]
\[
\left[v^m, z_k\right] + \sum_{i,j,k,l=1}^{n} [A_{ijkl} v^m_{x_{i} x_{j} x_{k} x_{l}}, (z_k)_{x_{i} x_{j}}] + \sum_{i,j,k=1}^{n} [L_{ijkl} v^m_{x_{i} x_{j} x_{k} x_{l}}, (z_k)_{x_{i} x_{j}}] + \sum_{i,j=1}^{n} [B_{ijkl} v^m_{x_{i} x_{j} x_{k} x_{l}}, z_k] +
\]
\[
+ \sum_{i=1}^{n} [Z_{ijkl} v^m_{x_{i} x_{j} x_{k} x_{l}}, z_k] + [D u^m, z_k] + [G v^m|^{s(x)} - 2 v^m, z_k] - \sum_{i,j=1}^{n} [E_{ijkl} u^m_{x_{i} x_{j}}, (z_k)_{x_{i} x_{j}}] +
\]
\[
+ \sum_{i=1}^{n} [H u^m_{x_{i} x_{j}}, z_k] + [I u^m, z_k] = [F, z_k].
\]
Tending \(m_j \to \infty\), we get
\[
\left[(R)_{t} + u_t, z_k\right] + \sum_{i,j=1}^{n} [a_{ij} u_t, (z_k)_{x_{j}}] + \sum_{i=1}^{n} [z_i u_t, z_k] + [d u_t, z_k] + [g|u|^{q(x)} - 2 u, z_k] +
\]
\[
+ \sum_{i,j=1}^{n} [b_{ij} v_{x_{i} x_{j}}, z_k] + \sum_{i=1}^{n} [l v_{x_{i} x_{j}}, z_k] + [h v, z_k] = [f, z_k],
\]
\[
\left[v_t, z_k\right] + \sum_{i,j,k,l=1}^{n} [A_{ijkl} v_{x_{i} x_{j} x_{k} x_{l}}, (z_k)_{x_{i} x_{j}}] + \sum_{i,j,k=1}^{n} [L_{ijkl} v_{x_{i} x_{j} x_{k} x_{l}}, (z_k)_{x_{i} x_{j}}] + \sum_{i,j=1}^{n} [B_{ijkl} v_{x_{i} x_{j} x_{k} x_{l}}, z_k] +
\]
\[
+ \sum_{i=1}^{n} [Z_{ijkl} v_{x_{i} x_{j} x_{k} x_{l}}, z_k] + [D u, z_k] + [G|v|^{s(x)} - 2 v, z_k] - \sum_{i,j=1}^{n} [E_{ijkl} u_{x_{i} x_{j}}, (z_k)_{x_{i} x_{j}}] +
\]
\[
+ \sum_{i=1}^{n} [H u_{x_{i} x_{j}}, z_k] + [I u, z_k] = [F, z_k].
\]
Hence, letting \(k \to \infty\) we obtain (8).
Let us mention that the uniqueness of a solution of problem (4)–(7) was not an object of our investigation. It worth mentioning that the uniqueness of a solution of a mixed problem for (5) is separately discovered in [2]. The uniqueness of a solution for the model doubly nonlinear equation of a type (4) is proved in [14].

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