ASYMPTOTIC APPROXIMATION OF A SOLUTION OF A QUASILINEAR PARABOLIC BOUNDARY-VALUE PROBLEM IN A TWO-LEVEL THICK JUNCTION OF TYPE 3:2:2


We consider a quasilinear parabolic boundary-value problem in a two-level thick junction $\Omega_\varepsilon$ of type $3 : 2 : 2$, which is the union of a cylinder $\Omega_0$ and a large number of $\varepsilon$-periodically situated thin discs with variable thickness. Different Robin boundary conditions with perturbed parameters are given on the surfaces of the thin discs. The leading terms of the asymptotic expansion are constructed and the corresponding estimate in Sobolev space is obtained.

1. Introduction. A thick junction of type $m : k : d$ is a union of some domain, which is called the junction’s body, and a large number of $\varepsilon$-periodically alternating thin domains, which are attached to some manifold (the joint zone) on the boundary of the junction’s body. The small parameter $\varepsilon$ characterizes the distance between neighboring thin domains and their thickness. The type $m : k : d$ of a thick junction refers, respectively, to the limiting dimensions (as $\varepsilon \to 0$) of the junction’s body, the joint zone and each of the attached thin domains. The subject of the investigation of boundary-value problems in thick junctions is the asymptotic behavior of solutions of such problems as $\varepsilon \to 0$, i.e. as the number of the attached thin domains infinitely increases as well as their thickness tends to zero.

The first researches in this direction were carried out in [10, 11, 15], where convergence theorems for the Green function of the Neumann problem for the Helmholtz equation in the junction’s body were proved. In these papers either the assumption about the convergence of certain components of the boundary-value problem was made, or explicit representations of

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1The author was partly supported by joint European grant EUMLS-FP7-People-2011-IRSES Project №295164

2010 Mathematics Subject Classification: 35B27, 35B40, 35C20, 35K60, 74K30.

Keywords: homogenization, quasilinear problem, parabolic problem, asymptotic approximation, thick junction.

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certain quantities were used, which was possible under certain configurations of the junction’s body (the half-space). In [17]–[21], [27] thick junctions were classified, asymptotic methods for the investigation of main boundary-value problems of mathematical physics in thick junctions of different types were developed, convergence theorems were proved, the first terms of asymptotic expansions were constructed, and the corresponding estimates were proved. It was shown that qualitative properties of solutions essentially depend on the junction’s type and the conditions given on the boundaries of the attached thin domains (see also [1, 3, 23]).

As an extension of the investigation, in papers [7, 8, 16] thick junctions of more complicated geometric structure were considered, namely multi-level thick junctions. A multi-level thick junction is a thick junction in which thin domains are divided into finitely many levels depending on their geometric structure and boundary conditions imposed on their surfaces. Besides, thin domains from each level \( \varepsilon \)-periodically alternate along the joint zone. In these papers linear boundary-value problems in thick junctions of types \( 2 : 1 : 1 \) and \( 3 : 2 : 1 \) were investigated in [5, 8]. Quasilinear parabolic problems in a two-level thick junction of type

\[
\varepsilon \left( x \right) = \begin{cases} 2 & 0 < x < l, \\ b_1(s) + h_i(s)/2 & l < x < 3l, \\ b_2(s) + h_i(s)/2 & 3l < x < 4l, \\ 0 & 4l < x < r. \end{cases}
\]

The leading terms of the asymptotic expansion for a solution of this problem are constructed and the asymptotic estimate in Sobolev space is proved. It should be noted that linear parabolic boundary-value problems in thick junctions of various types were investigated in [5, 8]. Quasilinear parabolic problems in a two-level thick junction of type \( 3 : 2 : 2 \) were considered in [24, 25], where only convergence theorems were proved.

2. Statement of the problem. Let \( 0 < d_0 < d_2 \leq d_1 \) and \( 0 < b_2 < b_1 < 1 \), and let

\[
h_i \colon [d_0, d_i] \to (0, 1), \quad i \in \{1, 2\}
\]

be piecewise smooth functions. Suppose that functions \( h_i \) satisfy the following conditions

\[
0 < b_i - \frac{h_i(s)}{2}, \quad b_i + \frac{h_i(s)}{2} < 1 \quad \forall s \in [d_0, d_i], \quad i \in \{1, 2\}, \quad b_2 + \frac{h_2(s)}{2} < b_1 - \frac{h_1(s)}{2} \quad \forall s \in [d_0, d_2].
\]

These inequalities imply that for all \( s \in [d_0, d_i] \) the intervals

\[
I_i(s) := \left( b_i - \frac{h_i(s)}{2}, b_i + \frac{h_i(s)}{2} \right), \quad i \in \{1, 2\},
\]

belong to the interval \( (0, 1) \), having no common points and do not adjoin.

We additionally assume that the functions \( h_1, h_2 \) are constant in some neighborhood of \( d_0 \), i.e. there exists \( \delta > 0 \) such that

\[
h_i(s) = h_i(d_0) \quad \text{for all} \quad s \in [d_0, d_0 + \delta], \quad i \in \{1, 2\}.
\]

Consider a model thick junction \( \Omega_\varepsilon \) of type \( 3 : 2 : 2 \) (see Fig. 1) that consists of the cylinder

\[
\Omega_0 = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_2 < l, \quad r := \sqrt{x_1^2 + x_3^2} < d_0 \}
\]

and \( 2N \) thin annular discs

\[
G^{(1)}_\varepsilon = \{ x \in \mathbb{R}^3 : |x_2 - \varepsilon(j + b_1)| < \varepsilon h_1(r)/2, \quad d_0 \leq r < d_1 \},
\]

\[
G^{(2)}_\varepsilon = \{ x \in \mathbb{R}^3 : |x_2 - \varepsilon(j + b_2)| < \varepsilon h_2(r)/2, \quad d_0 \leq r < d_2 \},
\]

where \( \{ (j + b_1) : j = 0, 1, \ldots, N \varepsilon_0 \} \subset (d_0, d_1) \) and \( \{ (j + b_2) : j = 0, 1, \ldots, N \varepsilon_0 \} \subset (d_0, d_2) \).
we assume that brackets denote the jump of enclosed quantities. For the right-hand sides of problem (1)
\[\frac{\partial}{\partial \nu} \nu = \partial \nu,\]
where \(j \in \{0, 1, \ldots, N - 1\}, \ \varepsilon = l/N, \ i.e. \ \Omega_\varepsilon = \Omega_0 \cup G_\varepsilon, \ G_\varepsilon = G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}, \ G_\varepsilon^{(1)} = \bigcup_{j=0}^{N-1} G_\varepsilon^{(1)}(j), \ G_\varepsilon^{(2)} = \bigcup_{j=0}^{N-1} G_\varepsilon^{(2)}(j).\) Here \(N\) is a large integer. Therefore, \(\varepsilon\) is a small parameter, which characterizes the distance between neighboring thin discs and their thickness.

Denote by \(S_\varepsilon^{(1)}\) and \(S_\varepsilon^{(2)}\) the union of the lateral surfaces of the thin discs of the first and the second levels, respectively, and by \(S\) the union of the bases of the cylinder \(\Omega_0\), i.e.

\[S_\varepsilon^{(i)} := \{x \in \partial G_\varepsilon^{(i)} : |x_2 - \varepsilon(j + b_i)| = \varepsilon h_i(r)/2, \ j \in \{0, 1, \ldots, N - 1\}, \ r \in (d_0, d_i), \ i \in \{1, 2\}, \ S^- = \{x \in \partial \Omega_0 : x_2 = 0\}, \ S^+ = \{x \in \partial \Omega_0 : x_2 = l\}, \ S = S^+ \cup S^-\]

We introduce the following notation

\[\overline{\Omega} = \overline{\Omega}_0 \cup \overline{D}_i, \ D_i = \{x \in \mathbb{R}^3 : 0 < x_2 < l, \ d_0 < r < d_i\}, \ i \in \{1, 2\}, \ Q_0^{(i)} = \{x \in \partial \Omega_i : r = d_i\}, \ i \in \{0, 1, 2\}, \ Q_\varepsilon^{(i)} = \{x \in \partial G_\varepsilon^{(i)} : r = d_i\}, \ i \in \{1, 2\}, \ \Theta_\varepsilon = \{x \in \partial \Omega_0 : x_2 = 0\}, \ i \in \{1, 2\}, \ \Theta_\varepsilon = \Theta_\varepsilon^{(1)} \cup \Theta_\varepsilon^{(2)}, \ Q_\varepsilon^{(0)} = Q_0^{(0)} \setminus \Theta_\varepsilon\]

In the thick junction \(\Omega_\varepsilon\) we consider the quasilinear parabolic boundary-value problem

\[
\begin{cases}
\partial_t u_\varepsilon(x, t) - \Delta u_\varepsilon(x, t) + \partial_\nu u_\varepsilon(x, t) + \partial_0 u_\varepsilon(x, t) = f_\varepsilon(x, t), & (x, t) \in \Omega_\varepsilon \times (0, T), \\
\partial_t u_\varepsilon(x, t) + \varepsilon \partial_1 u_\varepsilon(x, t) = \varepsilon g_\varepsilon(x, t), & (x, t) \in S_\varepsilon^{(1)} \times (0, T), \\
\partial_t u_\varepsilon(x, t) + \partial_1 u_\varepsilon(x, t) = 0, & (x, t) \in Q_\varepsilon^{(1)} \times (0, T), \\
\partial_t u_\varepsilon(x, t) + \varepsilon \partial_2 u_\varepsilon(x, t) = \varepsilon g_\varepsilon(x, t), & (x, t) \in S_\varepsilon^{(2)} \times (0, T), \\
\partial_t u_\varepsilon(x, t) = 0, & (x, t) \in Q_\varepsilon^{(0)} \times (0, T), \\
\partial_{x^p} u_\varepsilon(x, t)|_{S^-} = \partial_{x^p} u_\varepsilon(x, t)|_{S^+}, & p = 0, 1, \ t \in (0, T), \\
[u_\varepsilon]_{r=d_0} = [\partial_r u_\varepsilon]_{r=d_0} = 0, & (x, t) \in \Theta_\varepsilon \times (0, T), \\
u(x, 0) = 0, & x \in \Omega_\varepsilon.
\end{cases}
\]

Here \(\partial_\nu = \partial / \partial \nu\) is the outward normal derivative; \(\alpha, \ \beta \geq 1\) are parameters; the square brackets denote the jump of enclosed quantities. For the right-hand sides of problem (1) we assume that \(f_\varepsilon \in L^2(\Omega_\varepsilon \times (0, T)), \ g_\varepsilon \in L^2(D_1 \times (0, T))\), there exists a weak derivative \(\partial_{x^p} g_\varepsilon \in L^2(D_1 \times (0, T))\), and

\[\exists C_0 > 0 \ \exists \varepsilon_0 > 0 \ \forall \varepsilon \in (0, \varepsilon_0) : \|g_\varepsilon\|_{L^2(D_1 \times (0, T))} + \|\partial_{x^p} g_\varepsilon\|_{L^2(D_1 \times (0, T))} < C_0.\]
The functions \( \vartheta_i \) are Lipschitz-continuous (that is to say \( \vartheta_i \in W^{1,\infty}_\text{loc}(\mathbb{R}) \)) and

\[
\exists c_1, c_2 > 0: c_1 \leq \vartheta'_i(s) \leq c_2 \quad \text{for a.e. } s \in \mathbb{R}, \ i \in \{0, 1, 2\}. \tag{2}
\]

Consider the spaces \( H_{\varepsilon} = \{ \varphi \in H^1(\Omega_{\varepsilon}) : \varphi|_{S^-} = \varphi|_{S^+} \} \) and \( W_{\varepsilon} = \{ \varphi \in L^2(0, T; H_{\varepsilon}) : \partial_t \varphi = \varphi' \in L^2(0, T; H'_{\varepsilon}) \} \).

It is known (see, for instance, [9, §1 ch. IV]) that \( W_{\varepsilon} \subset C([0, T]; L^2(\Omega_{\varepsilon})) \).

A function \( u_{\varepsilon} \in L^2(0, T; H_{\varepsilon}) \) is a weak solution of problem (1) if for every function \( \varphi \in W_{\varepsilon} \) the following integral identity holds (see, e.g., [9, ch. IV])

\[
\int_{\Omega_{\varepsilon}} u_{\varepsilon}(x, T) \varphi(x, T) dx - \int_0^T \int_{\Omega_{\varepsilon}} u_{\varepsilon} \partial_t \varphi dx dt + \int_0^T \int_{\Omega_{\varepsilon}} (\nabla u_{\varepsilon} \cdot \nabla x \varphi + \vartheta_0(u_{\varepsilon}) \varphi) dx dt
+ \varepsilon \int_0^T \int_{S_{\varepsilon}^{(1)}} \partial_1(u_{\varepsilon}) \varphi \sigma_x dt + \int_0^T \int_{Q_{\varepsilon}^{(1)}} \vartheta_1(u_{\varepsilon}) \varphi \sigma_x dt + \varepsilon \alpha \int_0^T \int_{Y_{\varepsilon}^{(2)}} \partial_2(u_{\varepsilon}) \varphi \sigma_x dt
= \int_0^T \int_{\Omega_{\varepsilon}} f_{\varepsilon} \varphi dx dt + \varepsilon \beta \int_0^T \int_{S_{\varepsilon}^{(1)} \cup Y_{\varepsilon}^{(2)}} g_{\varepsilon} \varphi \sigma_x dt. \tag{3}
\]

Similarly as in [26] we can show that for any fixed \( \varepsilon > 0 \) there exists a unique weak solution of problem (1).

The aim is to study the asymptotic behavior of the solution of problem (1) as \( \varepsilon \to 0 \), i.e. as the number of the attached thin discs infinitely increases and their thickness tends to zero.

3. Formal Asymptotic Expansions for the Solution. In this section only, for formal calculations we assume that the functions \( f_{\varepsilon}, g_{\varepsilon} \) do not depend on \( \varepsilon \), i.e. \( f_{\varepsilon} = f_0 \) in \( \Omega_1 \times (0, T) \) and \( g_{\varepsilon} = g_0 \) on \( D_1 \times (0, T) \), and they are smooth in \( \overline{\Omega}_1 \times [0, T] \) and \( \overline{D}_1 \times [0, T] \), respectively.

3.1. Outer Expansions. We seek the leading terms of the asymptotic expansion for solution \( u_{\varepsilon} \), restricted to \( \Omega_0 \), in the form

\[
u_{\varepsilon}(x, t) \approx u_0^+(x, t) + \sum_{k \geq 1} \varepsilon^k u_k^+(x, t), \quad (x, t) \in \Omega_0 \times (0, T), \tag{4}
\]

and, restricted to the thin discs \( G_{\varepsilon}^{(i)}(j), \ j \in \{0, 1, \ldots, N - 1\} \), in the form

\[
u_{\varepsilon}(x, t) \approx u_0^{(i)}(x, t) + \sum_{k \geq 1} \varepsilon^k u_k^{(i)}(x, \xi_2 - j, t), \quad (x, t) \in G_{\varepsilon}^{(i)}(j) \times (0, T), \quad i \in \{1, 2\}, \tag{5}
\]

where \( \xi_2 = x_2/\varepsilon \).

Expansions (4) and (5) are usually called outer expansions.

With the help of Taylor’s formula we get

\[
u_0(u_{\varepsilon}(x, t)) = \nu_0(u_0^+(x, t)) + \mathcal{O}(\varepsilon), \quad \varepsilon \to 0, \quad (x, t) \in \Omega_0 \times (0, T). \tag{6}
\]

Plugging the series (4) into the first equation of problem (1), the boundary conditions on \( S \), and the initial condition, using (6) and collecting coefficients of the same powers of \( \varepsilon \), we get the following relations for function \( u_0^+ \)

\[
\begin{cases}
\partial_t u_0^+(x, t) - \Delta x u_0^+(x, t) + \nu_0(u_0^+(x, t)) = f_0(x, t), \quad (x, t) \in \Omega_0 \times (0, T), \\
\partial_{x_2} u_0^+(x, t)|_{S^-} = \partial_{x_2} u_0^+(x, t)|_{S^+}, \quad p \in \{0, 1\}, \quad t \in (0, T), \\
\partial_{x_2} u_0^+(x, t)|_{S^+} = \partial_{x_2} u_0^+(x, t)|_{S^+}, \quad p \in \{0, 1\}, \quad t \in (0, T), \\
u_0^+(x, 0) = 0, \quad x \in \Omega_0.
\end{cases}
\]
Now let us find the limit relations in domains $D_\varepsilon$, $i \in \{1, 2\}$, which are filled up by the thin discs from $i$-th level as $\varepsilon$ tends to zero. Assuming for a moment that functions $u_k^{i,-}$ are smooth, we write their Taylor series with respect to $x_2$ at the point $\varepsilon(j+b_i)$ and pass to the “rapid” variable $\xi_2 = x_2/\varepsilon$. Then (5) takes the form

$$u_\varepsilon(x, t) \approx u_0^{i,-}(x_1, \varepsilon(j+b_i), x_3, t) + \sum_{k \geq 1} \varepsilon^k V_{\varepsilon,k}^{i,j}(\tilde{x}, \xi_2, t), \quad (x, t) \in G_{\varepsilon}^{(i)}(j) \times (0, T), \quad (7)$$

where $\tilde{x} := (x_1, x_3)$, and

$$V_{\varepsilon,k}^{i,j}(\tilde{x}, \xi_2, t) = \sum_{m=0}^{k-1} \frac{(\xi_2 - j - b_i)^m}{m!} \frac{\partial^m u_k^{i,-}}{\partial x_2^m}(x_1, \varepsilon(j+b_i), x_3, \xi_2 - j, t) + \frac{(\xi_2 - j - b_i)^k}{k!} \frac{\partial^k u_0^{i,-}}{\partial x_2^k}(x_1, \varepsilon(j+b_i), x_3, t). \quad (8)$$

Further we will indicate arguments of functions only if their absence may cause confusion.

The outward unit normal to the lateral surfaces of the thin discs except a set of zero measure is as follows

$$\nu_\varepsilon(x) = \frac{1}{\sqrt{1 + 4 - 1\varepsilon^2|h'_{i}(r)|^2}} \left( -\varepsilon h'_{i}(r)x_1, \pm 1, -\varepsilon h'_{i}(r)x_3 \right), \quad x \in S_{\varepsilon}^{(i)}, \ i \in \{1, 2\}, \quad (9)$$

where “+” and “−” refer, respectively, to the left and the right parts of the lateral surface of each thin disc. Obviously, $1 + \varepsilon^2 4 - 1|h'_{i}(r)|^2)^{-\frac{1}{2}} = 1 + O(\varepsilon^2), \varepsilon \to 0$.

Again by Taylor’s formula we obtain

$$\vartheta_0(u_\varepsilon(x, t)) = \vartheta_0(u_0^{i,-}(x, t)|_{x_2=\varepsilon(j+b_i)}) + O(\varepsilon), \quad \varepsilon \to 0, \quad (x, t) \in G_{\varepsilon}^{(i)} \times (0, T). \quad (10)$$

Let us put (7) into (1) instead of $u_\varepsilon$. Taking into account (9), (10) and that the Laplace operator in the variables $(\tilde{x}, \xi_2)$ has the form $\Delta_{\tilde{x}} = \Delta_{\tilde{x}} + \varepsilon^{-2} \frac{\partial^2}{\partial \xi_2^2}$ and collecting coefficients of the same powers of $\varepsilon$, we arrive at one-dimensional boundary-value problems with respect to $\xi_2$ for functions $V_{\varepsilon,k}^{i,j}$.

Problems for $V_{\varepsilon,1}^{i,j}$ read

$$\left\{ \begin{array}{l}
\frac{\partial^2 \xi_2}{\partial \xi_2^2} V_{\varepsilon,1}^{i,j} = 0, \quad \xi_2 \in I_h_{i}(r)(j):=\left(-\frac{h_{i}(r)}{2} + j + b_i, \frac{h_{i}(r)}{2} + j + b_i\right), \\
\frac{\partial \xi_2}{\partial \xi_2} V_{\varepsilon,1}^{i,j} = 0, \quad \xi_2 = \pm \frac{h_{i}(r)}{2} + j + b_i,
\end{array} \right. \quad i \in \{1, 2\}, \quad (11)$$

where $\partial_{\xi_2} = \frac{\partial}{\partial \xi_2}$, $\partial_{\xi_2}^2 = \frac{\partial^2}{\partial \xi_2^2}$. Here the variables $\tilde{x}, t$ are regarded as parameters.

It follows from (11) that $V_{\varepsilon,1}^{i,j}$ do not depend on $\xi_2$. Therefore, $V_{\varepsilon,1}^{i,j}$ are equal to some functions $\varphi_{1}^{(i)}(x_1, \varepsilon(j+b_i), x_3, t), \ (x, t) \in G_{\varepsilon}^{(i)}(j) \times (0, T)$, which will be defined later. Then, due to (8) we have

$$\varphi_{1}^{(i)}(x_1, \varepsilon(j+b_i), x_3, \xi_2 - j, t) = \varphi_{1}^{(i)}(x_1, \varepsilon(j+b_i), x_3, t) - (\xi_2 - j - b_i) \frac{\partial \varphi_{1}^{(i)}}{\partial x_2}u_0^{i,-}(x_1, \varepsilon(j+b_i), x_3, t), \quad (x, t) \in G_{\varepsilon}^{(i)}(j) \times (0, T). \quad (12)$$

Boundary-value problems for $V_{\varepsilon,2}^{1,j}$ and $V_{\varepsilon,2}^{2,j}$ have the view

$$\left\{ \begin{array}{l}
-\frac{\partial^2 \xi_2}{\partial \xi_2^2} V_{\varepsilon,2}^{1,j} = (\partial \varphi_{1}^{(i)} - \Delta_{\tilde{x}} \varphi_{1}^{(i)} - \vartheta_0(u_0^{i,-}) + f_0)|_{x_2=\varepsilon(j+b_i)}, \quad \xi_2 \in I_h_{i}(r)(j), \\
\pm \frac{\partial \xi_2}{\partial \xi_2} V_{\varepsilon,2}^{1,j} = (2^{-1}\nabla_{\tilde{x}} h_{i} \cdot \nabla_{\tilde{x}} \varphi_{1}^{(i)} - \vartheta_1(u_0^{i,-}) + \delta_{\tilde{x}} g_0)|_{x_2=\varepsilon(j+b_i)}, \quad \xi_2 = \pm \frac{h_{i}(r)}{2} + j + b_i,
\end{array} \right. \quad (13)$$
The solvability conditions for problems (13) and (14) read

\[
\begin{align*}
&h_1 \partial_r u_0^{1-} - \text{div}_2 (h_1 \nabla_x u_0^{1-}) + h_1 \partial_0 (u_0^{1-}) + 2 \partial_1 (u_0^{1-}) = h_1 f_0 + 2 \delta_{\beta_1} g_0, \\
x_2 = \varepsilon (j + b_1), \ r \in (d_0, d_1), \ t \in (0, T),
\end{align*}
\]

respectively.

Putting (7) into the Robin boundary conditions on \(Q_{x}^{(i)}\), we get

\[
\begin{align*}
&\partial_r u_0^{1-} + \partial_1 (u_0^{1-}) = 0, \quad (x, t) \in Q_{x}^{(1)} \times (0, T), \quad x_2 = \varepsilon (j + b_1), \\
&\partial_r u_0^{2-} = 0, \quad (x, t) \in Q_{x}^{(2)} \times (0, T), \quad x_2 = \varepsilon (j + b_2).
\end{align*}
\]

Plugging (7) into the initial condition of problem (1), we find that

\[
u_0^{1-} (x, 0) = 0, \quad x \in G_{x}^{(i)}, \quad x_2 = \varepsilon (j + b_1), \quad i \in \{1, 2\}.
\]

In order to find conditions in the joint zone \(Q_{0}^{(0)}\) we use the method of matched asymptotic expansions for outer expansions (4), (7) and an inner expansion which will be constructed in the next subsection.

### 3.2. Inner Expansion

In a neighborhood of the joint zone \(Q_{0}^{(0)}\) we introduce the “rapid” coordinates \(\xi = (\xi_1, \xi_2)\), where \(\xi_1 = -(r - d_0) / \varepsilon\) and \(\xi_2 = x_2 / \varepsilon\). Here \((r, x_2, \theta) \in \mathbb{R}^3\) are the cylindric coordinates: \(r = \sqrt{x_1^2 + x_2^2}\), \(\tan(\theta) = x_3 / x_1\). The Laplace operator in the coordinates \((\xi_1, \xi_2, \theta)\) has the form

\[
\Delta_{\xi} = \varepsilon^{-2} \Delta_{\xi} - \varepsilon^{-1} \frac{1}{d_0 - \varepsilon \xi_1} \frac{\partial}{\partial \xi_1} + \frac{1}{(d_0 - \varepsilon \xi_1)^2} \frac{\partial^2}{\partial \theta^2}.
\]

We seek the leading terms of the inner expansion in a neighborhood of \(Q_{0}^{(0)}\) in the form

\[
u_r (x, t) \approx u_0^+(x, t)|_{r=d_0} + \varepsilon (Z_1(\xi) \partial_{\xi_2} u_0^+(x, t)|_{r=d_0} - \eta(x_2, t) \Xi_1(\xi) + (1 - \eta(x_2, t)) \Xi_2(\xi)) \partial_r u_0^+(x, t)|_{r=d_0}) + \ldots,
\]

where \(Z_1, \Xi_1, \Xi_2\) are functions, which are 1-periodic with respect to \(\xi_2\) and defined in the union \(\Pi = \Pi^+ \cup \Pi^- \cup \Pi^\circ\) of the seminfinite strips \(\Pi^+ = \{\xi \in \mathbb{R}^2: \xi_1 > 0, \xi_2 \in (0, 1)\}\), \(\Pi^- = \{\xi \in \mathbb{R}^2: \xi_1 \leq 0, \xi_2 \in I_i(d_0)\}, \ i \in \{1, 2\}\), (see definition of \(I_i(d_0)\)), \(\eta\ is a function, which will be defined from matching conditions.

Putting (21) into the differential equation of problem (1) with regard to (20) and into the corresponding boundary conditions and collecting coefficients of the same powers of \(\varepsilon\), we get the junction-layer problems for \(Z_1, \Xi_1, \Xi_2\). The functions \(\Xi_1\) and \(\Xi_2\) are solutions of the following homogeneous problem

\[
\begin{align*}
&\begin{cases}
- \Delta_\xi \Xi = 0, & \text{in } \Pi, \\
\partial_{\xi_2} \Xi = 0, & \text{on } (\partial \Pi^+ \cup \partial \Pi^-) \cap \{\xi \in \mathbb{R}^2: \xi_1 < 0\}, \\
\partial_{\xi_1} \Xi = 0, & \text{on } \partial \Pi \cap \{\xi \in \mathbb{R}^2: \xi_1 = 0\}, \\
\partial_{\xi_2} \Xi|_{\xi_2=0} = \partial_{\xi_2} \Xi|_{\xi_2=1}, & p \in \{0, 1\}, \ \xi_1 > 0.
\end{cases}
\end{align*}
\]
Main asymptotic relations for \( \Xi_1, \Xi_2 \) can be obtained from general results on the asymptotic behavior of solutions of elliptic problems in domains with different exits to infinity (see, for instance, [29]). However, for the domain \( \Pi \), we can define more exactly the asymptotic relations for junction-layer solutions \( \Xi_1, \Xi_2 \) in the same way as in [18, 19].

**Proposition 1.** There exist two solutions \( \Xi_1, \Xi_2 \in H^1_{x,loc}(\Pi) \) to problem (22), which have the following differentiable asymptotics

\[
\Xi_1 = \begin{cases} 
\xi_1 + O(\exp(-2\pi \xi_1)), & \xi_1 \to +\infty, \xi \in \Pi^+, \\
\alpha_1^{(1)} + O(\exp(\pi h_1^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty, \xi \in \Pi_1^{-}, \\
h_2^{-1}(d_0)\xi_1 + \alpha_1^{(2)} + O(\exp(\pi h_2^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty, \xi \in \Pi_2^{-}.
\end{cases}
\]

\[
\Xi_2 = \begin{cases} 
\xi_1 + O(\exp(-2\pi \xi_1)), & \xi_1 \to +\infty, \xi \in \Pi^+, \\
h_1^{-1}(d_0)\xi_1 + \alpha_2^{(1)} + O(\exp(\pi h_1^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty, \xi \in \Pi_1^{-}, \\
\alpha_2^{(2)} + O(\exp(\pi h_2^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty, \xi \in \Pi_2^{-}.
\end{cases}
\]  

Here \( H^1_{x,loc}(\Pi) = \{ u: \Pi \to \mathbb{R} : u(\xi_1,0) = u(\xi_1,1) \text{ for any } \xi_1 > 0, u \in H^1(\Pi_R) \text{ for any } R > 0 \} \), \( \Pi_R = \{ \xi \in \Pi : -R < \xi_1 < R \}; \alpha_1^{(i)}, \alpha_2^{(i)}, i \in \{1,2\}, \text{ are some constants} \).

Any other solution of problem (22), which has a polynomial growth at infinity, can be represented as a linear combination \( c_0 + c_1\Xi_1 + c_2\Xi_2 \).

The function \( Z_1 \) is a solution of the following problem

\[
\begin{cases} 
-\Delta_\xi Z = 0, & \text{in } \Pi, \\
\partial_{\xi_2} Z = -1, & \text{on } (\partial\Pi_1^{-} \cup \partial\Pi_2^{-}) \cap \{ \xi \in \mathbb{R}^2: \xi_1 < 0 \}, \\
\partial_\xi Z = 0, & \text{on } \partial\Pi \cap \{ \xi \in \mathbb{R}^2: \xi_1 = 0 \}, \\
\partial^p_{\xi_2} Z|_{\xi_2=0} = \partial^p_{\xi_2} Z|_{\xi_2=1}, & p = i \in \{0,1\}, \xi_1 > 0.
\end{cases}
\]

Similarly to [18, 19, 28] it is easy to verify that there exists a unique solution \( Z_1 \in H^1_{x,loc}(\Pi) \) with the following asymptotics

\[
Z = \begin{cases} 
\mathcal{O}(\exp(-2\pi \xi_1)), & \xi_1 \to +\infty, \xi \in \Pi^+, \\
-\xi_2 + b_1 + \alpha_2^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty, \xi \in \Pi_1^{-}, \\
-\xi_2 + b_2 + \alpha_3^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty, \xi \in \Pi_2^{-}.
\end{cases}
\]  

Now let us verify the matching conditions for outer expansions (4), (5) and inner expansion (21), namely, the leading terms of the asymptotics of the outer expansions as \( \xi_1 \to \pm 0 \) must coincide with the leading terms of the asymptotics of the inner expansion as \( \xi_1 \to \pm \infty \). Near the point \( (x_1, \varepsilon(j + b_1), x_3) \in \mathcal{Q}_0^{(0)} \) for any fixed \( t \in (0,T) \) the function \( u_0^+ \) has the following asymptotics

\[
u_0^+(x, t) \approx u_0^+(x_1, \varepsilon(j + b_1), x_3, t)|_{r=d_0} + \varepsilon(\xi_2 - j - b_1)\partial_{x_2} u_0^+(x_1, \varepsilon(j + b_1), x_3, t)|_{r=d_0} - \varepsilon\xi_1 \partial_{x_2} u_0^+(x_1, \varepsilon(j + b_1), x_3, t)|_{r=d_0} + \ldots \text{ as } \xi_1 \to 0+, (x, t) \in \Omega_0 \times (0,T).
\]  

Taking into account the asymptotics of \( Z_1, \Xi_1 \) and \( \Xi_2 \) as \( \xi_1 \to +\infty \), we see that the matching conditions are satisfied for expansions (4) and (21).
For any \( t \in (0, T) \) the asymptotics of (5) in the neighborhood of \((x_1, \varepsilon(j + b_i), x_3) \in Q_0^0\) are equal to
\[
\frac{u_{i1}^0(x_1, \varepsilon(j + b_i), x_3, t)|_{r=d_0} + \varepsilon(\varphi_1^{(i)}(x_1, \varepsilon(j + b_i), x_3, t)|_{r=d_0}}{-\xi_1 \partial_r u_{i1}^0(x_1, \varepsilon(j + b_i), x_3, t)|_{r=d_0} + \ldots}
\text{as} \xi_1 \to 0^-, \quad (x, t) \in G_\varepsilon^{(i)}(j) \times (0, T), \quad i \in \{1, 2\}.
\] (27)

It follows from (23), (24) and (26) that the first terms of the asymptotics of (21) in the neighborhood of \((x_1, \varepsilon(j + b_i), x_3) \in Q_0^0\) are
\[
\left(\alpha_1^{(1)}(\varepsilon(j + b_i), t) + (h_1^{-1}(d_0)\alpha_1^{(1)}(1 - \varepsilon(j + b_i), t))\partial_r u_{i1}^0(x_1, \varepsilon(j + b_i), x_3, t)|_{r=d_0}\right)
\text{as} \xi_1 \to -\infty, \quad (x, t) \in G_\varepsilon^{(i)}(j) \times (0, T),
\] (28)
\[
\left(-\alpha_1^{(1)}(\varepsilon(j + b_i), t) + (h_2^{-1}(d_0)\alpha_1^{(1)}(1 - \varepsilon(j + b_i), t))\partial_r u_{i1}^0(x_1, \varepsilon(j + b_i), x_3, t)|_{r=d_0}\right)
\text{as} \xi_1 \to -\infty, \quad (x, t) \in G_\varepsilon^{(i)}(j) \times (0, T).
\] (29)

Comparing the first terms of (27), (28) with (29), we get
\[
u_{i1}^0(x, t) = \frac{u_{i1}^0(x, t), \quad (x, t) \in Q_0^0 \times (0, T), \quad x_2 = \varepsilon(j + b_i), \quad i \in \{1, 2\}}{Q_0^0}. (30)
\]

Comparing the second terms of (27), (28) with (29), we find that
\[
\varphi_1^{(i)}(x, t) = \frac{\alpha_1^{(i)}\partial_x u_{i1}^0(x, t), \quad (x, t) \in Q_0^0 \times (0, T), \quad x_2 = \varepsilon(j + b_i), \quad i \in \{1, 2\}}{Q_0^0}. (31)
\]

and
\[
\eta(x_2, t) = \frac{h_2(d_0)\partial_r u_{i1}^0(x, t), \quad (x, t) \in Q_0^0}{h_1(d_0)\partial_r u_{i1}^0(x, t) + h_2(d_0)\partial_r u_{i1}^0(x, t) = \partial_r u_{i1}^0(x, t), \quad (x, t) \in Q_0^0 \times (0, T), \quad x_2 = \varepsilon(j + b_i).} (32)
\]

Since the points \(\{\varepsilon(j + b_i) : j \in \{0, 1, \ldots, N - 1\}, \quad i \in \{1, 2\}\) make up an \(\varepsilon\)-net on the segment \([0, l]\), we can extend equalities (12), (15), (16), (19) to the domains \(D_i\), equalities (17), (18) to \(Q_0^1\) and \(Q_0^2\), respectively, and equalities (30), (31) and (32) to \(Q_0^0\). As a result, from equalities (32) we derive the relation
\[
\eta(x_2, t) = \frac{h_2(d_0)\partial_r u_{i1}^0(x, t), \quad (x, t) \in D_1 \times (0, T), \quad x_2 \in (0, l), \quad t \in (0, T),}{h_1(d_0)\partial_r u_{i1}^0(x, t) + h_2(d_0)\partial_r u_{i1}^0(x, t) = \partial_r u_{i1}^0(x, t), \quad (x, t) \in Q_0^0 \times (0, T), \quad x_2 = \varepsilon(j + b_i).}
\]

By virtue of (30) and (31) we can define \(\varphi_1^{(i)}\) as follows
\[
\varphi_1^{(i)}(x, t) = \frac{\alpha_1^{(i)}\partial_x u_{i1}^0(x, t), \quad (x, t) \in D_1 \times (0, T), \quad i \in \{1, 2\}}{D_1}. (32)
\]

3.3. The homogenized problem. With the help of the first terms \(u_0^+, u_0^{1-}\) and \(u_0^{2-}\) of asymptotic expansions (4) and (5) we define the multi-sheeted function
\[
U_0(x, t) = \begin{cases} \frac{u_0^+(x, t), \quad (x, t) \in \Omega_0 \times (0, T),}{u_0^{1-}(x, t), \quad (x, t) \in D_1 \times (0, T),} \quad \frac{u_0^{2-}(x, t), \quad (x, t) \in D_2 \times (0, T),}{D_2} \end{cases}
\]
or in a short form \( \mathbf{U}_0 = (\mathbf{u}_0^+, \mathbf{u}_0^{1-}, \mathbf{u}_0^{2-}) \). It follows from the foregoing that the components of \( \mathbf{U}_0 \) must satisfy the relations

\[
\begin{align*}
\partial_t \mathbf{u}_0^+ - \Delta_x \mathbf{u}_0^+ + \partial_\nu (\mathbf{u}_0^+) &= f_0, \\
\partial_{x_2} \mathbf{u}_0^+ |_{S^-} &= \partial_{x_2} \mathbf{u}_0^+ |_{S^+}, \\
h_1(r) \partial_r \mathbf{u}_0^{1-} - \text{div}_x (h_1(r) \nabla_x \mathbf{u}_0^{1-}) + h_1(r) \partial_\nu (\mathbf{u}_0^{1-}) + 2 \partial_{x_1} (\mathbf{u}_0^{1-}) &= h_1(r) f_0 + 2 \delta_{\beta,1} g_0, \\
\partial_r \mathbf{u}_0^{1-} + \partial_\nu (\mathbf{u}_0^{1-}) &= 0, \\
h_2(r) \partial_r \mathbf{u}_0^{2-} - \text{div}_x (h_2(r) \nabla_x \mathbf{u}_0^{2-}) + h_2(r) \partial_\nu (\mathbf{u}_0^{2-}) + 2 \delta_{\beta,1} \partial_2 (\mathbf{u}_0^{2-}) &= h_2(r) f_0 + 2 \delta_{\beta,1} g_0, \\
\mathbf{u}_0^+ |_{Q_0^{(0)}} = \mathbf{u}_0^{1-} |_{Q_0^{(0)}} = \mathbf{u}_0^{2-} |_{Q_0^{(0)}}, \\
h_1(d_0) \partial_r \mathbf{u}_0^{1-} + h_2(d_0) \partial_2 \mathbf{u}_0^{2-} = \partial_r \mathbf{u}_0^+, \\
\mathbf{U}_0(x, 0) &= 0.
\end{align*}
\]

These relations form the homogenized problem for problem (1).

We introduce the space \( \mathcal{V}_0 := L^2(\Omega_0) \times L^2(D_1) \times L^2(D_2) \) of multi-sheeted functions \( \mathbf{u} = (u_0, u_1, u_2) \) defined as follows

\[
\mathbf{u}(x) = \begin{cases}
  u_0(x), & x \in \Omega_0, \\
  u_1(x), & x \in D_1, \\
  u_2(x), & x \in D_2.
\end{cases}
\]

The space \( \mathcal{V}_0 \) is equipped with natural inner product. Also we introduce the anisotropic Sobolev space of multi-sheeted functions

\[
\mathcal{H}_0 := \{ \mathbf{u} = (u_0, u_1, u_2) \in \mathcal{V}_0 : u_0 \in H^1(\Omega_0), \; u_0|_{S^-} = u_0|_{S^+}; \\
\exists \partial_x u_i \in L^2(D_i), \; j = 1, 3, \; i = 1, 2; \; u_0|_{Q_0^{(0)}} = u_1|_{Q_0^{(0)}} = u_2|_{Q_0^{(0)}} \}
\]

with the inner product

\[
(\mathbf{u}, \mathbf{v})_{\mathcal{H}_0} = \int_{\Omega_0} (\nabla_x u_0 \cdot \nabla_x v_0 + u_0 v_0) dx + \sum_{i=1}^{2} \int_{D_i} (\nabla \cdot \nabla u_i \cdot \nabla \nu v_i + u_i v_i) dx.
\]

It is obvious that \( \mathcal{H}_0 \) is continuously embedded in \( \mathcal{V}_0 \).

Consider the space

\[
\mathcal{W}_0 := \{ \varphi = (\varphi_0, \varphi_1, \varphi_2) \in L^2(0, T; \mathcal{H}_0), \; \exists \partial_t \varphi := \varphi' \in L^2(0, T; \mathcal{V}_0) \}.
\]

A function \( \mathbf{U}_0 = (u_0^+, u_0^{1-}, u_0^{2-}) \in L^2(0, T; \mathcal{H}_0) \) is a weak solution of problem (33) if for every function \( \varphi = (\varphi_0, \varphi_1, \varphi_2) \in \mathcal{W}_0 \) the integral identity

\[
\begin{align*}
\int_{\Omega_0} (u_0^+ \varphi_0)_{|t=T} dx + \sum_{i=1}^{2} \int_{D_i} (h_i u_0^{i-} \varphi_i)_{|t=T} dx - \int_{0}^{T} \left( \int_{\Omega_0} u_0^+ \partial_t \varphi_0 dx + \sum_{i=1}^{2} \int_{D_i} h_i u_0^{i-} \partial_t \varphi_i dx + \int_{\Omega_0} (\nabla_x u_0^+ \cdot \nabla_x \varphi_0 + \partial_\nu (u_0^+) \varphi_0) dx + \sum_{i=1}^{2} \int_{D_i} h_i (\nabla_x u_0^{i-} \cdot \nabla_x \varphi_i + \partial_\nu (u_0^{i-}) \varphi_i) dx + \right. \\
+ 2 \int_{D_1} \partial_1 (u_0^{1-}) \varphi_1 dx + h_1(d_0) \int_{Q_0^{(0)}} \partial_1 (u_0^{1-}) \varphi_1 d\sigma_x + 2 \delta_{0,1} \int_{D_2} \partial_2 (u_0^{2-}) \varphi_2 dx \right) dt =
\end{align*}
\]
\[
= \int_0^T \left( \int_{\Omega_t} f_0 \varphi_0 dx + \sum_{i=1}^2 \int_{D_i} (h_i f_0 + 2\delta_{\beta,i} g_0) \varphi_i dx \right) dt
\]

holds.

Similarly as in [26] we can prove the existence and the uniqueness of a weak solution of problem (33).

4. Approximation and asymptotic estimates. Let \( U_0 = (u_0^+, u_0^{1-}, u_0^{2-}) \) be the unique weak solution of problem (33). With the help of \( U_0 \) and the solutions \( Z_1, \Xi_1, \Xi_2 \) of junction-layer problems (22) and (25) we construct the main terms of expansions (4), (5) and (21). Consider the smooth cut-off function \( \chi_0(r) \), which is equal to 1 as \( |r - d_0| < \delta_0/2 \) and 0 as \( |r - d_0| > \delta_0 \), where \( \delta_0 \in (0, \delta) \) is some fixed number. Matching the outer expansions with the inner expansion with the help of \( \chi_0 \), we define the approximation function \( R_\varepsilon \)

\[
R_\varepsilon(x,t) := R_\varepsilon^+(x,t) = u_0^+(x,t) + \varepsilon \chi_0(r) N^+(\xi, x_2, \theta, t), \quad (x,t) \in \Omega_t \times (0,T),
\]

\[
R_\varepsilon(x,t) := R_\varepsilon^-(x,t) = u_0^{-}(x,t) + \varepsilon \left( \tilde{Y}_1 \left( \frac{x_2}{\varepsilon} \right) \partial_{x_2} u_0^{-}(x,t) + \chi_0(r) N_-^i(\xi, x_2, \theta, t) \right), \quad (x,t) \in G^{(i)}(j) \times (0,T), \quad i \in \{1, 2\}.
\]

Here

\[
N^+(\xi, x_2, \theta, t) = Z_1(\xi) \partial_{x_2} u_0^+|_{r=d_0} + (\xi - \eta(x_2,t)) \Xi_1(\xi) - (1 - \eta(x_2,t)) \Xi_2(\xi) \partial_{x_2} u_0^+|_{r=d_0},
\]

\[
N_-^i(\xi, x_2, \theta, t) = (Z_1(\xi) - \tilde{Y}_i(2\xi_2)) \partial_{x_2} u_0^{-}|_{r=d_0} + (\tilde{Y}_1(\xi, x_2, t) - \eta(x_2,t)) \Xi_1(\xi) - (1 - \eta(x_2,t)) \Xi_2(\xi) \partial_{x_2} u_0^{-}|_{r=d_0},
\]

where \( \tilde{Y}_i(s) := -s + [s] + b_i + \alpha_3^{(i)}, \) \( [s] \) is the integer part of \( s \in \mathbb{R}, \) \( i \in \{1, 2\} \), and

\[
\tilde{Y}_1(\xi, x_2, t) := h_1^{-1}(d_0) \xi_1(1 - \eta(x_2,t)), \quad \tilde{Y}_2(\xi, x_2, t) := h_2^{-1}(d_0) \xi_1 \eta(x_2,t),
\]

\[
\xi_1 \leq 0, \quad x_2 \in (0,l), \quad t \in (0,T).
\]

Obviously, \( R_\varepsilon \in W_\varepsilon \). Due to the initial condition of problem (33) we have \( R_\varepsilon|_{t=0} = 0 \) in \( \Omega_\varepsilon \).

**Theorem 1.** Let \( f_0(x,t), \) \( (x,t) \in \Omega_t \times [0,T], \) and \( g_0(x,t), \) \( (x,t) \in \Omega_t \times [0,T], \) be smooth functions such that \( \partial_{x_2}^2 f_0|_{s-} = \partial_{x_2}^2 f_0|_{s+} \) for all \( t \in [0,T], \) \( p \in \{0,1\}, \) \( f_0(x,0) = g_0(x,0) = 0. \)

Then for any \( \mu > 0 \) there exist positive constants \( \varepsilon_0, \varepsilon_1 \) such that

\[
\|u_\varepsilon - R_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + \max_{t \in [0,T]} \|u_\varepsilon(\cdot,t) - R_\varepsilon(\cdot,t)\|_{L^2(\Omega_\varepsilon)} \leq
\]

\[
\leq c_0(\|f_\varepsilon - f_0\|_{L^2(\Omega_t \times (0,T))} + \varepsilon^{1-\mu} + \varepsilon^{\delta_{3,1} (2-\alpha) + \alpha - 1} + \varepsilon^{\beta-1}) \|y_\varepsilon - g_0\|_{L^2(\Omega_\varepsilon \times (0,T))} \]

for all \( \varepsilon \in (0,\varepsilon_0), \) where \( u_\varepsilon \) is the solution of problem (1), and \( R_\varepsilon \) is defined by (34) and (35).

**Proof.** Discrepancies in domain \( \Omega_0 \). Notice that the first two relations in (33) and the assumptions of the theorem yield \( \partial_{x_2}^2 u_0^+|_{s-} = \partial_{x_2}^2 u_0^+|_{s+}. \) Then, according to the properties of \( Z_1, \Xi_1, \Xi_2 \) and \( u_0^+ \), the function \( R_\varepsilon^+ \) satisfies the boundary conditions of problem (1) on \( \partial \Omega_\varepsilon \cap \partial \Omega_0 \).

Problems (22) and (25) imply

\[
\Delta_\varepsilon N^+ = 0, \quad \Delta_\varepsilon N_-^i = 0 \quad \xi \in \Pi, \quad x_2 \in (0,l), \quad \theta \in [0,2\pi], \quad t \in [0,T], \quad i \in \{1, 2\}.
\]
Observe that the following equality holds
\[
\Delta \tilde{x}(\chi_0(r) N) = \text{div}\tilde{x}(N \nabla \tilde{x} \chi_0(r)) + \nabla \chi_0(r) \cdot \nabla \tilde{x} N + \chi_0(r) \Delta \tilde{x} N, \quad N = N(\xi, x_2, \theta). \tag{38}
\]
Using (20), (33), (37) and (38), we get
\[
\partial_t R^+_\varepsilon(x, t) - \Delta \varepsilon R^+_\varepsilon(x, t) - f_\varepsilon(x, t) = f_\varepsilon(x, t) - f_\varepsilon(x, t) + \varepsilon \chi_0(0) \partial_t N^+(\xi, x_2, \theta, t) - \\
- \vartheta_0(0) \vartheta_0 + \chi_0(0)(r^{-1}) \partial_{\xi^i} N^+(\xi, x_2, \theta, t) - 2 \partial_{\xi^i \xi^j}^2 N^+(\xi, x_2, \theta, t) - \\
- \varepsilon \vartheta_0(0) \partial_{x_2 x_2}^2 N^+(\xi, x_2, \theta, t) - \varepsilon \chi_0(0) \partial_{x_2 x_2}^2 N^+(\xi, x_2, \theta, t) - (x, t) \in \Omega_0 \times (0, T). \tag{39}
\]
We multiply (39) by a test function \( \psi \in W_\varepsilon \) such that \( \psi(x, T) = 0 \), integrate by parts in \( \Omega_0 \times (0, T) \) and take into account the properties of \( R^+_\varepsilon \). This yields
\[
\int_0^T \left( - \int_{\Omega_0} R^+_\varepsilon \partial_t \psi dx + \int_{\Omega_0} (\nabla_x R^+_\varepsilon \cdot \nabla_x \psi) dx - \int_{\Omega_0} \partial_t R^+_\varepsilon \psi dx dt - \int_{\Omega_0} f_\varepsilon \psi dx \right) dt = \\
= I^+_1(\varepsilon, \psi) + \ldots + I^+_5(\varepsilon, \psi) \tag{40}
\]
for all \( \psi \in W_\varepsilon, \psi(x, T) = 0 \), where
\[
I^+_1(\varepsilon, \psi) := \int_0^T \int_{\Omega_0} (f_0 - f_\varepsilon) \psi dx dt, \quad I^+_1(\varepsilon, \psi) := \varepsilon \int_0^T \int_{\Omega_0} \chi_0 \partial_{\partial \varepsilon} N^+ \psi dx dt,
\]
\[
I^+_2(\varepsilon, \psi) := \int_0^T \int_{\Omega_0} (\partial_t (R^+_\varepsilon) - \partial_t (u^0_\varepsilon)) \psi dx dt, \quad I^+_3(\varepsilon, \psi) := \int_0^T \int_{\Omega_0} \chi_0 r^{-1} \partial_{\xi^i} N^+ + \partial_{x_2 x_2}^2 N^+ \psi dx dt,
\]
\[
I^+_4(\varepsilon, \psi) := \varepsilon \int_0^T \int_{\Omega_0} \nabla_x \chi_0 \cdot \nabla_x \psi dx + \int_{\Omega_0} \chi_0' \partial_{\xi^i} N^+ \psi dx dt,
\]
\[
I^+_5(\varepsilon, \psi) := \varepsilon \int_0^T \left( \int_{\Omega_0} \chi_0 \partial_{x_2} N^+ \partial_{x_2} \psi dx + \int_{\Omega_0} r^{-2} \chi_0 \partial_{\partial \varepsilon} N^+ \partial_{\partial \varepsilon} \psi dx dt \right).
\]

Discrepancies in the thin discs. One can readily check that
\[
\partial_t R^1_\varepsilon = - \vartheta_1(u^1_\varepsilon) - \varepsilon \tilde{Y}_1 \frac{x_2}{\varepsilon} \partial_{x_2} \vartheta_1(u^1_\varepsilon), \quad (x, t) \in Q^1_\varepsilon \times (0, T),
\]
\[
\partial_t R^2_\varepsilon = 0, \quad (x, t) \in Q^2_\varepsilon \times (0, T), \quad (x, t) \in Q^2_\varepsilon \times (0, T), \quad i \in \{1, 2\}. \tag{41}
\]
Taking into account (9) and that functions \( h_i \) are constant on a neighborhood of \( d_0 \), we derive that
\[
\partial_t R^1_\varepsilon = \frac{\varepsilon}{\sqrt{1 + 4 - 4 \xi^2 \bar{r}'_i(r)^2}} \left( \pm \tilde{Y}_i \frac{x_2}{\varepsilon} \partial_{x_2}^2 u^i_0 \pm \chi_0 \partial_{x_2} (N^1_i |_{\xi = x_2/\varepsilon}) - \\
- \frac{1}{2} \nabla_x h_i \cdot \nabla_x (u^i_0 + \varepsilon \tilde{Y}_i \frac{x_2}{\varepsilon} \partial_{x_2} u^i_0) \right), \quad (x, t) \in S^1_\varepsilon \times (0, T), \quad i \in \{1, 2\}. \tag{43}
\]
where “+” and “−” refer to the left and the right parts of the lateral surfaces of the thin discs, respectively.
Consider the integral identity
\[
\int_{S^2_t(\varepsilon)} \frac{\varepsilon h_i(r)}{2\sqrt{1 + 4^{-1}\varepsilon^2|h_i'(r)|^2}} \varphi d\sigma_x = \int_{G_{2}^{(i)}} \varphi dx - \varepsilon \int_{G_{1}^{(i)}} Y_i \left( \frac{x_0}{\varepsilon} \right) \partial_{x_2} \varphi dx, \quad i \in \{1, 2\},
\]
where \(Y_i(s) = -s + [s] + b_i\) and \([s]\) is the integer part of \(s\), \(\varphi \in H^1(G_{3}^{(i)})\) is an arbitrary function. We multiply (44) by a test function \(\psi \in W_{\varepsilon}\), \(\psi(x,T) = 0\), and integrate by parts in \(G_{3}^{(i)} \times (0,T)\), using (45) and taking into account relations (41), (42), (43). This yields
\[
\int_0^T \left( -\int_{G_{1}^{(i)}} \partial_{t} \psi dx + \int_{G_{2}^{(i)}} \nabla_{x} R_{\varepsilon}^1 \cdot \nabla_{x} \psi + \theta_0(R_{\varepsilon}^1) \psi \right) dx + \varepsilon \int_{S^2_t(\varepsilon)} \varphi d\sigma_x + \int_{G_{1}^{(i)}} \partial_i(1)(1) \psi d\sigma_x + \int_{G_{2}^{(i)}} \partial_i(1) \psi d\sigma_x - \int_{G_{3}^{(i)}} f_\varepsilon \psi dx - \varepsilon^3 \int_{G_{3}^{(i)}} g_\varepsilon \psi d\sigma_x \right) dt = I^i_{1-}(\varepsilon, \psi) + \ldots + I^i_{8-}(\varepsilon, \psi),
\]
for all \(\psi \in W_{\varepsilon}\), \(\psi(x,T) = 0\), where
\[
I^i_{1-}(\varepsilon, \psi) := \int_0^T \int_{G_{2}^{(i)}} (f_\varepsilon - f_\varepsilon) \psi dx dt,
\]
\[
I^i_{1-}(\varepsilon, \psi) := \varepsilon \int_0^T \int_{G_{2}^{(i)}} (\tilde{Y}_i \left( \frac{x_0}{\varepsilon} \right) \partial_{x_2} u_{0}^{i-} + \chi_0(r) \partial_{t} N_{\varepsilon}^{i-}) \psi dx dt,
\]
\[
I^i_{2-}(\varepsilon, \psi) := \int_0^T \int_{G_{1}^{(i)}} (\theta_0(R_{\varepsilon}^1) - \theta_0(u_{0}^{i-})) \psi dx dt,
\]
\[
I^i_{3-}(\varepsilon, \psi) := \int_0^T \int_{G_{2}^{(i)}} \chi_0(r) \partial_{t} N_{\varepsilon}^{i-} \psi dx dt,
\]
\[
I^i_{4-}(\varepsilon, \psi) := \int_0^T \left( \varepsilon \int_{G_{2}^{(i)}} N_{\varepsilon}^{i-} \nabla_{x} \varphi dx + \int_{G_{2}^{(i)}} \chi_0 \partial_{t} N_{\varepsilon}^{i-} \psi dx \right) dt,
\]
\[
I^i_{5-}(\varepsilon, \psi) := \varepsilon \int_0^T \left( \int_{G_{2}^{(i)}} \chi_0 \partial_{x_2} N_{\varepsilon}^{i-} \partial_{x_2} \psi dx + \int_{G_{2}^{(i)}} \chi_0 \partial_{t} N_{\varepsilon}^{i-} \partial_{x_2} \psi dx \right) dt,
\]
\[
I^i_{6-}(\varepsilon, \psi) := \varepsilon \int_0^T \left( \int_{G_{2}^{(i)}} \tilde{Y}_i \left( \frac{x_0}{\varepsilon} \right) \partial_{x_2} \psi \nabla_{x} u_{0}^{i-} \partial_{x_2} \ln h_i dx + \right. \]
\[
\left. + \int_{G_{2}^{(i)}} \tilde{Y}_i \left( \frac{x_0}{\varepsilon} \right) \nabla_{x} (\partial_{x_2} u_{0}^{i-}) \partial_{x_2} \psi dx \right) dt, \quad i \in \{1, 2\}.
\]
Asymptotic estimates. After summing (40), (46) and (47) we see that the function \(R_\varepsilon\) defined by (34) and (35) satisfies the integral identity

\[
F_\varepsilon(\psi) = \int_0^T \left( -\int_{\Omega_\varepsilon} R_\varepsilon \partial t_1 \psi dx + \int_{\Omega_\varepsilon} (\nabla_x R_\varepsilon \cdot \nabla_x \psi + \partial_0(R_\varepsilon)\psi) dx + \varepsilon \int_{Q_\varepsilon^{(1)}} \partial_1(R_\varepsilon)\psi d\sigma_x + \right.
\]

\[
+ \int_{Q_\varepsilon^{(1)}} \partial_1(R_\varepsilon)\psi d\sigma_x + \varepsilon^\alpha \int_{Y_\varepsilon^{(2)}} \partial_2(R_\varepsilon)\psi d\sigma_x - \int_{\Omega_\varepsilon} f_\varepsilon dx - \varepsilon^\beta \int_{S_\varepsilon^{(2)} \cup Y_\varepsilon^{(2)}} g_\varepsilon \psi d\sigma_x \right) dt
\]

(48)

for all \(\psi \in W_\varepsilon, \psi|_{t=T} = 0\), where \(F_\varepsilon(\psi) := I_0^\varepsilon + \ldots + I_5^\varepsilon + I_6^\varepsilon + I_7^\varepsilon + I_8^\varepsilon\), \(I_k^\pm := I_k^\varepsilon^\pm + I_k^\varepsilon^-\), \(k \in \{0, 1, \ldots, 8\}\), \(I_m^\pm := I_m^\varepsilon^+ + I_m^\varepsilon^-, m \in \{0, 1, \ldots, 5\}\).

It follows from (3) and (48) that

\[
\int_0^T \left( -\int_{\Omega_\varepsilon} (R_\varepsilon - u_\varepsilon) \partial_t \psi dx + \int_{\Omega_\varepsilon} (\nabla_x (R_\varepsilon - u_\varepsilon) \cdot \nabla_x \psi + (\partial_0(R_\varepsilon) - \partial_0(u_\varepsilon))\psi) dx + \right.
\]

\[
+ \varepsilon \int_{S_\varepsilon^{(1)}} (\partial_1(R_\varepsilon) - \partial_1(u_\varepsilon))\psi d\sigma_x + \int_{Q_\varepsilon^{(1)}} (\partial_1(R_\varepsilon) - \partial_1(u_\varepsilon))\psi d\sigma_x + \right.
\]

\[
+ \varepsilon^\alpha \int_{Y_\varepsilon^{(2)}} (\partial_2(R_\varepsilon) - \partial_2(u_\varepsilon))\psi d\sigma_x \right) dt = F_\varepsilon(\psi)
\]

(49)

for all \(\psi \in W_\varepsilon, \psi|_{t=T} = 0\).

Now we are going to estimate \(F_\varepsilon(\psi)\). With the help of the Cauchy-Schwartz-Bunyakovskii inequality we obtain \(|I_0^\pm(\varepsilon, \psi)| \leq \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon \times (0,T))}\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}, |I_k^\pm(\varepsilon, \psi)| \leq C_1\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}^2\).
By (2), Taylor’s formula and the Cauchy-Schwartz-Bunyakovskii inequality we derive that

$$|I_2^+(\varepsilon, \psi)| \leq c_0 \varepsilon \left| \int_0^T \int_{\Omega_0} \chi_0 \mathcal{N}^+ \psi dxdt \right| \leq c_1 \varepsilon \|\psi\|_{L^2(0;T;H^1(\Omega_0))}.$$  

Similarly we estimate $I_2^-$. Thus, $|I_2^-(\varepsilon, \psi)| \leq C_2 \varepsilon \|\psi\|_{L^2(0;T;H^1(\Omega_0))}$.

Since the functions $\partial_{x_1} N^+, \partial_{x_2}^2 N^+$, $\partial_{x_2} N^-$, $\partial_{x_2}^2 N^-$ exponentially decrease as $|\xi_1| \to \infty$ (see (23), (24) and (26)), then from Lemma 3.1 in [6] we derive that

$$\forall \mu > 0 \ \exists C_3 > 0 \ \exists \varepsilon_0 > 0 \ \forall \varepsilon \in (0, \varepsilon_0): |I_2^-(\varepsilon, \psi)| \leq C_3 \varepsilon^{1-\mu} \|\psi\|_{L^2(0;T;H^1(\Omega_0))}.$$  

The integrals in $I_2^-(\varepsilon, \psi)$ are in fact over

$$(\text{supp}(\chi_0(r)) \cap \Omega_\varepsilon) \times (0, T) = \{x \in \Omega_\varepsilon: \delta_0/2 < |r - d_0| < \delta_0\} \times (0, T),$$

where, according to (22) and (25), the functions $N^+$, $\partial_{x_1} N^+$, $\partial_{x_1} N^-$, and $\partial_{x_2} N^-$ can be estimated by some constant $c_2$. Thus, $|I_2^-(\varepsilon, \psi)| \leq C_2 \varepsilon \|\psi\|_{L^2(0;T;H^1(\Omega_0))}$.

The integrals in $I_5^-$ are over $\{x \in \mathbb{R}^3: |r - d_0| < \delta_0\}$ and they can be estimated, extracting if necessary the exponentially decreasing part in the corresponding integrand and then using the Cauchy-Schwartz-Bunyakovskii inequality. Consider, for example, the integral

$$|\int_0^T \int_{G_\varepsilon^{(1)}} \chi_0 \partial_{x_2} N^+ \partial_{x_2} \psi dxdt| = \left| \int_0^T \int_{G_\varepsilon^{(1)}} \chi_0 (Z_1 - \tilde{Y}_1) \partial_{x_2} \psi dxdt \right| \leq c_3 \|\psi\|_{L^2(0;T;H^1(\Omega_0))} \left( \sqrt{\int_0^T \int_{G_\varepsilon^{(1)}} \chi_0 |Z_1 - \tilde{Y}_1|^2 dxdt} + \right.$$

$$+ \|\alpha_1^{(1)} \eta + \alpha_2^{(1)} (1 - \eta) + (\alpha_1^{(1)} - \alpha_2^{(1)}) \partial_{x_2} \eta\|_{L^2(G_\varepsilon^{(1)} \times (0, T))} + \right.\right.$$

$$\left. + \sqrt{\int_0^T \int_{G_\varepsilon^{(1)}} \chi_0 \left( h_1^{-1}(d_0) \xi_1 + (\xi_1 - \alpha_1^{(1)}) - (\xi_2 - \alpha_2^{(1)}) \right)^2 dxdt} \right) \leq$$

$$\leq c_4 \|\psi\|_{L^2(0;T;H^1(\Omega_0))} \sqrt{2\pi \lambda_0 \varepsilon} \|Z_1 - \tilde{Y}_1\|_{L^2(\Pi_1)} + \sqrt{|G_\varepsilon^{(1)}|} + \right.$$

$$\right. + \sqrt{2\pi \lambda_0 \varepsilon} \|h_1^{-1}(d_0) \xi_1 + (\xi_1 - \alpha_1^{(1)}) - (\xi_2 - \alpha_2^{(1)})\|_{L^2(\Pi_1)} + \right.$$

$$\right. + \sqrt{2\pi \lambda_0 \varepsilon} \|\eta(\xi_1 - \alpha_1^{(1)}) + (1 - \eta)(\xi_2 - h_1^{-1}(d_0) \xi_1 - \alpha_2^{(1)})\|_{L^2(\Pi_1)}.$$  

where $|G_\varepsilon^{(1)}|$ is the measure of $G_\varepsilon^{(1)}$. Relations (23), (24) and (26) show that the norms in the right-hand side of the last inequality are bounded in $\varepsilon$. Similarly we can estimate the rest of the integrals in $I_5^-(\varepsilon, \psi)$. As a result, we obtain $|I_5^-(\varepsilon, \psi)| \leq C_5 \varepsilon \|\psi\|_{L^2(0;T;H^1(\Omega_0))}$.

**Remark 2.** The constants $C_4$ and $C_5$ depend on

$$\sup_{x \in G_0^{(0)}, \ t \in (0, T)} \left| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} u_0^+(x, t) \right|, \ |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq 2, \ \alpha_k \geq 0, \ k \in \{1, 2, 3\}.$$
Extending homogenized problem (33) periodically in $x_2$ through the planes $\{x \in \mathbb{R}^3 : x_2 = 0\}$ and $\{x \in \mathbb{R}^3 : x_2 = 1\}$ and taking into account the assumptions for $f_0$ and $g_0$, by virtue of classical results on the smoothness of solutions to boundary-value problems we conclude that these quantities are bounded.

Since $f_0$ is smooth, one has $u_i^0 \in L^2(0, T; H^2(D_i)), i = 1, 2$. Consequently,

$$|I_6^0(\varepsilon, \psi)| \leq c_6\varepsilon \sum_{i=1}^{2} (\|u_i^0\|_{L^2(0, T; H^1(D_i))} + \|\partial_{x_2} u_i^0\|_{L^2(0, T; H^1(D_i))})\|\psi\|_{L^2(0, T; H^1(\Omega))} \leq C_6\varepsilon\|\psi\|_{L^2(0, T; H^1(\Omega))}.$$

In order to estimate $I_7^0$ we consider summand $I_7^0$ with $\alpha = 1$. Obviously, the module of the second integral in $I_7^0$ can be estimated by $c_6\varepsilon\|\psi\|_{L^2(0, T; H^1(\Omega))}$. Using (2), Taylor’s formula and the obvious equality

$$1 - \frac{1}{a} = \frac{a^2 - 1}{a^2 + a} \quad (a^2 + a \neq 0)$$

we derive that the absolute value of the sum of the first and third integrals in $I_7^0$ can be evaluated by

$$|I_7^0(\varepsilon, \psi)| = c_7\varepsilon \left| \frac{4^{-1}\varepsilon^3}{T} \int_0^T \int_{S^2} (\psi_{\varepsilon}^2) \frac{|h_2'(r)|^2 \partial_2(u_0^2 - \psi)}{1 + 4\varepsilon^2|h_2'(r)|^2 + \sqrt{1 + 4\varepsilon^2|h_2'(r)|^2}} d\sigma_x dt + \right|$$

$$+ c_7\varepsilon \left| \frac{T}{\varepsilon} \int_0^T \int_{S_2} (\psi_{\varepsilon}^2) \left( \frac{\partial_2(u_0^2 - \psi)}{1 + 4\varepsilon^2|h_2'(r)|^2 + \sqrt{1 + 4\varepsilon^2|h_2'(r)|^2}} \right) d\sigma_x dt \right|$$

$$=: J_1(\varepsilon, \psi) + J_2(\varepsilon, \psi) + J_3(\varepsilon, \psi).$$

With the help of (45) we obtain $J_1(\varepsilon, \psi) + J_2(\varepsilon, \psi) \leq C_8\varepsilon\|\psi\|_{L^2(0, T; H^1(\Omega))}$. Taking into account (2), properties of the trace operator and the fact that $f_0$ is smooth, we deduce $J_3(\varepsilon, \psi) \leq c_9\varepsilon\|\psi\|_{L^2(0, T; H^1(\Omega))}$. Thus, in the case when $\alpha = 1$ we have

$$|I_7^0(\varepsilon, \psi)| \leq C_{10}\varepsilon\|\psi\|_{L^2(0, T; H^1(\Omega))}.$$

In the case when $\alpha > 1$ by (45) we obtain $|I_7^0(\varepsilon, \psi)| \leq C_{11}\varepsilon^{\alpha - 1}\|\psi\|_{L^2(0, T; H^1(\Omega))}$. Similarly to $I_7^0(\varepsilon, \psi)$, we estimate $I_7^0(\varepsilon, \psi)$ and $I_7^0(\varepsilon, \psi)$. As a result, we get $|I_7^0(\varepsilon, \psi)| \leq C\varepsilon\|\psi\|_{L^2(0, T; H^1(\Omega))}$ and

$$|I_7^0(\varepsilon, \psi)| \leq C_9 \begin{cases} (\varepsilon + \|g_0 - g\|_{L^2(D_1 \times (0, T))})\|\psi\|_{L^2(0, T; H^1(\Omega))}, & \beta = 1, \\ \varepsilon^{\beta - 1}\|\psi\|_{L^2(0, T; H^1(\Omega))}, & \beta > 1. \end{cases}$$

Thus,

$$|F_\varepsilon(\psi)| \leq C_7(\|f_0 - f\|_{L^2(\Omega \times (0, T))} + \|g_0 - g\|_{L^2(\Omega \times (0, T))})\|\psi\|_{L^2(0, T; H^1(\Omega))} +$$

$$+ \varepsilon^{\beta - 1}\|g_0 - g\|_{L^2(\Omega \times (0, T))}\|\psi\|_{L^2(0, T; H^1(\Omega))},$$

where $\mu > 0$ is an arbitrary number. From the last estimate with the help of standard scheme we deduce inequality (36).
5. Discussion of the obtained results. As we can see from the obtained results, the homogenized problem (33) for problem (1) is a nonstandard boundary-value problems for multi-sheeted function $U_0$ in anisotropic Sobolev space $W_0^0$ (see Section ). This problem consists of three boundary-value problems (in domains $\Omega_0$ and $D_i$, $i \in \{1, 2\}$), connected with each other by the conjugation conditions (on $Q_0^{(0)}$).

The nonhomogeneous Robin boundary conditions on the lateral surfaces of the thin discs in problem (1) are transformed as $\varepsilon \rightarrow 0$ into new summands in the differential equations in domains $D_i$, $i \in \{1, 2\}$, in problem (33). These summands show us the influence of the perturbed parameters $\alpha$ and $\beta$. If $\alpha > 1$, then the summand $2\delta_{\alpha,1}\vartheta_1(u_0^2)$ vanishes. From physical point of view this means that the outer heat conduction coefficient is too small, and we can neglect this heat exchange. If $\beta > 1$, then summands $2\delta_{\beta,1}g_0$ vanish, which means that the temperature of the environment is too small, and we can consider it being equal to zero.

Functions $h_i$, $i \in \{1, 2\}$, which describe the relative thickness of the thin discs from the $i$-th level, are transformed into the coefficients of the differential equations in domains $D_i$, respectively. The variable $x_2$ is involved as a parameter in the boundary-value problems in $D_i$, $i \in \{1, 2\}$, which shows us the influence of the type of thick junction $\Omega_{\varepsilon}$ on the asymptotic behavior of solution $u_\varepsilon$.

From results proved in the present paper it follows that for applied problems in thick junctions we can use the homogenized problem (33), which is simpler, instead of the initial problem (1) with sufficient plausibility.

Acknowledges. The author is grateful to professor T. A. Mel’nyk for the statement of problem, attention during it’s solving and discussion of the obtained results.

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