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ASYMPTOTIC APPROXIMATION OF A SOLUTION OF A QUASILINEAR PARABOLIC BOUNDARY-VALUE PROBLEM IN A TWO-LEVEL THICK JUNCTION OF TYPE 3:2:2

D. Yu. Sadovyj. *Asymptotic approximation of a solution of a quasilinear parabolic boundary-value problem in a two-level thick junction of type 3:2:2*, Mat. Stud. **38** (2012), 51–67.

We consider a quasilinear parabolic boundary-value problem in a two-level thick junction Ω_ε of type $3:2:2$, which is the union of a cylinder Ω_0 and a large number of ε-periodically situated thin discs with variable thickness. Different Robin boundary conditions with perturbed parameters are given on the surfaces of the thin discs. The leading terms of the asymptotic expansion are constructed and the corresponding estimate in Sobolev space is obtained.

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Рассматривается квазилинейная параболическая краевая задача в двухуровневом густом соединении Ω_ε типа $3:2:2$, которое состоит из цилиндра Ω_0 и большого количества ε-периодически присоединенных тонких дисков переменной толщины. На поверхностях тонких дисков из обеих уровней задаются разные краевые условия третьего рода с возмущенными коэффициентами. Строятся главные члены асимптотического разложения и доказывается соответствующая оценка в пространстве Соболева.

1. Introduction. A thick junction of type $m:k:d$ is a union of some domain, which is called the junction's body, and a large number of ε-periodically alternating thin domains, which are attached to some manifold (the joint zone) on the boundary of the junction's body. The small parameter ε characterizes the distance between neighboring thin domains and their thickness. The type $m:k:d$ of a thick junction refers, respectively, to the limiting dimensions (as $\varepsilon \to 0$) of the junction's body, the joint zone and each of the attached thin domains. The subject of the investigation of boundary-value problems in thick junctions is the asymptotic behavior of solutions of such problems as $\varepsilon \to 0$, i.e. as the number of the attached thin domains infinitely increases as well as their thickness tends to zero.

The first researches in this direction were carried out in [10, 11, 15], where convergence theorems for the Green function of the Neumann problem for the Helmholz equation in the junction's body were proved. In these papers either the assumption about the convergence of certain components of the boundary-value problem was made, or explicit representations of

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certain quantities were used, which was possible under certain configurations of the junction's body (the half-space). In [17]–[21], [27] thick junctions were classified, asymptotic methods for the investigation of main boundary-value problems of mathematical physics in thick junctions of different types were developed, convergence theorems were proved, the first terms of asymptotic expansions were constructed, and the corresponding estimates were proved. It was shown that qualitative properties of solutions essentially depend on the junction's type and the conditions given on the boundaries of the attached thin domains (see also [1, 3, 23]).

As an extension of the investigation, in papers [7, 8, 16] thick junctions of more complicated geometric structure were considered, namely multi-level thick junctions. A multi-level thick junction is a thick junction in which thin domains are divided into finitely many levels depending on their geometric structure and boundary conditions imposed on their surfaces. Besides, thin domains from each level ε-periodically alternate along the joint zone. In these papers linear boundary-value problems in thick junctions of types $2:1:1$ and $3:2:1$ were considered. Moreover, there a new qualitative difference in the asymptotic behavior of solutions of boundary-value problems in multi-level thick junctions was noticed, namely the "multi-phase" effect in the domain that is filled up simultaneously by the thin domains from different levels.

The successful applying in nanotechnology and microelectronics of constructions, which have form of thick junctions (see [12]–[14]), has lead to effective studying of boundary-value problems in thick junctions of various types and more complicated structure (see also [1]–[4], [22, 23]).

In the present paper we consider quasilinear parabolic boundary-value problem in a two-level thick junction of type $3:2:2$, which consists of a cylinder Ω_0 and a large number of thin annular discs with varying thickness, which are ε-periodically attached to Ω_0. Different nonhomogeneous Robin boundary conditions are given on the surfaces of the thin discs from various levels. The leading terms of the asymptotic expansion for a solution of this problem are constructed and the asymptotic estimate in Sobolev space is proved. It should be noted that linear parabolic boundary-value problems in thick junctions of various types were investigated in [5, 8]. Quasilinear parabolic problems in a two-level thick junction of type $3:2:2$ were considered in [24, 25], where only convergence theorems were proved.

2. Statement of the problem. Let $0 < d_0 < d_2 \le d_1$ and $0 < b_2 < b_1 < 1$, and let $h_i\colon [d_0, d_i] \to (0,1)$, $i \in \{1,2\}$ be piecewise smooth functions. Suppose that functions h_i satisfy the following conditions

$$0 < b_i - \frac{h_i(s)}{2},\ b_i + \frac{h_i(s)}{2} < 1\ \forall s \in [d_0, d_i],\ i \in \{1,2\} \quad b_2 + \frac{h_2(s)}{2} < b_1 - \frac{h_1(s)}{2}\ \forall s \in [d_0, d_2].$$

These inequalities imply that for all $s \in [d_0, d_i]$ the intervals $I_i(s) := (b_i - h_i(s)/2, b_i + h_i(s)/2)$, $i \in \{1,2\}$, belong to the interval $(0,1)$, having no common points and do not adjoin.

We additionally assume that the functions h_1, h_2 are constant in some neighborhood of d_0, i.e. there exists $\delta > 0$ such that $h_i(s) = h_i(d_0)$ for all $s \in [d_0, d_0 + \delta]$, $i \in \{1,2\}$.

Consider a model thick junction Ω_ε of type $3:2:2$ (see Fig. 1) that consists of the cylinder $\Omega_0 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3\colon 0 < x_2 < l,\ r := \sqrt{x_1^2 + x_3^2} < d_0\}$ and $2N$ thin annular discs

$$G_\varepsilon^{(1)}(j) = \{x \in \mathbb{R}^3\colon |x_2 - \varepsilon(j + b_1)| < \varepsilon h_1(r)/2,\ d_0 \le r < d_1\},$$
$$G_\varepsilon^{(2)}(j) = \{x \in \mathbb{R}^3\colon |x_2 - \varepsilon(j + b_2)| < \varepsilon h_2(r)/2,\ d_0 \le r < d_2\},$$

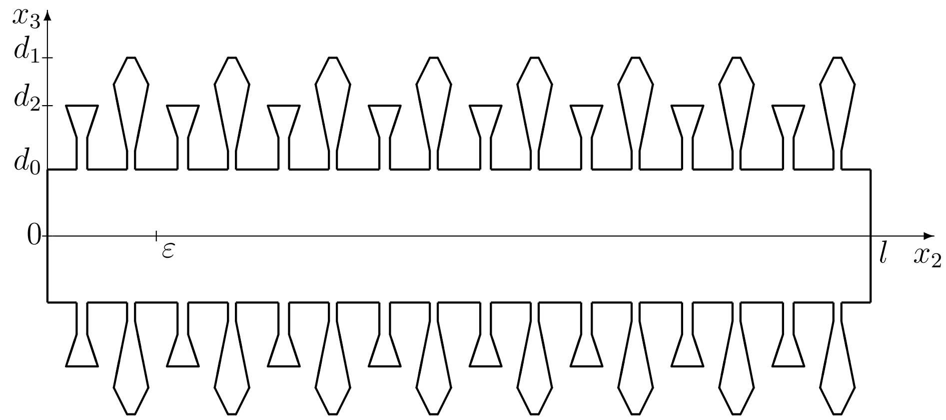

Fig. 1: The cross-section of thick junction Ω_ε of type $3:2:2$ ($N=8$).

where $j \in \{0,1,\dots,N-1\}$, $\varepsilon = l/N$, i.e. $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon$, $G_\varepsilon = G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$, $G_\varepsilon^{(1)} = \cup_{j=0}^{N-1} G_\varepsilon^{(1)}(j)$, $G_\varepsilon^{(2)} = \cup_{j=0}^{N-1} G_\varepsilon^{(2)}(j)$. Here N is a large integer. Therefore, ε is a small parameter, which characterizes the distance between neighboring thin discs and their thickness.

Denote by $S_\varepsilon^{(1)}$ and $S_\varepsilon^{(2)}$ the union of the lateral surfaces of the thin discs of the first and the second levels, respectively, and by S the union of the bases of the cylinder Ω_0, i.e.

$$S_\varepsilon^{(i)} := \{x \in \partial G_\varepsilon^{(i)} : |x_2 - \varepsilon(j+b_i)| = \varepsilon h_i(r)/2,\ j \in \{0,1,\dots,N-1\},\ r \in (d_0,d_i)\},\ i \in \{1,2\},$$
$$S^- = \{x \in \partial\Omega_0 : x_2 = 0\}, \quad S^+ = \{x \in \partial\Omega_0 : x_2 = l\}, \quad S = S^+ \cup S^-.$$

We introduce the following notation

$$\overline{\Omega}_i = \overline{\Omega}_0 \cup \overline{D}_i,\ D_i = \{x \in \mathbb{R}^3 : 0 < x_2 < l,\ d_0 < r < d_i\},\ i \in \{1,2\},$$
$$Q_0^{(i)} = \{x \in \partial\Omega_i : r = d_i\},\ i \in \{0,1,2\},\ Q_\varepsilon^{(i)} = \{x \in \partial G_\varepsilon^{(i)} : r = d_i\},\ i \in \{1,2\},$$
$$\Upsilon_\varepsilon^{(i)} = S_\varepsilon^{(i)} \cup Q_\varepsilon^{(i)},\ \Theta_\varepsilon^{(i)} = G_\varepsilon^{(i)} \cap \partial\Omega_0,\ i \in \{1,2\},\ \Theta_\varepsilon = \Theta_\varepsilon^{(1)} \cup \Theta_\varepsilon^{(2)},\ Q_\varepsilon^{(0)} = Q_0^{(0)} \setminus \Theta_\varepsilon.$$

In the thick junction Ω_ε we consider the quasilinear parabolic boundary-value problem

$$\begin{cases} \partial_t u_\varepsilon(x,t) - \Delta_x u_\varepsilon(x,t) + \vartheta_0(u_\varepsilon(x,t)) = f_\varepsilon(x,t), & (x,t) \in \Omega_\varepsilon \times (0,T), \\ \partial_\nu u_\varepsilon(x,t) + \varepsilon\vartheta_1(u_\varepsilon(x,t)) = \varepsilon^\beta g_\varepsilon(x,t), & (x,t) \in S_\varepsilon^{(1)} \times (0,T), \\ \partial_\nu u_\varepsilon(x,t) + \vartheta_1(u_\varepsilon(x,t)) = 0, & (x,t) \in Q_\varepsilon^{(1)} \times (0,T), \\ \partial_\nu u_\varepsilon(x,t) + \varepsilon^\alpha\vartheta_2(u_\varepsilon(x,t)) = \varepsilon^\beta g_\varepsilon(x,t), & (x,t) \in \Upsilon_\varepsilon^{(2)} \times (0,T), \\ \partial_\nu u_\varepsilon(x,t) = 0, & (x,t) \in Q_\varepsilon^{(0)} \times (0,T), \\ \partial_{x_2}^p u_\varepsilon(x,t)|_{S^-} = \partial_{x_2}^p u_\varepsilon(x,t)|_{S^+}, & p = 0,1,\ t \in (0,T), \\ [u_\varepsilon]|_{r=d_0} = [\partial_r u_\varepsilon]|_{r=d_0} = 0, & (x,t) \in \Theta_\varepsilon \times (0,T), \\ u(x,0) = 0, & x \in \Omega_\varepsilon. \end{cases} \tag{1}$$

Here $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; $\alpha,\ \beta \geq 1$ are parameters; the square brackets denote the jump of enclosed quantities. For the right-hand sides of problem (1) we assume that $f_\varepsilon \in L^2(\Omega_\varepsilon \times (0,T))$, $g_\varepsilon \in L^2(D_1 \times (0,T))$, there exists a weak derivative $\partial_{x_2} g_\varepsilon \in L^2(D_1 \times (0,T))$, and

$$\exists C_0 > 0 \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in (0,\varepsilon_0) : \|g_\varepsilon\|_{L^2(D_1\times(0,T))} + \|\partial_{x_2} g_\varepsilon\|_{L^2(D_1\times(0,T))} < C_0.$$

The functions ϑ_i are Lipschitz-continuous (that is to say $\vartheta_i \in W^{1,\infty}_{\rm loc}(\mathbb{R})$) and

$$\exists c_1, c_2 > 0\colon c_1 \le \vartheta_i'(s) \le c_2 \quad \text{for a.e. } s \in \mathbb{R},\ i \in \{0,1,2\}. \tag{2}$$

Consider the spaces $H_\varepsilon = \{\varphi \in H^1(\Omega_\varepsilon)\colon \varphi|_{S^-} = \varphi|_{S^+}\}$ and $W_\varepsilon = \{\varphi \in L^2(0,T;H_\varepsilon)\colon \partial_t\varphi{:=}\varphi' \in L^2(0,T;H^*_\varepsilon)\}$.

It is known (see, for instance, [9, §1 ch. IV]) that $W_\varepsilon \subset C([0,T];L^2(\Omega_\varepsilon))$.

A function $u_\varepsilon \in L^2(0,T;H_\varepsilon)$ is a weak solution of problem (1) if for every function $\varphi \in W_\varepsilon$ the following integral identity holds (see, e.g., [9, ch. IV])

$$\begin{aligned}
&\int_{\Omega_\varepsilon} u_\varepsilon(x,T)\varphi(x,T)dx - \int_0^T\int_{\Omega_\varepsilon} u_\varepsilon\partial_t\varphi dxdt + \int_0^T\int_{\Omega_\varepsilon}(\nabla_x u_\varepsilon\cdot\nabla_x\varphi + \vartheta_0(u_\varepsilon)\varphi)dxdt\\
&+\varepsilon\int_0^T\int_{S^{(1)}_\varepsilon}\vartheta_1(u_\varepsilon)\varphi d\sigma_x dt + \int_0^T\int_{Q^{(1)}_\varepsilon}\vartheta_1(u_\varepsilon)\varphi d\sigma_x dt + \varepsilon^\alpha\int_0^T\int_{\Upsilon^{(2)}_\varepsilon}\vartheta_2(u_\varepsilon)\varphi d\sigma_x dt\\
&\qquad = \int_0^T\int_{\Omega_\varepsilon} f_\varepsilon\varphi dxdt + \varepsilon^\beta\int_0^T\int_{S^{(1)}_\varepsilon\cup\Upsilon^{(2)}_\varepsilon} g_\varepsilon\varphi d\sigma_x dt.
\end{aligned} \tag{3}$$

Similarly as in [26] we can show that for any fixed $\varepsilon > 0$ there exists a unique weak solution of problem (1).

The aim is to study the asymptotic behavior of the solution of problem (1) as $\varepsilon \to 0$, i.e. as the number of the attached thin discs infinitely increases and their thickness tends to zero.

3. Formal Asymptotic Expansions for the Solution. In this section only, for formal calculations we assume that the functions $f_\varepsilon, g_\varepsilon$ do not depend on ε, i.e. $f_\varepsilon = f_0$ in $\Omega_1\times(0,T)$ and $g_\varepsilon = g_0$ on $D_1\times(0,T)$, and they are smooth in $\overline{\Omega}_1\times[0,T]$ and $\overline{D}_1\times[0,T]$, respectively.

3.1. Outer Expansions. We seek the leading terms of the asymptotic expansion for solution u_ε, restricted to Ω_0, in the form

$$u_\varepsilon(x,t) \approx u_0^+(x,t) + \sum_{k\ge1}\varepsilon^k u_k^+(x,t), \quad (x,t)\in\Omega_0\times(0,T), \tag{4}$$

and, restricted to the thin discs $G^{(i)}_\varepsilon(j),\ j\in\{0,1,\dots,N-1\}$, in the form

$$u_\varepsilon(x,t) \approx u_0^{i,-}(x,t) + \sum_{k\ge1}\varepsilon^k u_k^{i,-}(x,\xi_2 - j,t), \quad (x,t)\in G^{(i)}_\varepsilon(j)\times(0,T), \quad i\in\{1,2\}, \tag{5}$$

where $\xi_2 = x_2/\varepsilon$.

Expansions (4) and (5) are usually called *outer expansions*.

With the help of Taylor's formula we get

$$\vartheta_0(u_\varepsilon(x,t)) = \vartheta_0(u_0^+(x,t)) + \mathcal{O}(\varepsilon), \quad \varepsilon\to0,\ (x,t)\in\Omega_0\times(0,T). \tag{6}$$

Plugging the series (4) into the first equation of problem (1), the boundary conditions on S, and the initial condition, using (6) and collecting coefficients of the same powers of ε, we get the following relations for function u_0^+

$$\begin{cases}
\partial_t u_0^+(x,t) - \Delta_x u_0^+(x,t) + \vartheta_0(u_0^+(x,t)) = f_0(x,t), & (x,t)\in\Omega_0\times(0,T),\\
\partial^p_{x_2}u_0^+(x,t)|_{S^-} = \partial^p_{x_2}u_0^+(x,t)|_{S^+}, & p\in\{0,1\},\ t\in(0,T),\\
u_0^+(x,0) = 0, & x\in\Omega_0.
\end{cases}$$

Now let us find the limit relations in domains $D_i,\ i \in \{1,2\}$, which are filled up by the thin discs from i-th level as ε tends to zero. Assuming for a moment that functions $u_k^{i,-}$ are smooth, we write their Taylor series with respect to x_2 at the point $\varepsilon(j+b_i)$ and pass to the "rapid" variable $\xi_2 = x_2/\varepsilon$. Then (5) takes the form

$$u_\varepsilon(x,t) \approx u_0^{i,-}(x_1, \varepsilon(j+b_i), x_3, t) + \sum_{k\geq 1} \varepsilon^k V_{\varepsilon,k}^{i,j}(\tilde{x}, \xi_2, t), \quad (x,t) \in G_\varepsilon^{(i)}(j) \times (0,T), \tag{7}$$

where $\tilde{x}:=(x_1, x_3)$, and

$$V_{\varepsilon,k}^{i,j}(\tilde{x}, \xi_2, t) = \sum_{m=0}^{k-1} \frac{(\xi_2 - j - b_i)^m}{m!} \frac{\partial^m u_{k-m}^{i,-}}{\partial x_2^m}(x_1, \varepsilon(j+b_i), x_3, \xi_2 - j, t) + \\ + \frac{(\xi_2 - j - b_i)^k}{k!} \frac{\partial^k u_0^{i,-}}{\partial x_2^k}(x_1, \varepsilon(j+b_i), x_3, t). \tag{8}$$

Further we will indicate arguments of functions only if their absence may cause confusion.

The outward unit normal to the lateral surfaces of the thin discs except a set of zero measure is as follows

$$\nu_\varepsilon(x) = \frac{1}{\sqrt{1 + 4^{-1}\varepsilon^2 |h_i'(r)|^2}} \left(-\frac{\varepsilon h_i'(r) x_1}{2r}, \pm 1, -\frac{\varepsilon h_i'(r) x_3}{2r} \right), \quad x \in S_\varepsilon^{(i)},\ i \in \{1,2\}, \tag{9}$$

where "+" and "−" refer, respectively, to the left and the right parts of the lateral surface of each thin disc. Obviously, $(1 + \varepsilon^2 4^{-1} |h_i'(r)|^2)^{-\frac{1}{2}} = 1 + \mathcal{O}(\varepsilon^2),\ \varepsilon \to 0.$

Again by Taylor's formula we obtain

$$\vartheta_0(u_\varepsilon(x,t)) = \vartheta_0(u_0^{i,-}(x,t)|_{x_2 = \varepsilon(j+b_i)}) + \mathcal{O}(\varepsilon), \quad \varepsilon \to 0,\ (x,t) \in G_\varepsilon^{(i)} \times (0,T). \tag{10}$$

Let us put (7) into (1) instead of u_ε. Taking into account (9), (10) and that the Laplace operator in the variables $(\tilde{x}, \xi_2)$ has the form $\Delta_x = \Delta_{\tilde{x}} + \varepsilon^{-2} \frac{\partial^2}{\partial \xi_2^2}$ and collecting coefficients of the same powers of ε, we arrive at one-dimensional boundary-value problems with respect to ξ_2 for functions $V_{\varepsilon,k}^{i,j}$.

Problems for $V_{\varepsilon,1}^{i,j}$ read

$$\begin{cases} \partial^2_{\xi_2\xi_2} V_{\varepsilon,1}^{i,j} = 0, & \xi_2 \in I_{h_i(r)}(j) := (-\frac{h_i(r)}{2} + j + b_i, \frac{h_i(r)}{2} + j + b_i), \\ \partial_{\xi_2} V_{\varepsilon,1}^{i,j} = 0, & \xi_2 = \pm \frac{h_i(r)}{2} + j + b_i, \end{cases} \quad i \in \{1,2\}, \tag{11}$$

where $\partial_{\xi_2} = \frac{\partial}{\partial \xi_2},\ \partial^2_{\xi_2\xi_2} = \frac{\partial^2}{\partial \xi_2^2}$. Here the variables $\tilde{x}, t$ are regarded as parameters.

It follows from (11) that $V_{\varepsilon,1}^{i,j}$ do not depend on ξ_2. Therefore, $V_{\varepsilon,1}^{i,j}$ are equal to some functions $\varphi_1^{(i)}(x_1, \varepsilon(j+b_i), x_3, t),\ (x,t) \in G_\varepsilon^{(i)}(j) \times (0,T)$, which will be defined later. Then, due to (8) we have

$$u_1^{i,-}(x_1, \varepsilon(j+b_i), x_3, \xi_2 - j, t) = \varphi_1^{(i)}(x_1, \varepsilon(j+b_i), x_3, t) - \\ -(\xi_2 - j - b_i) \partial_{x_2} u_0^{i,-}(x_1, \varepsilon(j+b_i), x_3, t), \quad (x,t) \in G_\varepsilon^{(i)}(j) \times (0,T). \tag{12}$$

Boundary-value problems for $V_{\varepsilon,2}^{1,j}$ and $V_{\varepsilon,2}^{2,j}$ have the view

$$\begin{cases} -\partial^2_{\xi_2\xi_2} V_{\varepsilon,2}^{1,j} = (-\partial_t u_0^{1,-} + \Delta_{\tilde{x}} u_0^{1,-} - \vartheta_0(u_0^{1,-}) + f_0)|_{x_2 = \varepsilon(j+b_1)}, & \xi_2 \in I_{h_1(r)}(j), \\ \pm \partial_{\xi_2} V_{\varepsilon,2}^{1,j} = (2^{-1} \nabla_{\tilde{x}} h_1 \cdot \nabla_{\tilde{x}} u_0^{1,-} - \vartheta_1(u_0^{1,-}) + \delta_{\beta,1} g_0)|_{x_2 = \varepsilon(j+b_1)}, & \xi_2 = \pm \frac{h_1(r)}{2} + j + b_1, \end{cases} \tag{13}$$

$$\begin{cases} -\partial^2_{\xi_2\xi_2} V^{2,j}_{\varepsilon,2} = (-\partial_t u_0^{2,-} + \Delta_{\tilde{x}} u_0^{2,-} - \vartheta_0(u_0^{2,-}) + f_0)|_{x_2=\varepsilon(j+b_2)}, & \xi_2 \in I_{h_2(r)}(j), \\ \pm\partial_{\xi_2} V^{2,j}_{\varepsilon,2} = (2^{-1}\nabla_{\tilde{x}} h_2 \nabla_{\tilde{x}} u_0^{2,-} - \delta_{\alpha,1}\vartheta_2(u_0^{2,-}) + \delta_{\beta,1} g_0)|_{x_2=\varepsilon(j+b_2)}, & \xi_2 = \pm\frac{h_2(r)}{2} + j + b_2, \end{cases} \tag{14}$$

respectively, where $\delta_{k,n}$ is the Kronecker symbol.

The solvability conditions for problems (13) and (14) read

$$h_1\partial_t u_0^{1,-} - \mathrm{div}_{\tilde{x}}(h_1\nabla_{\tilde{x}} u_0^{1,-}) + h_1\vartheta_0(u_0^{1,-}) + 2\vartheta_1(u_0^{1,-}) = h_1 f_0 + 2\delta_{\beta,1} g_0,$$
$$x_2 = \varepsilon(j+b_1),\ r \in (d_0, d_1),\ t \in (0,T), \tag{15}$$

$$h_2\partial_t u_0^{2,-} - \mathrm{div}_{\tilde{x}}(h_2\nabla_{\tilde{x}} u_0^{2,-}) + h_2\vartheta_0(u_0^{2,-}) + 2\delta_{\alpha,1}\vartheta_2(u_0^{2,-}) = h_2 f_0 + 2\delta_{\beta,1} g_0,$$
$$x_2 = \varepsilon(j+b_2),\ r \in (d_0, d_2),\ t \in (0,T), \tag{16}$$

respectively.

Putting (7) into the Robin boundary conditions on $Q^{(i)}_\varepsilon$, we get

$$\partial_r u_0^{1,-} + \vartheta_1(u_0^{1,-}) = 0, \quad (x,t) \in Q^{(1)}_\varepsilon \times (0,T),\ x_2 = \varepsilon(j+b_1), \tag{17}$$

$$\partial_r u_0^{2,-} = 0, \quad (x,t) \in Q^{(2)}_\varepsilon \times (0,T),\ x_2 = \varepsilon(j+b_2). \tag{18}$$

Plugging (7) into the initial condition of problem (1), we find that

$$u_0^{i,-}(x,0) = 0, \quad x \in G^{(i)}_\varepsilon,\ x_2 = \varepsilon(j+b_i),\ i \in \{1,2\}. \tag{19}$$

In order to find conditions in the joint zone $Q^{(0)}_0$ we use the method of matched asymptotic expansions for outer expansions (4), (7) and an inner expansion which will be constructed in the next subsection.

3.2. Inner Expansion. In a neighborhood of the joint zone $Q^{(0)}_0$ we introduce the "rapid" coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = -(r-d_0)/\varepsilon$ and $\xi_2 = x_2/\varepsilon$. Here $(r, x_2, \theta) \in \mathbb{R}^3$ are the cylindric coordinates: $r = \sqrt{x_1^2 + x_3^2}$, $\tan(\theta) = x_3/x_1$. The Laplace operator in the coordinates (ξ_1, ξ_2, θ) has the form

$$\Delta_x = \varepsilon^{-2}\Delta_\xi - \varepsilon^{-1}\frac{1}{d_0 - \varepsilon\xi_1}\frac{\partial}{\partial\xi_1} + \frac{1}{(d_0-\varepsilon\xi_1)^2}\frac{\partial^2}{\partial\theta^2}. \tag{20}$$

We seek the leading terms of the inner expansion in a neighborhood of $Q^{(0)}_0$ in the form

$$u_\varepsilon(x,t) \approx u_0^+(x,t)|_{r=d_0} + \varepsilon\big(Z_1(\xi)\partial_{x_2} u_0^+(x,t)|_{r=d_0} -$$
$$-\big(\eta(x_2,t)\Xi_1(\xi) + (1-\eta(x_2,t))\Xi_2(\xi)\big)\partial_r u_0^+(x,t)|_{r=d_0}\big) + \ldots, \tag{21}$$

where $Z_1,\ \Xi_1,\ \Xi_2$ are functions, which are 1-periodic with respect to ξ_2 and defined in the union $\Pi{:=}\Pi^+ \cup \Pi_1^- \cup \Pi_2^-$ of the semiinfinite strips $\Pi^+ = \{\xi \in \mathbb{R}^2 \colon \xi_1 > 0,\ \xi_2 \in (0,1)\}$, $\Pi_i^- = \{\xi \in \mathbb{R}^2 \colon \xi_1 \le 0,\ \xi_2 \in I_i(d_0)\}$, $i \in \{1,2\}$, (see definition of $I_i(d_0)$), η is a function, which will be defined from matching conditions.

Putting (21) into the differential equation of problem (1) with regard to (20) and into the corresponding boundary conditions and collecting coefficients of the same powers of ε, we get the junction-layer problems for $Z_1,\ \Xi_1,\ \Xi_2$. The functions Ξ_1 and Ξ_2 are solutions of the following homogeneous problem

$$\begin{cases} -\Delta_\xi \Xi = 0, & \text{in } \Pi, \\ \partial_{\xi_2}\Xi = 0, & \text{on } (\partial\Pi_1^- \cup \partial\Pi_2^-) \cap \{\xi \in \mathbb{R}^2 \colon \xi_1 < 0\}, \\ \partial_{\xi_1}\Xi = 0, & \text{on } \partial\Pi \cap \{\xi \in \mathbb{R}^2 \colon \xi_1 = 0\}, \\ \partial^p_{\xi_2}\Xi|_{\xi_2=0} = \partial^p_{\xi_2}\Xi|_{\xi_2=1}, & p \in \{0,1\},\ \xi_1 > 0. \end{cases} \tag{22}$$

Main asymptotic relations for $\Xi_1,\ \Xi_2$ can be obtained from general results on the asymptotic behavior of solutions of elliptic problems in domains with different exits to infinity (see, for instance, [29]). However, for the domain Π, we can define more exactly the asymptotic relations for junction-layer solutions $\Xi_1,\ \Xi_2$ in the same way as in [18, 19].

Proposition 1. *There exist two solutions $\Xi_1,\ \Xi_2 \in H^1_{\sharp,\mathrm{loc}}(\Pi)$ to problem (22), which have the following differentiable asymptotics*

$$\Xi_1 = \begin{cases} \xi_1 + \mathcal{O}(\exp(-2\pi\xi_1)), & \xi_1 \to +\infty,\ \xi \in \Pi^+, \\ \alpha_1^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_1^-, \\ h_2^{-1}(d_0)\xi_1 + \alpha_1^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_2^-, \end{cases} \tag{23}$$

$$\Xi_2 = \begin{cases} \xi_1 + \mathcal{O}(\exp(-2\pi\xi_1)), & \xi_1 \to +\infty,\ \xi \in \Pi^+, \\ h_1^{-1}(d_0)\xi_1 + \alpha_2^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_1^-, \\ \alpha_2^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_2^-. \end{cases} \tag{24}$$

Here $H^1_{\sharp,\mathrm{loc}}(\Pi) = \{u\colon Pi \to \mathbb{R}\colon u(\xi_1,0) = u(\xi_1,1)$ for any $\xi_1 > 0,\ u \in H^1(\Pi_R)$ for any $R > 0\}$, $\Pi_R = \{\xi \in \Pi\colon -R < \xi_1 < R\}$; $\alpha_1^{(i)},\ \alpha_2^{(i)},\ i \in \{1,2\}$, are some constants.

Any other solution of problem (22), which has a polynomial growth at infinity, can be represented as a linear combination $c_0 + c_1\Xi_1 + c_2\Xi_2$.

The function Z_1 is a solution of the following problem

$$\begin{cases} -\Delta_\xi Z = 0, & \text{in } \Pi, \\ \partial_{\xi_2} Z = -1, & \text{on } (\partial\Pi_1^- \cup \partial\Pi_2^-) \cap \{\xi \in \mathbb{R}^2\colon \xi_1 < 0\}, \\ \partial_{\xi_1} Z = 0, & \text{on } \partial\Pi \cap \{\xi \in \mathbb{R}^2\colon \xi_1 = 0\}, \\ \partial^p_{\xi_2} Z|_{\xi_2=0} = \partial^p_{\xi_2} Z|_{\xi_2=1}, & p = i \in \{0,1\},\ \xi_1 > 0. \end{cases} \tag{25}$$

Similarly to [18, 19, 28] it is easy to verify that there exists a unique solution $Z_1 \in H^1_{\sharp,\mathrm{loc}}(\Pi)$ with the following asymptotics

$$Z = \begin{cases} \mathcal{O}(\exp(-2\pi\xi_1)), & \xi_1 \to +\infty,\ \xi \in \Pi^+, \\ -\xi_2 + b_1 + \alpha_3^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_1^-, \\ -\xi_2 + b_2 + \alpha_3^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}(d_0)\xi_1)), & \xi_1 \to -\infty,\ \xi \in \Pi_2^-. \end{cases} \tag{26}$$

Now let us verify the matching conditions for outer expansions (4), (5) and inner expansion (21), namely, the leading terms of the asymptotics of the outer expansions as $\xi_1 \to \pm 0$ must coincide with the leading terms of the asymptotics of the inner expansion as $\xi_1 \to \pm\infty$. Near the point $(x_1, \varepsilon(j+b_i), x_3) \in Q_0^{(0)}$ for any fixed $t \in (0,T)$ the function u_0^+ has the following asymptotics

$$u_0^+(x,t) \approx u_0^+(x_1,\varepsilon(j+b_i),x_3,t)|_{r=d_0} + \varepsilon(\xi_2 - j - b_i)\partial_{x_2} u_0^+(x_1,\varepsilon(j+b_i),x_3,t)|_{r=d_0} - \\ -\varepsilon\xi_1\partial_r u_0^+(x_1,\varepsilon(j+b_i),x_3,t)|_{r=d_0} + \ldots \quad \text{as } \xi_1 \to 0+,\ (x,t) \in \Omega_0 \times (0,T).$$

Taking into account the asymptotics of Z_1, Ξ_1 and Ξ_2 as $\xi_1 \to +\infty$, we see that the matching conditions are satisfied for expansions (4) and (21).

For any $t \in (0,T)$ the asymptotics of (5) in the neighborhood of $(x_1, \varepsilon(j+b_i), x_3) \in Q_0^{(0)}$ are equal to

$$\begin{gathered} u_0^{i,-}(x_1, \varepsilon(j+b_i), x_3, t)|_{r=d_0} + \varepsilon\big(\varphi_1^{(i)}(x_1, \varepsilon(j+b_i), x_3, t)|_{r=d_0} - \\ -\xi_1 \partial_r u_0^{i,-}(x_1, \varepsilon(j+b_i), x_3, t)|_{r=d_0}\big) + \dots \text{ as } \xi_1 \to 0-, \ (x,t) \in G_\varepsilon^{(i)}(j) \times (0,T), \ i \in \{1,2\}. \end{gathered} \tag{27}$$

It follows from (23), (24) and (26) that the first terms of the asymptotics of (21) in the neighborhood of $(x_1, \varepsilon(j+b_i), x_3) \in Q_0^{(0)}$ are

$$\begin{gathered} u_0^{+}(x_1, \varepsilon(j+b_1), x_3, t)|_{r=d_0} + \varepsilon\big(\alpha_3^{(1)} \partial_{x_2} u_0^{+}(x_1, \varepsilon(j+b_1), x_3, t)|_{r=d_0} - \\ -\big(\alpha_1^{(1)} \eta(\varepsilon(j+b_1), t) + (h_1^{-1}(d_0)\xi_1 + \alpha_2^{(1)})(1 - \eta(\varepsilon(j+b_1), t))\big) \partial_r u_0^{+}(x_1, \varepsilon(j+b_1), x_3, t)|_{r=d_0}\big) \\ \text{as } \xi_1 \to -\infty, \ (x,t) \in G_\varepsilon^{(1)}(j) \times (0,T), \end{gathered} \tag{28}$$

$$\begin{gathered} u_0^{+}(x_1, \varepsilon(j+b_2), x_3, t)|_{r=d_0} + \varepsilon\big(\alpha_3^{(2)} \partial_{x_2} u_0^{+}(x_1, \varepsilon(j+b_2), x_3, t)|_{r=d_0} - \\ -\big((h_2^{-1}(d_0)\xi_1 + \alpha_1^{(2)}) \eta(\varepsilon(j+b_2), t) + \alpha_2^{(2)}(1 - \eta(\varepsilon(j+b_2), t))\big) \partial_r u_0^{+}(x_1, \varepsilon(j+b_2), x_3, t)|_{r=d_0}\big) \\ \text{as } \xi_1 \to -\infty, \ (x,t) \in G_\varepsilon^{(2)}(j) \times (0,T). \end{gathered} \tag{29}$$

Comparing the first terms of (27), (28) with (29), we get

$$u_0^{+}(x,t) = u_0^{i,-}(x,t), \quad (x,t) \in Q_0^{(0)} \times (0,T), \ x_2 = \varepsilon(j+b_i), \ i \in \{1,2\}. \tag{30}$$

Comparing the second terms of (27), (28) with (29), we find that

$$\varphi_1^{(i)}(x,t) = \alpha_3^{(i)} \partial_{x_2} u_0^{+}(x,t), \quad (x,t) \in Q_0^{(0)} \times (0,T), \ x_2 = \varepsilon(j+b_i), \ i \in \{1,2\}, \tag{31}$$

and

$$\begin{aligned} (1-\eta) h_1^{-1}(d_0) \partial_r u_0^{+}(x,t) &= \partial_r u_0^{1,-}(x,t), \quad (x,t) \in Q_0^{(0)} \times (0,T), \ x_2 = \varepsilon(j+b_1), \\ \eta h_2^{-1}(d_0) \partial_r u_0^{+}(x,t) &= \partial_r u_0^{2,-}(x,t), \quad (x,t) \in Q_0^{(0)} \times (0,T), \ x_2 = \varepsilon(j+b_2). \end{aligned} \tag{32}$$

Since the points $\{\varepsilon(j+b_i) \colon j \in \{0,1,\dots,N-1\}, \ i \in \{1,2\}\}$ make up an ε-net on the segment $[0,l]$, we can extend equalities (12), (15), (16), (19) to the domains D_i, equalities (17), (18) to $Q_0^{(1)}$ and $Q_0^{(2)}$, respectively, and equalities (30), (31) and (32) to $Q_0^{(0)}$. As a result, from equalities (32) we derive the relation

$$\eta(x_2,t) = \frac{h_2(d_0)\partial_r u_0^{2,-}|_{r=d_0}}{h_1(d_0)\partial_r u_0^{1,-}|_{r=d_0} + h_2(d_0)\partial_r u_0^{2,-}|_{r=d_0}}, \quad x_2 \in (0,l), \ t \in (0,T),$$

and obtain $\partial_r u_0^{+} = h_1(d_0)\partial_r u_0^{1,-} + h_2(d_0)\partial_r u_0^{2,-}, \quad (x,t) \in Q_0^{(0)} \times (0,T)$.

By virtue of (30) and (31) we can define $\varphi_1^{(i)}$ as follows

$$\varphi_1^{(i)}(x,t) = \alpha_3^{(i)} \partial_{x_2} u_0^{i,-}(x,t), \quad (x,t) \in D_i \times (0,T), \ i \in \{1,2\}.$$

3.3. The homogenized problem. With the help of the first terms u_0^{+}, $u_0^{1,-}$ and $u_0^{2,-}$ of asymptotic expansions (4) and (5) we define the multi-sheeted function

$$\mathbf{U}_0(x,t) = \begin{cases} u_0^{+}(x,t), & (x,t) \in \Omega_0 \times (0,T), \\ u_0^{1,-}(x,t), & (x,t) \in D_1 \times (0,T), \\ u_0^{2,-}(x,t), & (x,t) \in D_2 \times (0,T), \end{cases}$$

or in a short form $\mathbf{U}_0 = (u_0^+, u_0^{1,-}, u_0^{2,-})$. It follows from the foregoing that the components of $\mathbf{U}_0$ must satisfy the relations

$$\begin{cases} \partial_t u_0^+ - \Delta_x u_0^+ + \vartheta_0(u_0^+) = f_0, & (x,t) \in \Omega_0 \times (0,T), \\ \partial_{x_2}^p u_0^+|_{S^-} = \partial_{x_2}^p u_0^+|_{S^+}, & p \in \{0,1\},\ t \in (0,T), \\ h_1(r)\partial_t u_0^{1,-} - \mathrm{div}_{\tilde{x}}(h_1(r)\nabla_{\tilde{x}} u_0^{1,-}) + & \\ +h_1(r)\vartheta_0(u_0^{1,-}) + 2\vartheta_1(u_0^{1,-}) = h_1(r)f_0 + 2\delta_{\beta,1}g_0, & (x,t) \in D_1 \times (0,T), \\ \partial_\nu u_0^{1,-} + \vartheta_1(u_0^{1,-}) = 0, & (x,t) \in Q_0^{(1)} \times (0,T), \\ h_2(r)\partial_t u_0^{2,-} - \mathrm{div}_{\tilde{x}}\big(h_2(r)\nabla_{\tilde{x}} u_0^{2,-}\big) + & \\ +h_2(r)\vartheta_0(u_0^{2,-}) + 2\delta_{\alpha,1}\vartheta_2(u_0^{2,-}) = h_2(r)f_0 + 2\delta_{\beta,1}g_0, & (x,t) \in D_2 \times (0,T), \\ \partial_\nu u_0^{2,-} = 0, & (x,t) \in Q_0^{(2)} \times (0,T), \\ u_0^+|_{Q_0^{(0)}} = u_0^{1,-}|_{Q_0^{(0)}} = u_0^{2,-}|_{Q_0^{(0)}}, & t \in (0,T), \\ h_1(d_0)\partial_r u_0^{1,-} + h_2(d_0)\partial_r u_0^{2,-} = \partial_r u_0^+, & (x,t) \in Q_0^{(0)} \times (0,T), \\ \mathbf{U}_0(x,0) = \mathbf{0}. & \end{cases} \tag{33}$$

These relations form *the homogenized problem* for problem (1).

We introduce the space $\mathcal{V}_0{:=}L^2(\Omega_0) \times L^2(D_1) \times L^2(D_2)$ of multi-sheeted functions $\mathbf{u} = (u_0, u_1, u_2)$ defined as follows

$$\mathbf{u}(x) = \begin{cases} u_0(x), & x \in \Omega_0, \\ u_1(x), & x \in D_1, \\ u_2(x), & x \in D_2. \end{cases}$$

The space $\mathcal{V}_0$ is equipped with natural inner product. Also we introduce the anisotropic Sobolev space of multi-sheeted functions

$$\mathcal{H}_0{:=}\{\mathbf{u} = (u_0, u_1, u_2) \in \mathcal{V}_0 \colon u_0 \in H^1(\Omega_0),\ u_0|_{S^-} = u_0|_{S^+};$$
$$\exists \partial_{x_j} u_i \in L^2(D_i),\ j = 1,3,\ i = 1,2;\ u_0|_{Q_0^{(0)}} = u_1|_{Q_0^{(0)}} = u_2|_{Q_0^{(0)}}\}$$

with the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathcal{H}_0} = \int_{\Omega_0} (\nabla_x u_0 \cdot \nabla_x v_0 + u_0 v_0)dx + \sum_{i=1}^{2} \int_{D_i} (\nabla_{\tilde{x}} u_i \cdot \nabla_{\tilde{x}} v_i + u_i v_i)dx.$$

It is obvious that $\mathcal{H}_0$ is continuously embedded in $\mathcal{V}_0$.

Consider the space

$$\mathcal{W}_0{:=}\{\varphi = (\varphi_0, \varphi_1, \varphi_2) \in L^2(0,T;\mathcal{H}_0),\ \exists \partial_t \varphi{:=}\varphi' \in L^2(0,T;\mathcal{V}_0)\}.$$

A function $\mathbf{U}_0 = (u_0^+, u_0^{1,-}, u_0^{1,-}) \in L^2(0,T;\mathcal{H}_0)$ is a weak solution of problem (33) if for every function $\varphi = (\varphi_0, \varphi_1, \varphi_2) \in \mathcal{W}_0$ the integral identity

$$\int_{\Omega_0} (u_0^+ \varphi_0)|_{t=T} dx + \sum_{i=1}^{2} \int_{D_i} (h_i u_0^{i,-} \varphi_i)|_{t=T} dx - \int_0^T \Bigg(\int_{\Omega_0} u_0^+ \partial_t \varphi_0 dx + \sum_{i=1}^{2} \int_{D_i} h_i u_0^{i,-} \partial_t \varphi_i dx +$$
$$+ \int_{\Omega_0} (\nabla_x u_0^+ \cdot \nabla_x \varphi_0 + \vartheta_0(u_0^+)\varphi_0)dx + \sum_{i=1}^{2} \int_{D_i} h_i (\nabla_{\tilde{x}} u_0^{i,-} \cdot \nabla_{\tilde{x}} \varphi_i + \vartheta_0(u_0^{i,-})\varphi_i)dx +$$
$$+ 2\int_{D_1} \vartheta_1(u_0^{1,-})\varphi_1 dx + h_1(d_0) \int_{Q_0^{(1)}} \vartheta_1(u_0^{1,-})\varphi_1 d\sigma_x + 2\delta_{\alpha,1} \int_{D_2} \vartheta_2(u_0^{2,-})\varphi_2 dx \Bigg) dt =$$

$$= \int_0^T \left(\int_{\Omega_0} f_0 \varphi_0 dx + \sum_{i=1}^{2} \int_{D_i} (h_i f_0 + 2\delta_{\beta,1} g_0)\varphi_i dx \right) dt$$

holds.

Similarly as in [26] we can prove the existence and the uniqueness of a weak solution of problem (33).

4. Approximation and asymptotic estimates. Let $\mathbf{U}_0 = (u_0^+, u_0^{1,-}, u_0^{2,-})$ be the unique weak solution of problem (33). With the help of $\mathbf{U}_0$ and the solutions $Z_1,\ \Xi_1,\ \Xi_2$ of junction-layer problems (22) and (25) we construct the main terms of expansions (4), (5) and (21). Consider the smooth cut-off function $\chi_0(r)$, which is equal to 1 as $|r-d_0| < \delta_0/2$ and 0 as $|r-d_0| > \delta_0$, where $\delta_0 \in (0,\delta)$ is some fixed number. Matching the outer expansions with the inner expansion with the help of χ_0, we define the approximation function R_ε

$$R_\varepsilon(x,t) := R_\varepsilon^+(x,t) = u_0^+(x,t) + \varepsilon \chi_0(r) \mathcal{N}^+(\xi, x_2, \theta, t), \quad (x,t) \in \Omega_0 \times (0,T), \tag{34}$$

$$R_\varepsilon(x,t) := R_\varepsilon^{i,-}(x,t) = u_0^{i,-}(x,t) + \varepsilon \left(\tilde{Y}_i \left(\frac{x_2}{\varepsilon} \right) \partial_{x_2} u_0^{i,-}(x,t) + \chi_0(r) \mathcal{N}^{i,-}(\xi, x_2, \theta, t) \right),$$
$$(x,t) \in G_\varepsilon^{(i)}(j) \times (0,T), \quad i \in \{1,2\}. \tag{35}$$

Here

$$\mathcal{N}^+(\xi, x_2, \theta, t) = Z_1(\xi) \partial_{x_2} u_0^+|_{r=d_0} + \big(\xi_1 - \eta(x_2,t)\Xi_1(\xi) - (1-\eta(x_2,t))\Xi_2(\xi)\big) \partial_r u_0^+|_{r=d_0},$$
$$\mathcal{N}^{i,-}(\xi, x_2, \theta, t) = (Z_1(\xi) - \tilde{Y}_i(\xi_2)) \partial_{x_2} u_0^+|_{r=d_0} +$$
$$+ \big(\mathcal{Y}_i(\xi_1, x_2, t) - \eta(x_2,t)\Xi_1(\xi) - (1-\eta(x_2,t))\Xi_2(\xi)\big) \partial_r u_0^+|_{r=d_0},$$

where $\tilde{Y}_i(s) := -s + [s] + b_i + \alpha_3^{(i)}$, $[s]$ is the integer part of $s \in \mathbb{R}$, $i \in \{1,2\}$, and

$$\mathcal{Y}_1(\xi_1, x_2, t) := h_1^{-1}(d_0)\xi_1(1-\eta(x_2,t)), \ \mathcal{Y}_2(\xi_1, x_2, t) := h_2^{-1}(d_0)\xi_1\eta(x_2,t),$$
$$\xi_1 \le 0, \ x_2 \in (0,l), \ t \in (0,T).$$

Obviously, $R_\varepsilon \in W_\varepsilon$. Due to the initial condition of problem (33) we have $R_\varepsilon|_{t=0} = 0$ in Ω_ε.

Theorem 1. *Let $f_0(x,t)$, $(x,t) \in \overline{\Omega}_1 \times [0,T]$, and $g_0(x,t)$, $(x,t) \in \overline{D}_1 \times [0,T]$, be smooth functions such that $\partial_{x_2}^p f_0|_{S^-} = \partial_{x_2}^p f_0|_{S^+}$ for all $t \in [0,T]$, $p \in \{0,1\}$, $f_0(x,0) = g_0(x,0) = 0$.*

Then for any $\mu > 0$ there exist positive constants ε_0, c_0 such that

$$\|u_\varepsilon - R_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + \max_{t\in[0,T]} \|u_\varepsilon(\cdot,t) - R_\varepsilon(\cdot,t)\|_{L^2(\Omega_\varepsilon)} \le$$
$$\le c_0(\|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon \times (0,T))} + \varepsilon^{1-\mu} + \varepsilon^{\delta_{\alpha,1}(2-\alpha)+\alpha-1} + \varepsilon^{\beta-1}\|g_\varepsilon - g_0\|^{\delta_{\beta,1}}_{L^2(D_1\times(0,T))}) \tag{36}$$

for all $\varepsilon \in (0,\varepsilon_0)$, where u_ε is the solution of problem (1), and R_ε is defined by (34) and (35).

Proof. Discrepancies in domain Ω_0. Notice that the first two relations in (33) and the assumptions of the theorem yield $\partial^2_{x_2x_2} u_0^+|_{S^-} = \partial^2_{x_2x_2} u_0^+|_{S^+}$. Then, according to the properties of $Z_1,\ \Xi_1,\ \Xi_2$ and u_0^+, the function R_ε^+ satisfies the boundary conditions of problem (1) on $\partial\Omega_\varepsilon \cap \partial\Omega_0$.

Problems (22) and (25) imply

$$\Delta_\xi \mathcal{N}^+ = 0, \ \Delta_\xi \mathcal{N}^{i,-} = 0 \quad \xi \in \Pi, \ x_2 \in (0,l), \ \theta \in [0,2\pi], \ t \in [0,T], \ i \in \{1,2\}. \tag{37}$$

Observe that the following equality holds

$$\Delta_{\tilde{x}}(\chi_0(r)\mathcal{N}) = \mathrm{div}_{\tilde{x}}(\mathcal{N}\nabla_{\tilde{x}}\chi_0(r)) + \nabla_{\tilde{x}}\chi_0(r)\cdot\nabla_{\tilde{x}}\mathcal{N} + \chi_0(r)\Delta_{\tilde{x}}\mathcal{N}, \quad \mathcal{N} = \mathcal{N}(\xi, x_2, \theta). \quad (38)$$

Using (20), (33), (37) and (38), we get

$$\begin{gathered}\partial_t R_\varepsilon^+(x,t) - \Delta_x R_\varepsilon^+(x,t) - f_\varepsilon(x,t) = f_0(x,t) - f_\varepsilon(x,t) + \varepsilon\chi_0(r)\partial_t\mathcal{N}^+(\xi, x_2, \theta, t) - \\ -\vartheta_0(u_0^+(x,t)) + \chi_0(r)(r^{-1}\partial_{\xi_1}\mathcal{N}^+(\xi, x_2, \theta, t) - 2\partial^2_{\xi_2 x_2}\mathcal{N}^+(\xi, x_2, \theta, t)) - \\ -\varepsilon\mathrm{div}_{\tilde{x}}(\mathcal{N}^+|_{\xi_1 = -(r-d_0)/\varepsilon}\nabla_{\tilde{x}}\chi_0(r)) + \chi_0'(r)\partial_{\xi_1}\mathcal{N}^+(\xi, x_2, \theta, t) - \\ -\varepsilon\chi_0(r)\partial^2_{x_2x_2}\mathcal{N}^+(\xi, x_2, \theta, t) - \varepsilon r^{-2}\chi_0(r)\partial^2_{\theta\theta}\mathcal{N}^+(\xi, x_2, \theta, t), \quad (x,t)\in\Omega_0\times(0,T). \quad (39)\end{gathered}$$

We multiply (39) by a test function $\psi \in W_\varepsilon$ such that $\psi(x,T) = 0$, integrate by parts in $\Omega_0 \times (0,T)$ and take into account the properties of R_ε^+. This yields

$$\int_0^T \Big(-\int_{\Omega_0} R_\varepsilon^+ \partial_t\psi dx + \int_{\Omega_0} (\nabla_x R_\varepsilon^+ \cdot \nabla_x \psi + \vartheta_0(R_\varepsilon^+)\psi) dx - \int_{\Theta_\varepsilon} \partial_r R_\varepsilon^+ \psi d\sigma_x - \int_{\Omega_0} f_\varepsilon\psi dx \Big) dt = \\ = I_0^+(\varepsilon,\psi) + \ldots + I_5^+(\varepsilon,\psi) \quad (40)$$

for all $\psi \in W_\varepsilon,\ \psi(x,T) = 0$, where

$$\begin{gathered} I_0^+(\varepsilon,\psi) := \int_0^T\int_{\Omega_0}(f_0 - f_\varepsilon)\psi dx dt,\ I_1^+(\varepsilon,\psi) := \varepsilon\int_0^T\int_{\Omega_0}\chi_0\partial_t\mathcal{N}^+\psi dx dt, \\ I_2^+(\varepsilon,\psi) := \int_0^T\int_{\Omega_0}(\vartheta_0(R_\varepsilon^+) - \vartheta_0(u_0^+))\psi dx dt,\ I_3^+(\varepsilon,\psi) := \int_0^T\int_{\Omega_0}\chi_0(r^{-1}\partial_{\xi_1}\mathcal{N}^+ - \partial^2_{x_2\xi_2}\mathcal{N}^+)\psi dx dt, \\ I_4^+(\varepsilon,\psi) := \int_0^T\Big(\varepsilon\int_{\Omega_0}\mathcal{N}^+\nabla_{\tilde{x}}\chi_0\cdot\nabla_{\tilde{x}}\psi dx + \int_{\Omega_0}\chi_0'\partial_{\xi_1}\mathcal{N}^+\psi dx\Big)dt, \\ I_5^+(\varepsilon,\psi) := \varepsilon\int_0^T\Big(\int_{\Omega_0}\chi_0\partial_{x_2}\mathcal{N}^+\partial_{x_2}\psi dx + \int_{\Omega_0} r^{-2}\chi_0\partial_\theta\mathcal{N}^+\partial_\theta\psi dx\Big)dt. \end{gathered}$$

Discrepancies in the thin discs. One can readily check that

$$\begin{gathered}\partial_r R_\varepsilon^{1,-} = -\vartheta_1(u_0^{1,-}) - \varepsilon\tilde{Y}_1\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}\vartheta_1(u_0^{1,-}), (x,t)\in Q_\varepsilon^{(1)}\times(0,T), \\ \partial_r R_\varepsilon^{2,-} = 0, (x,t)\in Q_\varepsilon^{(2)}\times(0,T), \quad (41)\end{gathered}$$

$$\partial_r R_\varepsilon^{i,-} = \varepsilon\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\partial^2_{rx_2}u_0^{i,-} + \partial_r R_\varepsilon^+, \quad (x,t)\in\Theta_\varepsilon^{(i)}\times(0,T),\ i\in\{1,2\}. \quad (42)$$

Taking into account (9) and that functions h_i are constant on a neighborhood of d_0, we derive that

$$\partial_\nu R_\varepsilon^{i,-} = \frac{\varepsilon}{\sqrt{1 + 4^{-1}\varepsilon^2|h_i'(r)|^2}}\Big(\pm\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\partial^2_{x_2x_2}u_0^{i,-} \pm \chi_0\frac{\partial}{\partial x_2}(\mathcal{N}^{i,-}|_{\xi_2 = x_2/\varepsilon}) - \\ -\frac{1}{2}\nabla_{\tilde{x}}h_i\cdot\nabla_{\tilde{x}}(u_0^{i,-} + \varepsilon\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}u_0^{i,-})\Big), \quad (x,t)\in S_\varepsilon^{(i)}\times(0,T),\ i\in\{1,2\}, \quad (43)$$

where “+” and “−” refer to the left and the right parts of the lateral surfaces of the thin discs, respectively.

Relations (20), (33), (37) and (38) yield

$$\partial_t R_\varepsilon^{i,-}(x,t) - \Delta_x R_\varepsilon^{i,-}(x,t) - f_\varepsilon(x,t) = f_0(x,t) - f_\varepsilon(x,t) + \varepsilon(\tilde{Y}_i(x_2/\varepsilon)\partial^2_{tx_2} u_0^{i,-} +$$
$$+\chi_0(r)\partial_t \mathcal{N}^{i,-}(\xi,x_2,\theta,t)) - \vartheta_0(u_0^{i,-}) + \chi_0(r)(r^{-1}\partial_{\xi_1}\mathcal{N}^{i,-}(\xi,x_2,\theta,t) - 2\partial^2_{\xi_2 x_2}\mathcal{N}^{i,-}(\xi,x_2,\theta,t))-$$
$$-\varepsilon \mathrm{div}_{\tilde{x}}(\mathcal{N}^{i,-}|_{\xi_1=-(r-d_0)/\varepsilon}\nabla_{\tilde{x}}\chi_0(r)) + \chi_0'(r)\partial_{\xi_1}\mathcal{N}^{i,-}(\xi,x_2,\theta,t) - \varepsilon\chi_0(r)\partial^2_{x_2x_2}\mathcal{N}^{i,-}(\xi,x_2,\theta,t)-$$
$$-\varepsilon\chi_0(r)r^{-2}\partial^2_{\theta\theta}\mathcal{N}^{i,-}(\xi,x_2,\theta,t) + \nabla_{\tilde{x}}(\ln h_i(r))\cdot\nabla_{\tilde{x}}u_0^{i,-} - \varepsilon\mathrm{div}_x\big(\tilde{Y}_i(x_2/\varepsilon)\nabla_x(\partial_{x_2}u_0^{i,-})\big)-$$
$$-2(1-\delta_{i,2}(1-\delta_{\alpha,1}))h_i^{-1}(r)\vartheta_i(u_0^{i,-}) + 2\delta_{\beta,1}h_i^{-1}(r)g_0(x,t), \quad (x,t)\in G_\varepsilon^{(i)}\times(0,T). \tag{44}$$

Consider the integral identity

$$\int_{S_\varepsilon^{(i)}} \frac{\varepsilon h_i(r)}{2\sqrt{1+4^{-1}\varepsilon^2|h_i'(r)|^2}}\varphi d\sigma_x = \int_{G_\varepsilon^{(i)}}\varphi dx - \varepsilon\int_{G_\varepsilon^{(i)}} Y_i\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}\varphi dx, \quad i\in\{1,2\}, \tag{45}$$

where $Y_i(s) = -s + [s] + b_i$ and $[s]$ is the integer part of s, $\varphi \in H^1(G_\varepsilon^{(i)})$ is an arbitrary function. We multiply (44) by a test function $\psi \in W_\varepsilon,\ \psi(x,T)=0$, and integrate by parts in $G_\varepsilon^{(i)}\times(0,T)$, using (45) and taking into account relations (41), (42), (43). This yields

$$\int_0^T \Big(-\int_{G_\varepsilon^{(i)}} R_\varepsilon^{1,-}\partial_t\psi dx + \int_{G_\varepsilon^{(1)}}(\nabla_x R_\varepsilon^{1,-}\cdot\nabla_x\psi + \vartheta_0(R_\varepsilon^{1,-})\psi)dx + \varepsilon\int_{S_\varepsilon^{(1)}}\vartheta_1(R_\varepsilon^{1,-})\psi d\sigma_x +$$
$$+\int_{Q_\varepsilon^{(1)}}\vartheta_1(R_\varepsilon^{1,-})\psi d\sigma_x + \int_{\Theta_\varepsilon^{(1)}}\partial_r R_\varepsilon^{+}\psi d\sigma_x - \int_{G_\varepsilon^{(1)}} f_\varepsilon\psi dx - \varepsilon^\beta\int_{S_\varepsilon^{(1)}} g_\varepsilon\psi d\sigma_x\Big)dt =$$
$$= I_0^{1,-}(\varepsilon,\psi) + \ldots + I_8^{1,-}(\varepsilon,\psi), \tag{46}$$

$$\int_0^T \Big(-\int_{G_\varepsilon^{(2)}} R_\varepsilon^{2,-}\partial_t\psi dx + \int_{G_\varepsilon^{(2)}}(\nabla_x R_\varepsilon^{2,-}\cdot\nabla_x\psi + \vartheta_0(R_\varepsilon^{1,-})\psi)dx + \varepsilon^\alpha\int_{\Upsilon_\varepsilon^{(2)}}\vartheta_2(R_\varepsilon^{2,-})\psi d\sigma_x +$$
$$+\int_{\Theta_\varepsilon^{(2)}}\partial_r R_\varepsilon^{+}\psi d\sigma_x - \int_{G_\varepsilon^{(2)}} f_\varepsilon\psi dx - \varepsilon^\beta\int_{\Upsilon_\varepsilon^{(2)}} g_\varepsilon\psi d\sigma_x\Big)dt = I_0^{2,-}(\varepsilon,\psi)+\ldots+I_8^{2,-}(\varepsilon,\psi) \tag{47}$$

for all $\psi\in W_\varepsilon,\ \psi(x,T)=0$, where

$$I_0^{i,-}(\varepsilon,\psi) := \int_0^T\int_{G_\varepsilon^{(i)}}(f_0-f_\varepsilon)\psi dxdt,$$

$$I_1^{i,-}(\varepsilon,\psi) := \varepsilon\int_0^T\int_{G_\varepsilon^{(i)}}(\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\partial^2_{x_2t}u_0^{i,-} + \chi_0(r)\partial_t\mathcal{N}^{i,-})\psi dxdt,$$

$$I_2^{i,-}(\varepsilon,\psi) := \int_0^T\int_{G_\varepsilon^{(i)}}(\vartheta_0(R_\varepsilon^{i,-}) - \vartheta_0(u_0^{i,-}))\psi dxdt,$$

$$I_3^{i,-}(\varepsilon,\psi) := \int_0^T\int_{G_\varepsilon^{(i)}}\chi_0(r^{-1}\partial_{\xi_1}\mathcal{N}^{i,-} - \partial^2_{x_2\xi_2}\mathcal{N}^{i,-})\psi dxdt,$$

$$I_4^{i,-}(\varepsilon,\psi) := \int_0^T\left(\varepsilon\int_{G_\varepsilon^{(i)}}\mathcal{N}^{i,-}\nabla_{\tilde{x}}\chi_0\cdot\nabla_{\tilde{x}}\psi dx + \int_{G_\varepsilon^{(i)}}\chi_0'\partial_{\xi_1}\mathcal{N}^{i,-}\psi dx\right)dt,$$

$$I_5^{i,-}(\varepsilon,\psi) := \varepsilon\int_0^T\left(\int_{G_\varepsilon^{(i)}}\chi_0\partial_{x_2}\mathcal{N}^{i,-}\partial_{x_2}\psi dx + \int_{G_\varepsilon^{(i)}} r^{-2}\chi_0\partial_\theta\mathcal{N}^{i,-}\partial_\theta\psi dx\right)dt,$$

$$I_6^{i,-}(\varepsilon,\psi) := \varepsilon\int_0^T\left(\int_{G_\varepsilon^{(i)}} Y_i\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}(\psi\nabla_{\tilde{x}}u_0^{i,-}\cdot\nabla_{\tilde{x}}\ln h_i)dx +\right.$$
$$\left.+\int_{G_\varepsilon^{(i)}}\tilde{Y}_i\left(\frac{x_2}{\varepsilon}\right)\nabla_x(\partial_{x_2}u_0^{i,-})\cdot\nabla_x\psi dx\right)dt, \quad i\in\{1,2\},$$

$$I_7^{1,-}(\varepsilon,\psi):=\int_0^T\left(-\varepsilon\int_{S_\varepsilon^{(1)}}\frac{\vartheta_1(u_0^{1,-})\psi}{\sqrt{1+4^{-1}\varepsilon^2|h_1'(r)|^2}}d\sigma_x-2\varepsilon\int_{G_\varepsilon^{(1)}}Y_1\left(\frac{x_2}{\varepsilon}\right)h_1^{-1}\partial_{x_2}(\vartheta_1(u_0^{1,-})\psi)dx+\right.$$
$$\left.+\varepsilon\int_{S_\varepsilon^{(1)}}\vartheta_1(R_\varepsilon^{1,-})\psi d\sigma_x+\int_{Q_\varepsilon^{(1)}}(\vartheta_1(R_\varepsilon^{1,-})-\vartheta_1(u_0^{1,-})-\varepsilon\tilde{Y}_1\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}\vartheta_1(u_0^{1,-}))\psi d\sigma_x\right)dt,$$
$$I_7^{2,-}(\varepsilon,\psi):=\int_0^T\left(-\varepsilon\delta_{\alpha,1}\int_{S_\varepsilon^{(2)}}\frac{\vartheta_2(u_0^{2,-})\psi}{\sqrt{1+4^{-1}\varepsilon^2|h_2'(r)|^2}}d\sigma_x-\right.$$
$$\left.-2\varepsilon\delta_{\alpha,1}\int_{G_\varepsilon^{(2)}}Y_2\left(\frac{x_2}{\varepsilon}\right)h_2^{-1}\partial_{x_2}(\vartheta_2(u_0^{2,-})\psi)dx+\varepsilon^\alpha\int_{\Upsilon_\varepsilon^{(2)}}\vartheta_2(R_\varepsilon^{2,-})\psi d\sigma_x\right)dt,$$
$$I_8^{1,-}(\varepsilon,\psi):=\int_0^T\left(\varepsilon\delta_{\beta,1}\int_{S_\varepsilon^{(1)}}\frac{g_0\psi}{\sqrt{1+4^{-1}\varepsilon^2|h_1'(r)|^2}}d\sigma_x+\right.$$
$$\left.+2\varepsilon\delta_{\beta,1}\int_{G_\varepsilon^{(1)}}Y_1\left(\frac{x_2}{\varepsilon}\right)h_1^{-1}\partial_{x_2}(g_0\psi)dx-\varepsilon^\beta\int_{S_\varepsilon^{(1)}}g_\varepsilon\psi d\sigma_x\right)dt,$$
$$I_8^{2,-}(\varepsilon,\psi):=\int_0^T\left(\varepsilon\delta_{\beta,1}\int_{S_\varepsilon^{(2)}}\frac{g_0\psi}{\sqrt{1+4^{-1}\varepsilon^2|h_2'(r)|^2}}d\sigma_x+\right.$$
$$\left.+2\varepsilon\delta_{\beta,1}\int_{G_\varepsilon^{(2)}}Y_2\left(\frac{x_2}{\varepsilon}\right)h_2^{-1}\partial_{x_2}(g_0\psi)dx-\varepsilon^\beta\int_{\Upsilon_\varepsilon^{(2)}}g_\varepsilon\psi d\sigma_x\right)dt.$$

Asymptotic estimates. After summing (40), (46) and (47) we see that the function R_ε defined by (34) and (35) satisfies the integral identity

$$F_\varepsilon(\psi)=\int_0^T\left(-\int_{\Omega_\varepsilon}R_\varepsilon\partial_t\psi dx+\int_{\Omega_\varepsilon}(\nabla_xR_\varepsilon\cdot\nabla_x\psi+\vartheta_0(R_\varepsilon)\psi)dx+\varepsilon\int_{S_\varepsilon^{(1)}}\vartheta_1(R_\varepsilon)\psi d\sigma_x+\right.$$
$$\left.+\int_{Q_\varepsilon^{(1)}}\vartheta_1(R_\varepsilon)\psi d\sigma_x+\varepsilon^\alpha\int_{\Upsilon_\varepsilon^{(2)}}\vartheta_2(R_\varepsilon)\psi d\sigma_x-\int_{\Omega_\varepsilon}f_\varepsilon\psi dx-\varepsilon^\beta\int_{S_\varepsilon^{(1)}\cup\Upsilon_\varepsilon^{(2)}}g_\varepsilon\psi d\sigma_x\right)dt \tag{48}$$

for all $\psi\in W_\varepsilon,\ \psi|_{t=T}=0$, where $F_\varepsilon(\psi):=I_0^\pm+\ldots+I_5^\pm+I_6^-+I_7^-+I_8^-,\ I_k^-:=I_k^{1,-}+I_k^{2,-},\ k\in\{0,1,\ldots,8\},\ I_m^\pm:=I_m^++I_m^-,\ m\in\{0,1,\ldots,5\}$.

It follows from (3) and (48) that

$$\int_0^T\left(-\int_{\Omega_\varepsilon}(R_\varepsilon-u_\varepsilon)\partial_t\psi dx+\int_{\Omega_\varepsilon}(\nabla_x(R_\varepsilon-u_\varepsilon)\cdot\nabla_x\psi+(\vartheta_0(R_\varepsilon)-\vartheta_0(u_\varepsilon))\psi)dx+\right.$$
$$+\varepsilon\int_{S_\varepsilon^{(1)}}(\vartheta_1(R_\varepsilon)-\vartheta_1(u_\varepsilon))\psi d\sigma_x+\int_{Q_\varepsilon^{(1)}}(\vartheta_1(R_\varepsilon)-\vartheta_1(u_\varepsilon))\psi d\sigma_x+$$
$$\left.+\varepsilon^\alpha\int_{\Upsilon_\varepsilon^{(2)}}(\vartheta_2(R_\varepsilon)-\vartheta_2(u_\varepsilon))\psi d\sigma_x\right)dt=F_\varepsilon(\psi) \tag{49}$$

for all $\psi\in W_\varepsilon,\ \psi|_{t=T}=0$.

Now we are going to estimate $F_\varepsilon(\psi)$. With the help of the Cauchy-Schwartz-Bunyakovskii inequality we obtain $|I_0^\pm(\varepsilon,\psi)|\le\|f_\varepsilon-f_0\|_{L^2(\Omega_\varepsilon\times(0,T))}\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$, $|I_1^\pm(\varepsilon,\psi)|\le C_1\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$.

Remark 1. Here and further all constants c_i, C_i in asymptotic estimates are independent of ε.

By (2), Taylor's formula and the Cauchy-Schwartz-Bunyakovskii inequality we derive that

$$|I_2^+(\varepsilon,\psi)| \le c_0\varepsilon \left| \int_0^T \int_{\Omega_0} \chi_0 \mathcal{N}^+ \psi dxdt \right| \le c_1\varepsilon \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}.$$

Similarly we estimate I_2^-. Thus, $|I_2^\pm(\varepsilon,\psi)| \le C_2\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$.

Since the functions $\partial_{\xi_1}\mathcal{N}^+,\ \partial^2_{x_2\xi_2}\mathcal{N}^+,\ \partial_{\xi_1}\mathcal{N}^{i,-},\ \partial^2_{x_2\xi_2}\mathcal{N}^{i,-}$ exponentially decrease as $|\xi_1| \to \infty$ (see (23), (24) and (26)), then from Lemma 3.1 in [6] we derive that

$$\forall\mu>0 \quad \exists C_3>0 \quad \exists\varepsilon_0>0 \quad \forall\varepsilon\in(0,\varepsilon_0)\colon |I_3^\pm(\varepsilon,\psi)| \le C_3\varepsilon^{1-\mu}\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}.$$

The integrals in $I_4^\pm(\varepsilon,\psi)$ are in fact over

$$(\mathrm{supp}(\chi_0'(r))\cap\Omega_\varepsilon)\times(0,T) = \{x\in\Omega_\varepsilon\colon \delta_0/2 < |r-d_0| < \delta_0\}\times(0,T),$$

where, according to (22) and (25), the functions $\mathcal{N}^+,\ \partial_{\xi_1}\mathcal{N}^+,\ \partial_{\xi_1}\mathcal{N}^{i,-}$ are exponentially small, and $\mathcal{N}^{i,-}$ can be estimated by some constant c_2. Thus, $|I_4^\pm(\varepsilon,\psi)| \le C_4\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$.

The integrals in $I_5^\pm$ are over $\{x\in\mathbb{R}^3\colon |r-d_0|<\delta_0\}$ and they can be estimated, extracting if necessary the exponentially decreasing part in the corresponding integrand and then using the Cauchy-Schwartz-Bunyakovskii inequality. Consider, for example, the integral

$$\begin{aligned}
&\left|\int_0^T\int_{G_\varepsilon^{(1)}} \chi_0\partial_{x_2}\mathcal{N}^{1,-}\partial_{x_2}\psi dxdt\right| = \left|\int_0^T\int_{G_\varepsilon^{(1)}} \chi_0\Big((Z_1-\tilde{Y}_1)\partial^2_{x_2x_2}u_0^+|_{r=d_0} -\right.\\
&\qquad -(h_1^{-1}(d_0)\xi_1+\Xi_1-\Xi_2)\partial_{x_2}\eta\partial_r u_0^+|_{r=d_0}+\\
&\qquad \left.+(h_1^{-1}(d_0)\xi_1(1-\eta)-\eta\Xi_1-(1-\eta)\Xi_2)\partial^2_{x_2r}u_0^+|_{r=d_0}\Big)\partial_{x_2}\psi dxdt\right| \le\\
&\le c_3\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}\Bigg(\sqrt{\int_0^T\int_{G_\varepsilon^{(1)}}\chi_0|Z_1-\tilde{Y}_1|^2dxdt}\,+\\
&\qquad +\|\alpha_1^{(1)}\eta+\alpha_2^{(1)}(1-\eta)+(\alpha_1^{(1)}-\alpha_2^{(1)})\partial_{x_2}\eta\|_{L^2(G_\varepsilon^{(1)}\times(0,T))}+\\
&\qquad +\sqrt{\int_0^T\int_{G_\varepsilon^{(1)}}\chi_0\big|h_1^{-1}(d_0)\xi_1+(\Xi_1-\alpha_1^{(1)})-(\Xi_2-\alpha_2^{(1)})\big|^2dxdt}\,+\\
&\qquad +\sqrt{\int_0^T\int_{G_\varepsilon^{(1)}}\chi_0\big|\eta(\Xi_1-\alpha_1^{(1)})+(1-\eta)(\Xi_2-h_1^{-1}(d_0)\xi_1-\alpha_2^{(1)})\big|^2dxdt}\Bigg) \le\\
&\le c_4\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}\Big(\sqrt{2\pi l d_0\varepsilon}\|Z_1-\tilde{Y}_1\|_{L^2(\Pi_1^-)}+\sqrt{|G_\varepsilon^{(1)}|}\,+\\
&\qquad +\sqrt{2\pi l d_0\varepsilon}\|h_1^{-1}(d_0)\xi_1+(\Xi_1-\alpha_1^{(1)})-(\Xi_2-\alpha_2^{(1)})\|_{L^2(\Pi_1^-)}+\\
&\qquad +\sqrt{2\pi l d_0\varepsilon}\|\eta(\Xi_1-\alpha_1^{(1)})+(1-\eta)(\Xi_2-h_1^{-1}(d_0)\xi_1-\alpha_2^{(1)})\|_{L^2(\Pi_1^-)}\Big),
\end{aligned}$$

where $|G_\varepsilon^{(1)}|$ is the measure of $G_\varepsilon^{(1)}$. Relations (23), (24) and (26) show that the norms in the right-hand side of the last inequality are bounded in ε. Similarly we can estimate the rest of the integrals in $I_5^\pm(\varepsilon,\psi)$. As a result, we obtain $|I_5^\pm(\varepsilon,\psi)| \le C_5\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$.

Remark 2. The constants C_4 and C_5 depend on

$$\sup_{x\in Q_0^{(0)},\ t\in(0,T)}\left|\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}u_0^+(x,t)\right|, \quad |\alpha|=\alpha_1+\alpha_2+\alpha_3\le 2,\ \alpha_k\ge 0,\ k\in\{1,2,3\}.$$

Extending homogenized problem (33) periodically in x_2 through the planes $\{x \in \mathbb{R}^3 \colon x_2 = 0\}$ and $\{x \in \mathbb{R}^3 \colon x_2 = l\}$ and taking into account the assumptions for f_0 and g_0, by virtue of classical results on the smoothness of solutions to boundary-value problems we conclude that these quantities are bounded.

Since f_0 is smooth, one has $u_0^{i,-} \in L^2(0,T;H^2(D_i)),\ i = 1,2$. Consequently,

$$|I_6^-(\varepsilon,\psi)| \le c_5\varepsilon \sum_{i=1}^{2} (\|u_0^{i,-}\|_{L^2(0,T;H^1(D_i))} + \|\partial_{x_2} u_0^{i,-}\|_{L^2(0,T;H^1(D_i))}) \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \le$$
$$\le C_6\varepsilon \|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}.$$

In order to estimate I_7^- we consider summand $I_7^{2,-}$ with $\alpha = 1$. Obviously, the module of the second integral in $I_7^{2,-}$ can be estimated by $c_6\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$. Using (2), Taylor's formula and the obvious equality

$$1 - \frac{1}{a} = \frac{a^2-1}{a^2+a} \qquad (a^2 + a \neq 0)$$

we derive that the absolute value of the sum of the first and third integrals in $I_7^{2,-}$ can be evaluated by

$$\left| 4^{-1}\varepsilon^3 \int_0^T \int_{S_\varepsilon^{(2)}} \frac{|h_2'(r)|^2 \vartheta_2(u_0^{2,-})\psi}{1 + 4^{-1}\varepsilon^2|h_2'(r)|^2 + \sqrt{1 + 4^{-1}\varepsilon^2 |h_2'(r)|^2}} d\sigma_x dt \right| +$$
$$+ c_7\varepsilon^2 \left| \int_0^T \int_{S_\varepsilon^{(2)}} (\tilde{Y}_2\left(\frac{x_2}{\varepsilon}\right) \partial_{x_2} u_0^{2,-} + \chi_0 \mathcal{N}^{2,-})\psi dx dt \right| + \varepsilon \left| \int_0^T \int_{Q_\varepsilon^{(2)}} \vartheta_2(R_\varepsilon)\psi d\sigma_x dt \right|$$
$$=: J_1(\varepsilon,\psi) + J_2(\varepsilon,\psi) + J_3(\varepsilon,\psi).$$

With the help of (45) we obtain $J_1(\varepsilon,\psi) + J_2(\varepsilon,\psi) \le c_8\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$. Taking into account (2), properties of the trace operator and the fact that f_0 is smooth, we deduce $J_3(\varepsilon,\psi) \le c_9\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$. Thus, in the case when $\alpha = 1$ we have

$$|I_7^{2,-}(\varepsilon,\psi)| \le c_{10}\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}.$$

In the case when $\alpha > 1$ by (45) we obtain $|I_7^{2,-}(\varepsilon,\psi)| \le c_{11}\varepsilon^{\alpha-1}\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$. Similarly to $I_7^{2,-}(\varepsilon,\psi)$, we estimate $I_7^{1,-}(\varepsilon,\psi)$ and $I_8^-(\varepsilon,\psi)$. As a result, we get $|I_7^{1,-}(\varepsilon,\psi)| \le C_7\varepsilon\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$ and

$$|I_8^-(\varepsilon,\psi)| \le C_8 \begin{cases} (\varepsilon + \|g_0 - g_\varepsilon\|_{L^2(D_1\times(0,T))})\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}, & \beta = 1, \\ \varepsilon^{\beta-1}\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))}, & \beta > 1. \end{cases}$$

Thus,

$$|F_\varepsilon(\psi)| \le C_7(\|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon\times(0,T))} + \varepsilon^{1-\mu} + \varepsilon^{\delta_{\alpha,1}(2-\alpha)+\alpha-1} +$$
$$+ \varepsilon^{\beta-1}\|g_\varepsilon - g_0\|^{\delta_{\beta,1}}_{L^2(D_1\times(0,T))})\|\psi\|_{L^2(0,T;H^1(\Omega_\varepsilon))},$$

where $\mu > 0$ is an arbitrary number. From the last estimate with the help of standard scheme we deduce inequality (36). $\square$

5. Discussion of the obtained results. As we can see from the obtained results, the homogenized problem (33) for problem (1) is a nonstandard boundary-value problems for multi-sheeted function $\mathbf{U}_0$ in anisotropic Sobolev space $\mathcal{W}_0$ (see Section). This problem consists of three boundary-value problems (in domains Ω_0 and $D_i,\ i \in \{1,2\}$), connected with each other by the conjugation conditions (on $Q_0^{(0)}$).

The nonhomogeneous Robin boundary conditions on the lateral surfaces of the thin discs in problem (1) are transformed as $\varepsilon \to 0$ into new summands in the differential equations in domains $D_i,\ i \in \{1,2\}$, in problem (33). These summands show us the influence of the perturbed parameters α and β. If $\alpha > 1$, then the summand $2\delta_{\alpha,1}\vartheta_1(u_0^{2,-})$ vanishes. From physical point of view this means that the outer heat conduction coefficient is too small, and we can neglect this heat exchange. If $\beta > 1$, then summands $2\delta_{\beta,1}g_0$ vanish, which means that the temperature of the environment is too small, and we can consider it being equal to zero.

Functions $h_i, i \in \{1,2\}$, which describe the relative thickness of the thin discs from the i-th level, are transformed into the coefficients of the differential equations in domains D_i, respectively. The variable x_2 is involved as a parameter in the boundary-value problems in $D_i, i \in \{1,2\}$, which shows us the influence of the type of thick junction Ω_ε on the asymptotic behavior of solution u_ε.

From results proved in the present paper it follows that for applied problems in thick junctions we can use the homogenized problem (33), which is simpler, instead of the initial problem (1) with sufficient plausibility.

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