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WIMAN'S TYPE INEQUALITIES WITHOUT EXCEPTIONAL SETS FOR RANDOM ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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In the paper we consider entire functions $f\colon \mathbb{C}^p\to\mathbb{C},\ p\geq 2$, defined by power series $f(z)=f(z_1,\dots,z_p)=\sum_{\|n\|=0}^{+\infty}a_nz^n,\ z^n=z_1^{n_1}\cdot\ldots\cdot z_p^{n_p},\ n=(n_1,\dots,n_p)$. For $r=(r_1,\dots,r_p)\in\mathbb{R}_+^p$ we set

$$M_f(r)=\max\{|f(z)|\colon |z_i|\leq r_i, i\in\{1,\dots,p\}\},\ \mu_f(r)=\max\{|a_n|r^n\colon n\in\mathbb{Z}_+^p\},$$
$$r^{\vee}=\max\{r_i\colon i\in\{1,\dots,p\}\},\ r^{\wedge}=\min\{r_i\colon i\in\{1,\dots,p\}\}$$

and let l be a log-convex real function on $(1,+\infty)$ such that $\ln t=o(l(t)),\ t\to+\infty$. Then for any entire transcendental function f with $\ln M_f(r)\leq l(r^{\vee}),\ r^{\wedge}\to+\infty$, the inequality

$$\varlimsup_{r^{\wedge}\to+\infty}\frac{\ln M_f(r)-\ln\mu_f(r)}{\ln\ln\mu_f(r)}\leq\alpha$$

holds if and only if $\varlimsup_{t\to+\infty}(\ln l(t)/\ln\ln t)\leq 1+\alpha/p$. Similar theorems are proved for random entire functions of several complex variables.

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В статье рассматриваются целые функции $f\colon \mathbb{C}^p\to\mathbb{C},\ p\geq 2$, определенные степенным рядом $f(z)=f(z_1,\dots,z_p)=\sum_{\|n\|=0}^{+\infty}a_nz^n,\ z^n=z_1^{n_1}\cdot\ldots\cdot z_p^{n_p},\ n=(n_1,\dots,n_p)$. Пусть для $r=(r_1,\dots,r_p)\in\mathbb{R}_+^p$

$$M_f(r)=\max\{|f(z)|\colon |z_i|\leq r_i, i\in\{1,\dots,p\}\},\ \mu_f(r)=\max\{|a_n|r^n\colon n\in\mathbb{Z}_+^p\},$$
$$r^{\vee}=\max\{r_i\colon i\in\{1,\dots,p\}\},\ r^{\wedge}=\min\{r_i\colon i\in\{1,\dots,p\}\},$$

и l выпуклая относительно логарифма на $(1,+\infty)$ вещественная функция, $\ln t=o(l(t))$, $t\to+\infty$. Доказано, что для того чтобы для любой целой трансцендентной функции f, удовлетворяющей условию $\ln M_f(r)\leq l(r^{\vee}),\ r^{\wedge}\to+\infty$, выполнялось соотношение

$$\varlimsup_{r^{\wedge}\to+\infty}\frac{\ln M_f(r)-\ln\mu_f(r)}{\ln\ln\mu_f(r)}\leq\alpha$$

необходимо и достаточно, чтобы $\varlimsup_{t\to+\infty}(\ln l(t)/\ln\ln t)\leq 1+\alpha/p$. Похожие теоремы доказано для случайных целых функций многих комплексных переменных.

1. Introduction. By classical Wiman-Valiron's theorem (see [1]) for all entire nonconstant functions and all $\varepsilon>0$ the following inequality

$$M_f(r)\leq\mu_f(r)\ln^{1/2+\varepsilon}\mu_f(r)\tag{1}$$

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holds for $r > 1$ outside an exceptional set $E_f(\varepsilon)$ of finite logarithmic measure, i.e. $\int_{E_f(\varepsilon)} \frac{dr}{r} < +\infty$. In general, we cannot replace power 1/2 in inequality (1) with a smaller number, because for the entire function $f(z) = e^z$ we have

$$\lim_{r\to+\infty} \frac{M_f(r)}{\mu_f(r)\ln^{1/2}\mu_f(r)} = \sqrt{2\pi}.$$

Let us assume, that the exceptional set in Wiman's inequality is absent ([2]) whenever

$$\varlimsup_{r\to+\infty} \frac{\ln M_f(r) - \ln \mu_f(r)}{\ln\ln \mu_f(r)} \le \frac{1}{2}. \tag{2}$$

It is known from ([3], [4]), that for any entire function $f \in \mathcal{E}$ of finite order, i.e.

$$\rho(f) = \varlimsup_{r\to+\infty} \frac{\ln\ln M_f(r)}{\ln r} < +\infty,$$

we have $\ln M_f(r) \sim \ln \mu_f(r),\ r \to +\infty$.

Therefore, the following **question** naturally arises: under what conditions on the function $\ln \mu_f(r)$ relation (2) holds?

Let $\mathcal{H}$ be the class of right-continuous real functions h on $(1, +\infty)$, for which $h(x) \nearrow +\infty,\ x \to +\infty$. Let

$$\triangle(h) = \varlimsup_{t\to+\infty} \frac{\ln h(t)}{\ln\ln t}.$$

We will consider the subclass $\mathcal{L}$ of $\mathcal{H}$ which consists of log-convex functions l, such that $\ln r = o(l(r)),\ r \to +\infty$.

The following theorems give conditions on the functions $\nu_f(r)$, $\ln \mu_f(r)$ and $\ln M_f(r)$, under which the inequality

$$\varlimsup_{r\to+\infty} \frac{\ln M_f(r) - \ln \mu_f(r)}{\ln\ln \mu_f(r)} \le \alpha \tag{3}$$

holds.

Theorem 1 ([2])**.** *Let $\alpha \in (0, +\infty)$, $h \in \mathcal{H}$. For any entire function $f \in \mathcal{E}$ such that $\nu_f(r) \le h(r),\ r \to +\infty$ inequality (3) holds if and only if $\triangle(h) \le \alpha$.*

Theorem 2 ([2])**.** *Let $\alpha \in (0, +\infty)$, $l \in \mathcal{L}$. For any entire function $f \in \mathcal{E}$ such that $\ln M_f(r) \le l(r),\ r \to +\infty$ inequality (3) holds if and only if $\triangle(l) \le 1 + \alpha$.*

Let $\Omega = [0, 1]$ and P be the Lebesgue measure on $\mathbb{R}$. We consider the Steinhaus probability space $(\Omega, \mathcal{A}, P)$, where $\mathcal{A}$ is the σ-algebra of Lebesgue measurable subsets of Ω. Let $(\xi_n(\omega))$ be some sequence of random variables defined on this space. Consider

$$f(z, \omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) a_n z^n. \tag{4}$$

In [5] P. Erdös and A. Renyi proved, that if $(\xi_n) = (\varepsilon_n)$ is a Rademacher sequence, then for any function f for all $\delta > 0$ and almost surely (a.s.) maximum modulus of random function $f(z, \omega)$ satisfies the inequality

$$M_f(r, \omega) \le \mu_f(r) \ln^{1/4} \mu_f(r) \{\ln\ln \mu_f(r)\}^{1+\delta},\ r \le 1,\ r \notin E_f(\delta, \omega),$$

where $E_f(\delta,\omega)$ is a set of finite logarithmic measure.

A sequence $(\xi_n(\omega))$ of real random variables is called ([6]) a *multiplicative system* ($\mathcal{MS}$) if $\mathbf{M}\,\xi_{i_1}\xi_{i_1}\ldots\xi_{i_k}=0$, for all $i_1<i_2<\ldots<i_k$ and $k\geq 1$, where $\mathbf{M}\,\xi$ is the expected value of a random variable ξ.

Let Ξ_0 be the class of complex sequences of random variables $(\xi_n(\omega))$ such that both sequences $\{\operatorname{Re}\xi_n\},\ \{\operatorname{Im}\xi_n\}\in\mathcal{MS}$ and $|\xi_n|=1$ for all $n\geq 0$. In [7], [8] it is proved that the statement of P. Erdös and A. Renyi is fulfilled for any sequence $(\xi_n(\omega))\in\Xi_0$.

In [2] we find a condition on the functions $\nu_f(r)$, $\ln\mu_f(r)$, $\ln M_f(r)$ under which the inequality

$$\varlimsup_{r\to+\infty}\frac{\ln M_f(r,\omega)-\ln\mu_f(r)}{\ln\ln\mu_f(r)}\leq\alpha \tag{5}$$

holds.

Theorem 3 ([2])**.** *Let $(\xi_n)\in\Xi_0,\ \alpha\in(0,+\infty),\ h\in\mathcal{H}$. For any entire function $f\in\mathcal{E}$ such that $\nu_f(r)\leq h(r),\ r\to+\infty$ inequality (5) holds a.s. if and only if $\triangle(h)\leq 2\alpha$.*

Theorem 4 ([2])**.** *Let $(\xi_n)\in\Xi_0,\ \alpha\in(0,+\infty),\ l\in\mathcal{L}$. For any entire function $f\in\mathcal{E}$ such that $\ln M_f(r)\leq l(r),\ r\to+\infty$ inequality (5) holds a.s. if and only if $\triangle(l)\leq 1+2\alpha$.*

2. Wiman's type inequalities without exceptional sets for entire functions of several variables. In this paper we consider entire functions $f\colon\mathbb{C}^p\to\mathbb{C}$, $p\geq 2$, defined by power series

$$f(z)=f(z_1,\ldots,z_p)=\sum_{\|n\|=0}^{+\infty}a_nz^n, \tag{6}$$

where $n=(n_1,\ldots,n_p)\in\mathbb{Z}_+^p$, $z^n=z_1^{n_1}\cdots z_p^{n_p}$, $\|n\|=\sum_{i=1}^p n_i$.

For $r=(r_1,\ldots,r_p)\in\mathbb{R}_+^p$ we set

$$r^\vee=\max\{r_i\colon i\in\{1,\ldots,p\}\},\ r^\wedge=\min\{r_i\colon i\in\{1,\ldots,p\}\},$$
$$M_f(r)=\max\{|f(z)|\colon |z_i|\leq r_i, i\in\{1,\ldots,p\}\},$$
$$\mathfrak{M}_f(r)=\sum_{\|n\|=0}^{+\infty}|a_n|r^n,\ \mu_f(r)=\max\{|a_n|r^n\colon n\in\mathbb{Z}_+^p\}.$$

By T^p we denote the class of entire functions of form (6), for which the entire function $f_j(z_j)\overset{def}{=}f(z_1,\ldots,z_{j-1},z_j,z_{j+1},\ldots,z_p)$ for each $j\in\{1,\ldots,p\}$ is a transcendental entire function of the variable z_j for at least one value of $(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_p)\in\mathbb{C}^{p-1}$.

For every $j\in\{1,\ldots,p\}$ and $(r_1,\ldots,r_{j-1},r_j,r_{j+1},\ldots,r_p)\in\mathbb{R}_+^p$ we set

$$f_j^+(r_j)=\mathfrak{M}_f(r_1,\ldots,r_{j-1},r_j,r_{j+1},\ldots,r_p),$$
$$\mathbb{R}_j^{p-1}=\mathbb{R}_+^{p-1}\setminus\bigcup_{k\neq j}\{(r_1,\ldots,r_{j-1},r_{j+1},\ldots,r_p)\in\mathbb{R}_+^{p-1}\colon r_k=0\}.$$

Proposition 1. *If for some $j\in\{1,\ldots,p\}$ and $(r_1,\ldots,r_{j-1},r_{j+1},\ldots,r_p)\in\mathbb{R}_+^{p-1}$ the entire function $f_j^+(r_j)$ is transcendental then for all $(r_1,\ldots,r_{j-1},r_{j+1},\ldots,r_p)\in\mathbb{R}_j^{p-1}$, we obtain*

$$\ln t=o(\ln\mu_g(r_1,\ldots,r_{j-1},t,r_{j+1},\ldots,r_p))\quad(t\to+\infty),$$

where $\mu_g(t)$ if the maximal term of multiple power series for $g(r)=\mathfrak{M}_f(r)$.

Proof. For any fixed $(r_1,\dots,r_{j-1},r_{j+1},\dots,r_p)\in\mathbb{R}^{p-1}_j$ we get

$$\ln\mu_g(r_1,\dots,r_{j-1},t,r_{j+1},\dots,r_p)\ge\sum_{k\ne j}n_k\ln r_k+$$
$$+\max\{\ln|a_n|+n_j\ln t\colon n=(n_1,\dots,n_j,\dots n_p),\ n_j\in\mathbb{Z}_+\},$$

where $\widetilde{n}=(n_1,\dots,n_{j-1},n_{j+1},\dots,n_p)\in\mathbb{Z}^{p-1}_+$ is any collection such that

$$\#\{n=(n_1,\dots,n_j,\dots n_p)\colon a_n\ne 0\}=+\infty.$$

The existence of such a collection $\widetilde{n}$ follows from the assumption of Proposition 1. It is known that if the function

$$g_j(t)=\sum_{s=0}^{+\infty}|a_{(n_1,\dots,n_{j-1},s,n_{j+1},\dots n_p)}|t^s$$

is transcendental then ([2]) $\ln t=o(\ln\mu_{g_j}(t))$ $(t\to+\infty)$.

So, for all fixed $(r_1,\dots,r_{j-1},r_{j+1},\dots,r_p)\in\mathbb{R}^{p-1}_j$ and $\widetilde{n}$ we have

$$0\le\varlimsup_{t\to+\infty}\frac{\ln t}{\ln\mu_g(r_1,\dots,r_{j-1},t,r_{j+1},\dots,r_p)}\le\varlimsup_{t\to+\infty}\frac{\ln t}{\sum\limits_{k\ne j}n_k\ln r_k+\ln\mu_{g_j}(t)}=0.\qquad\square$$

Proposition 2. *For any entire function $f\in T^p$ we get*

$$\ln r^{\vee}=o(\ln\mu_f(r)),\quad \ln r^{\vee}=o(\ln M_f(r))\quad (r^{\wedge}\to+\infty).\tag{7}$$

Proof. Using Cauchy's inequality $\mu_f(r)\le M_f(r)$, it is sufficient to prove the first relation.

Since the entire functions $f_j(z_j)$ are transcendental for every $j\in\{1,\dots,p\}$, we obtain

$$0\le\varlimsup_{r^{\wedge}\to+\infty}\frac{\ln r^{\vee}}{\ln\mu_f(r)}=\varlimsup_{r^{\wedge}\to+\infty}\frac{\ln r^{\vee}}{\ln\mu_g(r)}\le\varlimsup_{r^{\wedge}\to+\infty}\sum_{j=1}^{p}\frac{\ln r_j}{\ln\mu_g(r)}\le$$
$$\le\varlimsup_{r^{\wedge}\to+\infty}\sum_{j=1}^{p}\frac{\ln r_j}{\ln\mu_g(r_1,\dots,r_{j-1},r_j,r_{j+1},\dots,r_p)}.$$

It remains to apply Proposition 1 p times. $\square$

If there exists $j\in\{1,\dots,p\}$ such that the entire function $f(z_1,\dots,z_j,\dots,z_p)$ is a polynomial in the variable z_j for some fixed values of another variables, then Proposition 2 does not hold. Indeed, if we consider the function $f(z)=\sum_{i=1}^{p-1}e^{z_i}+z_p$ we get

$$\varlimsup_{r^{\wedge}\to+\infty}\frac{\ln r^{\vee}}{\ln M_f(r)}=\varlimsup_{r^{\wedge}\to+\infty}\frac{\ln r^{\vee}}{\ln(\sum_{i=1}^{p-1}e^{r_i}+r_p)}\ge$$
$$\ge\varlimsup_{r^{\wedge}\to+\infty}\frac{\ln r_p}{\ln(\sum_{i=1}^{p-1}e^{r_i}+r_p)}\ge\varlimsup_{n\to+\infty}\frac{\ln r_p^{(n)}}{\ln(\sum_{i=1}^{p-1}e^{r_i^{(n)}}+r_p^{(n)})}=1,$$

where $r_j^{(n)}\overset{def}{=}\ln r_p^{(n)}\to+\infty$ $(n\to+\infty)$, $j\in\{1,\dots,p-1\}$.

So, the following more general statement holds.

Proposition 3. *Let $J \subset \{1, \dots, p\}$ be such that $J \neq \{1, \dots, p\}$ and $\#J = j \geq 1$. If an entire function f is such that for all $j \in J$ the function $P_j(z_j) \overset{def}{=} f(1, \dots, 1, z_j, 1, \dots, 1)$ is a polynomial and for $j \notin J$ the entire function $f(1, \dots, 1, z_j, 1, \dots, 1)$ is transcendental, then*

$$\varlimsup_{r^{\wedge} \to +\infty} \frac{\ln r^{\vee}}{\ln M_f(r)} > 0,$$

i.e. relations (7) cannot hold.

Proof. Let $d_s = \deg P_s$ $(s \in J)$. Without loss of generality, we may assume that $J = \{1, 2, \dots, j\}$ and $d_s \neq 0$ $(1 \leq s \leq j)$. It is easy to see that for all $r \in \mathbb{R}^p_+$ we have

$$M_f(r) \leq \sum_{\|n\|=0}^{+\infty} |a_n| r^n \leq 2^p \mu_f(2r).$$

Applying the assumptions of Proposition 3 we obtain

$$\mu_f(2r) \leq 2^{d_1 + \cdots d_j} r_1^{d_1} \cdots r_j^{d_j} \mu_f(1, \dots, 1, 2r_{j+1}, \dots, 2r_p).$$

If we choose $r \in \mathbb{R}^p_+$ such that

$$r_1^{d_1} = \cdots = r_j^{d_j} = \exp\{\Phi(t)\|d\|\} \quad (t \geq 1),$$

where $\|d\| = \sum_{s=1}^{j} d_s$,

$$r_{j+1} = \cdots = r_p = \varphi(\ln t),$$

and φ is an inverse function to the function

$$\Phi(u) \overset{def}{=} \ln \mu_f(1, \dots, 1, 2r_{j+1}, \dots, 2r_p)\Big|_{r_{j+1} = \cdots = r_p = u},$$

then we get $\ln M_f(r) \leq C + j\|d\|\Phi(t) + \ln t$.

From transcendence of the function $f(1, \dots, 1, z_{j+1}, \dots, z_p)$ we obtain $\ln t = o(\Phi(t))$ $(t \to +\infty)$. Finally,

$$\varlimsup_{r^{\wedge} \to +\infty} \frac{\ln r^{\vee}}{\ln M_f(r)} \geq \varlimsup_{\substack{r^{\wedge} \to +\infty \\ r \in \Gamma}} \frac{\ln r^{\vee}}{C + j\|d\|\Phi(t) + \ln t} \geq \varlimsup_{t \to +\infty} \frac{\Phi(t)\|d\|/d_{s_0}}{C + j\|d\|\Phi(t) + \ln t} = \frac{1}{j d_{s_0}} > 0,$$

where $d_{s_0} = \min_{1 \leq k \leq j} d_k \geq 1$, $\Gamma = \{r = (r_1, \dots, r_p)\colon r_1^{d_1} = \cdots = r_j^{d_j} = \exp\{\Phi(t)\|d\|\},\ r_{j+1} = \cdots = r_p = \varphi(\ln t)\ (t \geq 1)\}$ and $\ln r^{\vee} = \|d\|/d_{s_0}\Phi(t)$ for $r \in \Gamma$. □

For $r = (r_1, \dots, r_p) \in \mathbb{R}^p_+$ we denote

$$\mathcal{N}_f(r) \overset{def}{=} \mathcal{N}(r_1, \dots, r_p) = \max\{\|N\|\colon N \in \mathbb{Z}^p_+,\ |a_N| r^N = \mu_f(r)\}.$$

The multiindex $N(r) = (N_1(r), \dots, N_p(r)) \in \mathbb{Z}^p_+$ we call the *central multiindex* of series (6) at the point $r = (r_1, \dots, r_p) \in \mathbb{R}^p_+$, if

1) $|a_N| r^N = \mu_f(r),\ N = N(r)$,
2) $\sum_{i=1}^{p} N_i(r) = \mathcal{N}_f(r)$.

It is obvious that there can be several central multiindexes at each point $r = (r_1, \dots, r_p) \in \mathbb{R}^p_+$. There is always a finite quantity of multiindexes and $\mathcal{N}_f(r)$ is their common "height".

The following theorem give conditions on the function $\mathcal{N}_f(r)$ under which the following inequality holds

$$\varlimsup_{r^\wedge \to +\infty} \frac{\ln M_f(r) - \ln \mu_f(r)}{\ln \ln \mu_f(r)} \le \alpha. \tag{8}$$

Theorem 5. *Let $\alpha \in (0, +\infty)$, $h \in \mathcal{H}$. Then for any entire function $f \in T^p$ such that $\mathcal{N}_f(r) \le ph(r^\vee)$, $r^\wedge \to +\infty$, inequality (8) holds if and only if*

$$\Delta(h) \stackrel{def}{=} \varlimsup_{t \to +\infty} \frac{\ln h(t)}{\ln \ln t} \le \frac{\alpha}{p}.$$

Proof. Sufficiency. We suppose that $\alpha \in (0, +\infty)$ and $\Delta(h) \le \alpha/p$. Now we prove that for the function $f \in T^p$ condition (8) holds. By the definition of the central multiindex $n \in \mathbb{Z}^p_+$ we have $|a_n|(2r_1)^{n_1} \dots (2r_2)^{n_p} \le |a_{N(2r)}|(2r_1)^{N_1(2r)} \dots (2r_p)^{N_p(r)} = |a_{N(2r)}| r_1^{N_1(r)} \dots r_p^{N_p(r)} 2^{\mathcal{N}_f(2r)} \le \mu_f(r) 2^{\mathcal{N}_f(2r)}$. Therefore the inequality

$$|a_n| r^n \le \mu_f(r) 2^{\mathcal{N}_f(2r) - \|n\|} \tag{9}$$

holds for all $n \in \mathbb{Z}^p_+$. Now we can estimate the maximum modulus of the function $f(z)$ from above. So,

$$G_0(f, r) \stackrel{def}{=} \sum_{n_1=0}^{\mathcal{N}_f(2r)-1} \dots \sum_{n_p=0}^{\mathcal{N}_f(2r)-1} |a_n| r_1^{n_1} \dots r_p^{n_p} \le \mathcal{N}_f^p(2r) \mu_f(r).$$

For each $m \in \{1, \dots, p\}$, $j \in \{1, \dots, C_p^m\}$ and any set $A_j \subset \{1, \dots, p\}$ of the power $\# A_j = m$, we set

$$V_j^{(m)} = \big\{n \in \mathbb{Z}^p_+ \colon n_i \in [\mathcal{N}_f^p(2r), +\infty), i \in A_j;\ n_i \in [0, \mathcal{N}_f^p(2r) - 1],\ i \in \{1, \dots, p\} \setminus A_j\big\},$$

where $\cup_j A_j = \{1, \dots, p\}$, and $A_{j_1} \ne A_{j_2}$, if $j_1 \ne j_2$. From inequality (9) we obtain that for every $m \in \{1, \dots, p\}$

$$G_m(f, r) \stackrel{def}{=} \sum_{j=1}^{C_p^m} \sum_{n \in V_j^{(m)}} |a_n| r_1^{n_1} \dots r_p^{n_p} \le$$

$$\le \sum_{j=1}^{C_p^m} \sum_{n \in V_j^{(m)}} \mu_f(r) 2^{\mathcal{N}_f(2r) - \|n\|} = C_p^m \mu_f(r) \sum_{n \in V_j^{(m)}} 2^{\mathcal{N}_f(2r) - \|n\|} =$$

$$= C_p^m \mu_f(r) \sum_{n_1 = \mathcal{N}_f(2r)}^{+\infty} \dots \sum_{n_m = \mathcal{N}_f(2r)}^{+\infty} \sum_{n_{m+1}=0}^{\mathcal{N}_f(2r)-1} \dots \sum_{n_p=0}^{\mathcal{N}_f(2r)-1} 2^{\mathcal{N}_f(2r) - \|n\|} =$$

$$= C_p^m \mu_f(r) \sum_{n_1=0}^{+\infty} \dots \sum_{n_m=0}^{+\infty} \sum_{n_{m+1}=0}^{\mathcal{N}_f(2r)-1} \dots \sum_{n_p=0}^{\mathcal{N}_f(2r)-1} 2^{-(m-1)\mathcal{N}_f(2r) - \|n\|} \le C_p^m \mu_f(r) 2^{-(m-1)\mathcal{N}_f(2r)} 2^p. \tag{10}$$

Therefore, letting $r^\wedge \to +\infty$ we obtain

$$M_f(r) \le G_0(f, r) + \sum_{m=1}^{p} G_m(f, r) \le \mathcal{N}_f^p(2r) \mu_f(r) + \sum_{m=1}^{p} C_p^m \mu_f(r) 2^{-(m-1)\mathcal{N}_f(2r)} 2^p \le$$

$$\le 2^p \mathcal{N}_f^p(2r) \mu_f(r), \quad \ln M_f(r) - \ln \mu_f(r) \le p \ln(2\mathcal{N}_f(2r))\ (r^\wedge \to +\infty). \tag{11}$$

Using Proposition 3 for the function $f \in T^p$ we have $\ln r^\vee = o(\ln \mu_f(r))$, $r^\wedge \to +\infty$. Therefore, $\ln r^\vee \le \ln \mu_f(r)$, $r^\wedge \ge t_0$, and

$$\overline{\lim_{r^\wedge \to +\infty}} \frac{\ln M_f(r) - \ln \mu_f(r)}{\ln\ln \mu_f(r)} \le \overline{\lim_{r^\wedge \to +\infty}} \frac{\ln(2^p \mathcal{N}_f^p(2r))}{\ln\ln r^\vee} =$$
$$= p \overline{\lim_{r^\wedge \to +\infty}} \frac{\ln \mathcal{N}_f(r)}{\ln\ln r^\vee} \le p \overline{\lim_{r^\wedge \to +\infty}} \frac{\ln(ph(r^\vee))}{\ln\ln r^\vee} = p \overline{\lim_{t \to +\infty}} \frac{\ln h(t)}{\ln\ln t} \le \alpha.$$

Necessity. Here we assume that $\Delta(h) > \alpha/p$. Let us construct the function $f \in T^p$ such that $\mathcal{N}_f(r) \le ph(r^\vee)$, $r^\wedge \to +\infty$, and inequality (8) does not hold.

Let $\beta \in (\alpha/p, \Delta(h))$. Then

$$\overline{\lim_{r \to +\infty}} \frac{h(r)}{\ln^\beta r} = +\infty.$$

For such a function $h(r)$ in [2] the transcendental entire function was constructed

$$g_0(r) = \sum_{n=0}^{+\infty} a_n z^n \quad a_n \ge 0, \tag{12}$$

and the sequence of real numbers (c_n) increasing to $+\infty$ such that

$$\nu_{g_0}(r) \le h(r),\ r > c_0,\quad M_{g_0}(c_n) \ge \mu_{g_0}(c_n) \ln^\beta \mu_{g_0}(c_n),\ n \in \mathbb{N}.$$

Consider the function $g\colon \mathbb{C}^p \to \mathbb{C}$ given by

$$g(z) = g(z_1, \ldots, z_p) = \prod_{i=1}^{p} g_0(z_i). \tag{13}$$

It is obvious, that $g \in T^p$. Then

$$M_g(r) = \prod_{i=1}^{p} M_{g_0}(r_i),\ \mu_g(r) = \prod_{i=1}^{p} \mu_{g_0}(r_i),\ \mathcal{N}_g(r) = \sum_{i=1}^{p} \nu_{g_0}(r_i) \le \sum_{i=1}^{p} h(r_i) \le ph(r^\vee).$$

Therefore, at the sequence $r_j = (c_j, \ldots, c_j)$ we get

$$M_g(r_j) = M_{g_0}^p(c_j) \ge (\mu_{g_0}(c_j) \ln^\beta \mu_{g_0}(c_j))^p = \mu_g(r_j) \ln^{p\beta} \sqrt[p]{\mu_g(r_j)} = \left(\frac{1}{p}\right)^{\beta p} \mu_g(r_j) \ln^{p\beta} \mu_g(r_j).$$

Since $p\beta > \alpha$ and the sequence (c_j) increases to $+\infty$, we obtain

$$\overline{\lim_{r^\wedge \to +\infty}} \frac{\ln M_g(r) - \ln \mu_g(r)}{\ln\ln \mu_g(r)} \ge \overline{\lim_{j \to +\infty}} \frac{\ln M_g(r_j) - \ln \mu_g(r_j)}{\ln\ln \mu_g(r_j)} \ge p\beta > \alpha,$$

i.e. inequality (8) does not hold. □

For the function $h_1(r) = h(r)/p$ we obtain the following statement.

Theorem 6. *Let $\alpha \in (0, +\infty)$, $h_1 \in \mathcal{H}$. For any entire function $f \in T^p$ such that $\mathcal{N}_f(r) \le h_1(r^\vee)$, $r^\wedge \to +\infty$ inequality (8) holds if and only if $\Delta(h_1) \le \alpha/p$.*

In the following theorem we find condition on a function $\ln\mu_f(r)$ under which inequality (8) holds.

Theorem 7. *Let $\alpha\in(0,+\infty)$, $l\in\mathcal{L}$. For any entire function $f\in T^p$ such that $\ln\mu_f(r)\le l(r^\vee)$, $r^\wedge\to+\infty$, inequality (8) holds if and only if $\Delta(l)\le 1+\alpha/p$.*

Proof. Let $h(r)=rl'(r), r>2r_0$ and $l(r_0)=\ln\mu_f(r_0,\ldots,r_0)>0$ for some $r_0>0$. Then from [2] we have $\Delta(l)=1+\Delta(h)$. It remains to prove the equivalence of the conditions

1) $\mathcal{N}_f(r)\le r^\vee l'(r^\vee)$, $r^\wedge\to+\infty$,
2) $\ln\mu_f(r)\le l(r^\vee)$, $r^\wedge\to+\infty$.

1) $\Rightarrow$ 2) From [10] we have for the function $f(z,\ldots,z)$

$$\ln\mu_f(r,\ldots,r)=\ln\mu_f(r_0,\ldots,r_0)+\int_{r_0}^{r}\frac{\mathcal{N}_f(t,\ldots,t)}{t}dt\le l(r).$$

Since the function $\ln\mu_f(r)$ is increasing with respect to each variable, $\ln\mu_f(r)\le l(r^\vee)$, $r^\wedge\to+\infty$.

2) $\Rightarrow$ 1) If condition 1) holds, then we have $\ln\mu_f(\alpha_1 r,\ldots,\alpha_p r)\le l(r)$, $r\to+\infty$ for all $\alpha_i\in[0,1]$ and $i\in\{1,\ldots,p\}$. So, the function $f^*(z)=f(\alpha_1 z,\ldots,\alpha_p z)$ satisfies

$$\ln\mu_{f^*}(r)=\ln\mu_f(\alpha_1 r,\ldots,\alpha_p r),\ \nu_{f^*}(r)=\mathcal{N}_f(\alpha_1 r,\ldots,\alpha_p r).$$

Finally, $\nu_{f^*}(r)\le rl'(r)$, $r\to+\infty$. Then for $\forall\alpha_i\in[0,1]$, $i\in\{1,\ldots,p\}$ we get

$$\mathcal{N}_f(\alpha_1 r,\ldots,\alpha_p r)\le rl'(r),\ r\to+\infty,$$

or $\mathcal{N}_f(r)\le r^\vee l'(r^\vee)=h(r^\vee)$, $r^\wedge\to+\infty$. □

If for entire function $f\in T^p$ we have $\ln\mu_f(r)\le l(r^\vee)$, $r^\wedge\to+\infty$, and $\Delta(l)<+\infty$, then by theorem 7 we get $\ln M_f(r)\sim\ln\mu_f(r)$, $r^\wedge\to+\infty$.

Theorem 8. *Let $\alpha\in(0,+\infty)$, $l\in\mathcal{L}$. For any entire function $f\in T^p$ such that $\ln M_f(r)\le l(r^\vee)$, $r^\wedge\to+\infty$, inequality (8) holds if and only if $\Delta(l)\le 1+\alpha/p$.*

3. Wiman's type inequalities without exceptional sets for random entire functions of several variables. Let Ξ be the class of the sequences of random variables (ξ_n) such that $(\operatorname{Re}\xi_n)$, $(\operatorname{Im}\xi_n)\in\mathcal{MS}$ and $|\xi_n|=1$ (a.s.) for all $n\in\mathbb{Z}_+^p$.

Now we consider the random function

$$f(z,\omega)=\sum_{\|n\|=0}^{+\infty}\xi_n(\omega)a_n z_1^{n_1}\ldots z_p^{n_p},\tag{14}$$

which is defined by the entire function

$$f(z)=\sum_{\|n\|=0}^{+\infty}a_n z_1^{n_1}\ldots z_p^{n_p}$$

and by sequence of random variables $(\xi_n)\in\Xi$.

By $M_f(r,\omega)$ we denote the maximum modulus of the entire random function $f(z,\omega)$ of the form (14). The following theorems give conditions on the functions $\mathcal{N}_f(r),\ln\mu_f(r)$, $\ln M_f(r)$ such that the following inequality holds

$$\varlimsup_{r^\wedge\to+\infty}\frac{\ln M_f(r,\omega)-\ln\mu_f(r)}{\ln\ln\mu_f(r)}\le\beta,\ \beta>0.\tag{15}$$

Theorem 9. *Let* $\beta \in (0, +\infty)$, $h \in \mathcal{H}$ *and* $(\xi_n) \in \Xi$. *For any entire function* $f \in T^p$ *such that* $\mathcal{N}_f(r) \le h(r^\vee)$, $r^\wedge \to +\infty$, *inequality (15) holds a.s. if and only if* $\Delta(h) \le 2\beta/p$.

Theorem 10. *Let* $\beta \in (0, +\infty)$, $l \in \mathcal{H}$ *and* $(\xi_n) \in \Xi$. *For any entire function* $f \in T^p$ *such that* $\ln \mu_f(r) \le l(r^\vee)$, $r^\wedge \to +\infty$, *inequality (15) holds a.s. if and only if* $\Delta(l) \le 1 + 2\beta/p$.

Theorem 11. *Let* $\beta \in (0, +\infty)$, $l \in \mathcal{H}$ *and* $(\xi_n) \in \Xi$. *For any entire function* $f \in T^p$ *such that* $\ln M_f(r) \le l(r^\vee)$, $r^\wedge \to +\infty$, *inequality (15) holds a.s. if and only if* $\Delta(l) \le 1 + 2\beta/p$.

To prove these theorems we need the following lemmas.

Lemma 1 ([8])**.** *If* $X = (X_n(t))$ *is* $\mathcal{MS}$ *and uniformly bounded by number 1, then for all* $b_i \in \mathbb{R}, i \in \{1, \dots, p\}$ *and* $\alpha > 0$

$$\mathbf{M} \exp\left(\alpha \left|\sum_{n=0}^{m} b_n X_n\right|\right) \le 2\exp(\alpha^2 S_m^2),$$

where $S_m = (\sum_{n=0}^{m} |b_n|^2)^{1/2}$.

Lemma 2 (**Modified Cauchy inequality**)**.** *Let* $f(z)$ *analytic function in the polydisc* $G = \{z \in \mathbb{C}^p \colon |z_i| \le r_i,\ i \in \{1, \dots, p\}\}$. *If* $|\operatorname{Re} f(z)| \le M$ *for all* $z \in G$, *then*

$$|a_n| \le \sqrt{2} M / r_1^{n_1} \dots r_p^{n_p}, \tag{16}$$

where a_n *are Taylor's coefficients of the function* f *and* $n = (n_1, \dots, n_p)$.

Proof. In the case $p = 1$ modified Cauchy inequality we find in [11]. When $p = 2$ we can find this inequality in [12, 13].

So, for entire function of one variable of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\{z\colon |z| \le r\}$, $r > 0$ such that $u = |\operatorname{Re} f(z)| \le B$ we obtain ([11])

$$|a_n| r^n \le \sqrt{2}\sqrt{B^2 - (\operatorname{Re} a_0)^2} \le \sqrt{2} B. \tag{17}$$

Let $p = 2$ and $f(z) = \sum_{n+m=0}^{\infty} a_{nm} z_1^n z_2^m$ analytic in $\{z \in \mathbb{C}^2 \colon |z_1| \le r_1,\ |z_2| \le r_2\}$ and $\overline{B} = \sup\{|\operatorname{Re} f(z_1, z_2)| \colon |z_1| \le r_1, |z_2| \le r_2\}$. Then

$$f(z_1, z_2) = \sum_{m=0}^{\infty} \varphi_m(z_1) z_2^m, \tag{18}$$

where $\varphi_m(z_1) = \sum_{n=0}^{\infty} a_{nm} z_1^n$ is convergent power series. Let us fix z_1 and denote $M(z_1) = \sup_{|z_2| \le r_2} |\operatorname{Re} f(z_1, z_2)|$. From modified Cauchy inequality (17) for the series (18) with $B = M(z_1)$ we have $|\psi_m(z_1)| r_2^m \le \sqrt{2} M(z_1)$. Therefore from Cauchy inequality we get

$$|a_{nm}| r_1^n r_2^m \le r_2^m \sup_{|z_1| \le r_1} |\psi_m(z_1)| \le \sqrt{2} \cdot \overline{B}.$$

We will prove modified Cauchy inequality for analytic function of p complex variables. We denote $M(z_1, \dots, z_{p-1}) = \sup\{|\operatorname{Re} f(z)| \colon |z_p| \le r_p\}$. If we apply inequality (17) to the function

$$\varphi(z_p) = f(z) = f(z_1, \dots, z_p) = \sum_{j=0}^{+\infty} \psi_j(z_1, \dots, z_{p-1}) z_p^j,$$

where

$$\psi_j(z_1,\ldots,z_{p-1})=\sum_{\|m'_j\|=j}^{+\infty}a_{m'_j}z_1^{m_1}\cdots z_{p-1}^{m_{p-1}},\quad m'_j=(m_1,\ldots,m_{p-1},j),$$

then we obtain $|\psi_j(z_1,\ldots,z_{p-1})|r_p^j\le\sqrt{2}M(z_1,\ldots,z_{p-1})$. Therefore

$$r_p^j\sup\Big\{|\psi_j(z_1,\ldots,z_{p-1})|\colon |z_1|\le r_1,\ldots,|z_{p-1}|\le r_{p-1}\Big\}\le\sqrt{2}B.$$

It remains to apply classical Cauchy inequality

$$\sup\{|\psi_j(z_1,\ldots,z_{p-1})|\colon |z_1|\le r_1,\ldots,|z_{p-1}|\le r_{p-1}\}\ge|a_{m'_j}|r_1^{m_1}\ldots r_{p-1}^{m_{p-1}}.\qquad\square$$

Lemma 3. *If $p\ge 2$ and*

$$q(\psi)=\sum_{\|n\|=0}^{N}b_n\cos(n_1\psi_1+\ldots+n_p\psi_p+\varphi_n),\ \ \psi=(\psi_1,\ldots,\psi_p),$$

where $\{\varphi_n\}\subset\mathbb{R}$, $\{b_n\colon 0\le\|n\|\le N\}\subset\mathbb{R}$ and $N\ge p$, then there exists a hypercube $K\subset[0,2\pi]^p$ with a side of length $(2N)^{-p-1}$ such that for all $\psi^0\in K$ we have

$$|q(\psi^0)|>\max\{|q(\psi)|\colon \psi\in[0,2\pi]^p\}/2.$$

Proof. Let $p\ge 2$ and

$$g(z)=\sum_{\|n\|=0}^{N}b_ne^{i\varphi_n}z_1^{n_1}\ldots z_p^{n_p}.$$

It is clear, that $\operatorname{Re}g(e^{i\psi_1},\ldots,e^{i\psi_p})=q(\psi_1,\ldots,\psi_p)$. By modified Cauchy's inequality we obtain $|b_n|\le\sqrt{2}\max\{|\operatorname{Re}g(z)|\colon |z_1|=|z_2|=\ldots=|z_p|=1\}\le 2\max\{|q(\psi)|\colon \psi\in[0,2\pi]^p\}$. Therefore for all $i\in\{1,\ldots,p\}$ we get

$$\max\left\{\left|\frac{\partial q(\psi)}{\partial\psi_i}\right|\colon \psi\in[0,2\pi]^p\right\}=\max\left\{\left|\sum_{n_i=1}^{N}\sum_{\|n\|-n_i=0}^{N-n_i}n_ib_n\sin\left(\varphi_n+\sum_{j=1}^{p}n_i\psi_i\right)\right|\colon \psi\in[0,2\pi]^p\right\}\le$$
$$\le\sum_{n_i=1}^{N}\sum_{\|n\|-n_i=0}^{N-n_i}n_i|b_n|\le 2\max\{|q(\psi)|\colon \psi\in[0,2\pi]^p\}\cdot\sum_{n_i=1}^{N}\left(n_i\sum_{\|n\|-n_i=0}^{N-n_i}1\right).\qquad(19)$$

Now we estimate the last sum. Since the equation $\sum_{j=1}^{N}x_j-x_i=k$ has $\overline{C_{N-1}^k}$ distinct roots,

$$\sum_{n_i=1}^{N}\left(n_i\sum_{\|n\|-n_i=0}^{N-n_i}1\right)=\sum_{n_i=1}^{N}\left(n_i\sum_{k=0}^{N-n_i}\overline{C_{p-1}^k}\right)=\sum_{n_i=1}^{N}\left(n_i\sum_{k=0}^{N-n_i}C_{p+k-2}^k\right)=$$
$$=\sum_{n_i=1}^{N}\left(n_i\sum_{k=0}^{N-n_i}\frac{(k+1)(k+2)\ldots(k+p-2)}{(p-2)!}\right)=$$

$$= \frac{1}{(p-2)!}\sum_{n_i=1}^{N}\left(n_i\sum_{k=0}^{N-n_i}(k+1)(k+2)\dots(k+p-2)\right)\le$$
$$\le \frac{1}{(p-2)!}\sum_{n_i=1}^{N}\left(n_i\sum_{k=0}^{N-n_i}(N+p-2)^{p-2}\right)=\frac{(N+p-2)^{p-2}}{(p-2)!}\sum_{n_i=1}^{N}\left(n_i\sum_{k=0}^{N-n_i}1\right)=$$
$$=\frac{(N+p-2)^{p-2}}{(p-2)!}\sum_{n_i=1}^{N}n_i(N-n_i+1)=\frac{(N+p-2)^{p-2}}{(p-2)!}\left((N+1)\sum_{n_i=1}^{N}n_i-\sum_{n_i=1}^{N}n_i^2\right)=$$
$$=\frac{(N+p-2)^{p-2}}{(p-2)!}\left((N+1)\frac{N(N+1)}{2}-\frac{N(N+1)(2N+1)}{6}\right)=$$
$$=\frac{(N+p-2)^{p-2}}{(p-2)!}\cdot\frac{N(N+1)}{6}(3N+3-2N-1)=$$
$$=\frac{(N+p-2)^{p-2}N(N+1)(N+2)}{6(p-2)!}\le\frac{(N+p)^{p+1}}{2p}\le\frac{(2N)^{p+1}}{2p}$$

for $N\ge p$. So, for all $i\in\{1,\dots,p\}$ and $N\ge p$ we get

$$\max\left\{\left|\frac{\partial q(\psi)}{\partial\psi_i}\right|:\psi\in[0,2\pi]^p\right\}\le\frac{(2N)^{p+1}}{p}\max\{|q(\psi)|:\psi\in[0,2\pi]^p\}. \qquad (20)$$

Let $\psi^*=(\psi_1^*,\dots\psi_p^*)\in[0,2\pi]^p$ be a point such that $\max\{|q(\psi)|:\psi\in[0,2\pi]^p\}=|q(\psi^*)|$. Using Lagrange's theorem, from (20) we obtain

$$|q(\psi)-q(\psi^*)|\le|q(\psi_1,\psi_2,\dots,\psi_p)-q(\psi_1^*,\psi_2,\dots,\psi_p)|+$$
$$+|q(\psi_1^*,\psi_2,\dots,\psi_p)-q(\psi_1^*,\psi_2^*,\dots,\psi_p)|+\dots+|q(\psi_1^*,\psi_2^*,\dots,\psi_p)-q(\psi_1^*,\psi_2^*,\dots,\psi_p^*)|\le$$
$$\le\sum_{j=1}^{p}|\psi_j^*-\psi_j|\max\left\{\left|\frac{\partial q(\psi)}{\partial\psi_j}\right|:\psi\in[0,2\pi]^p\right\}\le$$
$$\le\frac{(2N)^{p+1}}{p}\max\{|q(\psi)|:\psi\in[0,2\pi]^p\}\sum_{j=1}^{p}|\psi_j^*-\psi_j|.$$

Then for $\psi^0\in K=\prod_{i=1}^{p}[\psi_i^*-\frac{1}{2(2N)^{p+1}},\psi_i^*+\frac{1}{2(2N)^{p+1}}]$ we obtain

$$|q(\psi^*)|-|q(\psi^0)|\le|q(\psi^*)-q(\psi^0)|\le$$
$$\le\frac{(2N)^{p+1}}{p}p\frac{1}{2(2N)^{p+1}}\max\{|q(\psi)|:\psi\in[0,2\pi]^p\}\le\frac{1}{2}|q(\psi^*)|$$

and $|q(\psi^0)|\ge\frac{1}{2}|q(\psi^*)|=\frac{1}{2}\max\{|q(\psi)|:\psi\in[0,2\pi]^p\}$. □

Lemma 4. *Let $X=(X_n(t))$ be $\mathcal{MS}$ uniformly bound by number 1. Then for all $\beta>0$ there exists a constant $A_{\beta p}>0$, which depends on p and β, such that for all $N\ge N_1(p)=\max\{p,4\pi\}$ and $\{c_n:\|n\|\le N\}\subset\mathbb{C}$ we have*

$$P\left\{t:\ \max\left\{\left|\sum_{\|n\|=0}^{N}c_nX_n(t)e^{in_1\psi_1}\dots e^{in_p\psi_p}\right|:\psi\in[0,2\pi]^p\right\}\ge A_{\beta p}S_N\ln^{\frac{1}{2}}N\right\}\le N^{-\beta},$$

where $S_N^2=\sum_{\|n\|=0}^{N}|c_n|^2,\ p\ge2.$

Proof. Let

$$q(\psi,t)=\operatorname{Re}\Bigg(\sum_{\|n\|=0}^{N} c_nX_n(t)e^{in_1\psi_1}\dots e^{in_p\psi_p}\Bigg),\quad \psi=(\psi_1,\dots,\psi_p).$$

Using lemma 3 with $b_n=|c_n|$ and $\varphi_n=\arg c_n$ in the case when $c_n\neq 0$, we obtain that for $N\geq p$ and $t\in[0,1]$ there exists a hypercube $K(t)$ with a side of length $(2N)^{-p-1}$ such that for all $\psi^0\in K(t)$ we have $\max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\}\leq 2|q(\psi^0,t)|$. So, for all $\alpha>0$ we get

$$\exp(\alpha\max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\})\leq((2N)^{p+1})^p\underset{K(t)}{\int\cdots\int}\exp(2\alpha|q(\psi,t)|)d\psi_1\dots d\psi_p\leq$$
$$\leq 2^pN^{p^2+p}\underset{[0,2\pi]^p}{\int\cdots\int}\exp(2\alpha|q(\psi,t)|)d\psi_1\dots d\psi_p.$$

Applying Markov's inequality $P\{t\colon \eta(t)\geq a\}\leq\frac{\mathbf{M}\eta}{a}$ to the random variable

$$\eta=\exp(2\alpha\max\{q(\psi,t)\colon \psi\in[0,2\pi]^p\}),$$

where $a=e^{2\alpha\lambda},\ \lambda>0$, we get

$$P\{t\colon\ \max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\lambda\}=$$
$$=P\{t\colon\ \exp(2\alpha\max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\})\geq e^{2\alpha\lambda}\}\leq$$
$$\leq e^{-2\alpha\lambda}\,\mathbf{M}(\exp(2\alpha\max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\}))\leq$$
$$\leq 2^pN^{p^2+p}e^{-2\alpha\lambda}\,\mathbf{M}\Bigg(\underset{[0,2\pi]^p}{\int\cdots\int}\exp(2\alpha|q(\psi,t)|d\psi_1\dots d\psi_p\Bigg).$$

Applying the Fubini theorem and Lemma 1 to the previous integral, we get

$$P\{t\colon\ \max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\lambda\}\leq$$
$$\leq 2^pN^{p^2+p}e^{-2\alpha\lambda}\,\mathbf{M}\Bigg(\underset{[0,2\pi]^p}{\int\cdots\int}(\exp(2\alpha|q(\psi,t)|))d\psi_1\dots d\psi_p\Bigg)\leq$$
$$\leq 2^pN^{p^2+p}e^{-2\alpha\lambda}\underset{[0,2\pi]^p}{\int\cdots\int}\exp\Bigg(4\alpha^2\sum_{\|n\|=0}^{N}|b_n|^2\cos^2(n\psi_1+\ldots+n_p\psi_p+\arg b_n)\Bigg)d\psi_1\dots d\psi_p\leq$$
$$\leq(4\pi)^pN^{p^2+p}\exp\Bigg\{4\alpha^2\sum_{\|n\|=0}^{N}|b_n|^2-2\alpha\lambda\Bigg\}.$$

When $\alpha=\lambda/(4S_N^2)$, we obtain for $N\geq\max\{p,4\pi\}$

$$P\{t\colon\ \max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\lambda\}\leq N^{p^2+2p}\exp\left\{-\frac{\lambda^2}{4S_N^2}\right\}. \tag{21}$$

Since

$$q_1(\psi,t)=\operatorname{Im}\Bigg(\sum_{\|n\|=0}^{N}b_nX_n(t)e^{in\psi_1}\dots e^{in_p\psi_p}\Bigg)=\operatorname{Re}\Bigg(-\sum_{\|n\|=0}^{N}ib_nX_n(t)e^{in\psi_1}\dots e^{in_p\psi_p}\Bigg),$$

from the above we get

$$P\{t\colon \max\{|q_1(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\lambda\}\leq N^{p^2+2p}\exp\left\{-\frac{\lambda^2}{4S_N^2}\right\}. \tag{22}$$

We can see that

$$\{t\colon \max\{|q(\psi,t)+iq_1(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\lambda\}\subset$$
$$\subset\{t\colon \max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\}+\max\{|q_1(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\lambda\}\subset$$
$$\subset\left\{t\colon \max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\frac{\lambda}{2}\right\}\bigcup\left\{t\colon \max\{|q_1(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\frac{\lambda}{2}\right\}.$$

From inequalities (21) and (22) we have

$$P\left\{t\colon \max\left\{\left|\sum_{\|n\|=0}^{N}c_nX_n(t)e^{in\psi_1}\dots e^{in_p\psi_p}\right|\colon \psi\in[0,2\pi]^p\right\}\geq\lambda\right\}\leq$$
$$\leq P\{t\colon \max\{|q(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\lambda/2\}+P\{t\colon \max\{|q_1(\psi,t)|\colon \psi\in[0,2\pi]^p\}\geq\lambda/2\}\leq$$
$$\leq 2N^{p^2+2p}\exp\left\{-\frac{\lambda^2}{16S_N^2}\right\}\leq N^{(p+1)^2}\exp\left\{-\frac{\lambda^2}{16S_N^2}\right\}.$$

Let $\lambda=A_{\beta p}S_N(\ln N)^{1/2},\ A_{\beta p}=4\sqrt{(p+1)^2+\beta}$. So,

$$\left\{t\colon \max\left\{\left|\sum_{\|n\|=0}^{N}c_nX_n(t)e^{in\psi_1}\dots e^{in_p\psi_p}\right|\colon \psi\in[0,2\pi]^p\right\}\geq A_{\beta p}S_N(\ln N)^{1/2}\right\}\leq$$
$$\leq N^{(p+1)^2}\exp\left\{-\frac{1}{16}A_{\beta,p}^2\ln N\right\}=\exp\left\{(p+1)^2\ln N-((p+1)^2+\beta)\ln N\right\}=N^{-\beta}.\qquad\square$$

Proof of theorem 9. Necessity. Let $\Delta(h)>\gamma>2\beta/p$. We will construct the function $f\in T^p$ such that $\mathcal{N}_f(r)\leq h(r^\vee),\ r^\wedge\to+\infty$, and for all ω inequality (15) does not hold.

Indeed, for the function $g(z)$, constructed in the proof of the necessity of Theorem 5, as noted in [2], for all ω we have

$$M_g(c_n,\dots,c_n,\omega)\geq S_g(c_n,\dots,c_n,\omega)=S_g(c_n,\dots,c_n)=S_{g_0}^p(c_n)\geq$$
$$\geq(\mu_{g_0}(c_n)\ln^{\gamma/2}\mu_{g_0}(c_n))^p=\mu_g(c_n,\dots,c_n)\ln^{\frac{\gamma p}{2}}\sqrt[p]{\mu_f(c_n,\dots,c_n)}=$$
$$=\left(\frac{1}{p}\right)^{\frac{\gamma p}{2}}\mu_f(c_n,\dots,c_n)\ln^{\frac{\gamma p}{2}}\mu_f(c_n,\dots,c_n).$$

Therefore,

$$\overline{\lim_{r^\wedge\to+\infty}}\frac{\ln M_f(r,\omega)-\ln\mu_f(r)}{\ln\ln\mu_f(r)}\geq\overline{\lim_{n\to+\infty}}\frac{\ln M_f(c_n,\dots,c_n,\omega)-\ln\mu_f(c_n,\dots,c_n)}{\ln\ln\mu_f(c_n,\dots,c_n)}\geq\frac{\gamma p}{2}>\beta.$$

Sufficiency. Suppose that $f\in T^p,\Delta(h)\leq 2\beta/p$ and $\{\xi_n\}\in\Xi$. So, we will prove that almost surely for the function $f(z,\omega)$ inequality (15) holds. Then

$$\overline{\lim_{r^\wedge\to+\infty}}\frac{\ln\mathcal{N}_f(2r)}{\ln\ln r^\vee}\leq\frac{2\beta}{p}.$$

Since for the function $f\in T^p$ the relation $\ln r^{\vee}=o(\ln\mu_f(r))$, $r^{\wedge}\to+\infty$ holds, there exists a continuous decreasing in each variable function $d(r)=d(r_1,\dots,r_p)\to 0$, $r^{\wedge}\to+\infty$ such that

$$p\mathcal{N}_f(2r)\le\{\ln\mu_f(r)\}^{2\beta/p+d(r)},\ r^{\wedge}\to+\infty.$$

Suppose that $r_i^{(0)}\ge 1$, $i\in\{1,\dots,p\}$ are real numbers such that $\ln\mu_f(r_0)=\ln\mu_f(r_1^{(0)},\dots,r_p^{(0)})=q$, where $q=[2^{p/2\beta}]+1$. Now we define the sequence (r_k) such that

$$\lim_{n\to+\infty}\min_{1\le i\le p}r_n^{(i)}=+\infty \text{ and } \ln\mu_f(r_k)=q+k,\ k\ge 0.$$

Also we denote $G_k=\{r\in\mathbb{R}^p_+\colon q+k\le\ln\mu_f(r)\le q+k+1\}$, $N(r_k)=\{\ln\mu_f(r_k)\}^{2\beta/p+d(r_k)}$. So, we consider the sequence of events (A_k) such that for all $k\ge 0$ we get

$$A_k=\Bigg\{\omega\colon \max_{\varphi\in[0,2\pi]^p}\Bigg|\sum_{\|n\|=0}^{[N(r_k)]}\Bigg(|a_n|\Big(r_1^{(k)}\Big)^{n}\dots\Big(r_p^{(k)}\Big)^{n_p}\xi_n(\omega)\prod_{j=1}^{p}e^{in_j\varphi_j}\Bigg)\Bigg|\ge$$
$$\ge C\ln^{1/2}N\Bigg(\sum_{\|n\|=0}^{[N(r_k)]}|a_n|^2\Big(r_1^{(k)}\Big)^{2n_1}\dots\Big(r_p^{(k)}\Big)^{2n_p}\Bigg)^{1/2}\Bigg\},$$

where C is a constant $C(\beta_1)$ from Lemma 4 when $\beta_1=p/\beta$. By this lemma we have

$$\sum_{k=0}^{+\infty}P(A_k)\le\sum_{k=0}^{+\infty}[N(r_k)]^{-p/\beta}\le\sum_{k=0}^{+\infty}\{\ln\mu_f(r_k)\}^{-2\left(1+d(r_k)\frac{p}{\beta}\right)}\le\sum_{k=0}^{+\infty}(q+k)^{-2\left(1+d(r_k)\frac{p}{\beta}\right)}<+\infty.$$

Now using the Borel-Kantelly lemma, the infinite quantity of the events A_k may occur with probability zero. So, almost surely for $k\ge k_0(\omega)$ we get

$$I_f(r_k,\omega)=\max_{\varphi\in[0,2\pi]^p}\Bigg|\sum_{\|n\|=0}^{[N(r_k)]}\Bigg(|a_n|\Big(r_1^{(k)}\Big)^{n_1}\dots\Big(r_p^{(k)}\Big)^{n_p}\xi_n(\omega)\prod_{j=1}^{p}e^{in_j\varphi_j}\Bigg)\Bigg|<$$
$$<C\ln^{1/2}N(r_k)\Bigg(\sum_{\|n\|=0}^{[N(r_k)]}|a_n|^2\Big(r_1^{(k)}\Big)^{2n_1}\dots\Big(r_p^{(k)}\Big)^{2n_p}\Bigg)^{1/2}.$$

Using (10) we can estimate from above the maximum modulus of random analytic function $f(z,\omega)$ on the sequence (r_k). So, for $k\ge k_0(\omega)$ and almost all ω we obtain

$$M_f(r_k,\omega)\le\max\Bigg\{\sum_{\|n\|=0}^{p\mathcal{N}_f(2r_k)}a_nz_1^{n_1}\dots z_p^{n_p}\colon|z_i|=r_i^{(k)},i\in\{1,\dots,p\}\Bigg\}+$$
$$+\sum_{n\notin[0,p\mathcal{N}_f(2r_k)]^p}|a_n|\Big(r_1^{(k)}\Big)^{n_1}\dots\Big(r_p^{(k)}\Big)^{n_p}\le$$
$$\le I_f(r_k,\omega)+\sum_{n\notin[0,\mathcal{N}_f(2r_k)]^p}2^{\mathcal{N}_f(2r_k)-\|n\|}\le C\ln^{1/2}N(r_k)\Bigg(\sum_{\|n\|=0}^{[N(r_k)]}|a_n|^2\Big(r_1^{(k)}\Big)^{2n_1}\dots\Big(r_p^{(k)}\Big)^{2n_p}\Bigg)^{1/2}+$$
$$+\sum_{m=1}^{p}C_p^m\mu_f(r_k)2^{-(m-1)\mathcal{N}_f(2r_k)}2^p\le C_1\ln^{1/2}N(r_k)(N(r_k))^{p/2}\mu_f(r_k)+$$
$$+2p2^p\mu_f(r_k)\le\mu_f(r_k)(\ln\mu_f(r_k))^{\beta+pd(r_k)/2}\ln\ln\mu_f(r_k),$$

as $k \to +\infty$.

Therefore for $r \in G_k$ and $k \geq k_0(\omega)$ almost surely

$$M_f(r,\omega) \leq M_f(r_{k+1},\omega) \leq e\mu_f(r_k)\{1+\ln\mu_f(r_k)\}^{\beta+pd(r_{k+1})/2}\ln\{1+\ln\mu_f(r_k)\} \leq \\ \leq e\mu_f(r)\{1+\ln\mu_f(r)\}^{\beta+pd(r)/2}\ln\{1+\ln\mu_f(r)\}.$$

Since $f \in T^p$, exists $r_0 > 0$ such that $\mathbb{R}^p_+\backslash[0,r_0]^p \subset \bigcup_{k=0}^{+\infty} G_k$. Then for almost all ω relation (15) holds. □

4. Maximum modulus of entire function of finite logarithmic order and arguments of coefficients of multiple power series. We consider a function $f \in T^p$ of finite logarithmic order, i.e.

$$\rho(f) = \varlimsup_{r^\wedge\to+\infty}\frac{\ln\ln M_f(r)}{\ln\ln r^\vee} < +\infty.$$

In this chapter we will give an answer to the following question: under what conditions on the function $M_f(r)$ the inequality

$$\varlimsup_{r^\wedge\to+\infty}\frac{\ln\mathfrak{M}_f(r)-\ln M_f(r)}{\ln\ln M_f(r)} \leq \eta \tag{23}$$

holds? An answer to this question is given in Theorem 12.

Theorem 12. *Let $\eta \in (0,+\infty)$, $l \in \mathcal{L}$. For any entire function $f \in T^p$ such that $\ln M_f(r) \leq l(r^\vee)$, $r^\wedge \to +\infty$, inequality (23) holds if and only if $\triangle(l) \leq 1+2\eta/p$.*

Proof. Sufficiency. If $h(t) = tl'(t), t \to +\infty$, then $\mathcal{N}_f(r) \leq h(r^\vee)$, $r^\wedge \to +\infty$, and $\triangle(h) \leq 2\eta/p$. Now using inequality (10) and Cauchy-Bunyakovsky's inequality we get

$$\mathfrak{M}_f(r) \leq \sum_{n_1=0}^{\mathcal{N}_f(2r)-1}\dots\sum_{n_p=0}^{\mathcal{N}_f(2r)-1}|a_n|r_1^{n_1}\dots r_p^{n_p} + \sum_{n\notin[0,\mathcal{N}_f(2r_k))^p}|a_n|r_1^{n_1}\dots r_p^{n_p} \leq \\ \leq \mathcal{N}_f^{p/2}(2r)S_f(r) + 2^p 2p\mu_f(r) \leq 2\mathcal{N}_f^{p/2}(2r)M_f(r).$$

Since $f \in T^p$, one has $\ln r^\vee = o(\ln M_f(r))$, $r^\wedge \to +\infty$. We note that $\triangle(h) \leq 2\eta/p$, i.e. $\ln\mathcal{N}_f(r) \leq 2\eta\ln\ln r^\vee/p$ as $r^\wedge \to +\infty$. Then

$$\ln\mathfrak{M}_f(r) - \ln M_f(r) \leq \frac{p}{2}\ln\mathcal{N}_f(2r) + 1 \leq \eta\ln\ln r^\vee + 1 < \eta\ln\ln M_f(r).$$

Necessity. We assume that $\triangle(l) > 1+2\eta/p$. Now we prove that there exists a function $f \in T^p$ such that inequality (23) does not hold. Let $\triangle(l) = 1+2\gamma/p$, $\gamma > \eta$, $\varepsilon = (\gamma-\eta)/3$. Then there exists a function $f \in T^p$ such that, on the one hand, (see Theorem 11) by

$$M_f(r) \leq \mu_f(r)\ln^{\gamma+\varepsilon}\mu_f(r),\ r^\wedge \to +\infty$$

and, on the other hand, (see Theorem 8) there exists a sequence $(r_k) = (r_1^{(k)},\dots,r_p^{(k)})$ such that $r_i^{(k)} \nearrow +\infty$, $k \to +\infty$, $i \in \{1,\dots,p\}$ and

$$\mathfrak{M}_f(r_k) \geq M_f(r_k) \geq \mu_f(r_k)\ln^{2\gamma-\varepsilon}\mu_f(r_k),\ r^\wedge \to +\infty.$$

Without loss of generality, we may assume that $\triangle(l) < +\infty$. As follows from Theorem 8 by this condition we have $\ln M_f(r) \sim \ln\mu_f(r)$, $r^\wedge \to +\infty$ and

$$2\mathfrak{M}_f(r_k) \geq 2M_f(r_k)\ln^{\gamma-2\varepsilon}\mu_f(r_k) \geq M_f(r_k)\ln^{\gamma-2\varepsilon}M_f(r_k) = M_f(r_k)\ln^{\eta+\varepsilon}M_f(r_k),$$

i.e. inequality (23) does not hold. □

If for entire function $f\in T^p$ we have $\ln M_f(r)\le l(r^\vee),\ r^\wedge\to+\infty$, and $\Delta(l)<+\infty$, then by theorem 12 we get $\ln M_f(r)\sim\ln\mathfrak{M}_f(r),\ r^\wedge\to+\infty$.

Theorem 13. *Let $\eta\in(0,+\infty),\ l\in\mathcal{L}$. For any entire function $f\in T^p$ such that $\ln\mathfrak{M}_f(r)\le l(r^\vee),\ r^\wedge\to+\infty$, inequality (23) holds if and only if $\triangle(l)\le 1+2\eta/p$.*

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