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BALLEANS AND FILTERS

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A ballean (equivalently, a coarse structure) is an asymptotic counterpart of a uniform topological space. We introduce three new constructions (namely, a ballean-filter mix, a ballean-ideal mix and a filter product of directed sets) to give some balleans with extremal properties. In particular, we construct a non-metrizable Fréchet group ballean.

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Боллеан (эквивалентно, грубая структура) — это асимптотический двойник равномерного топологического пространства. Предлагаются три новые конструкции (смесь боллеана и фильтра, смесь боллеана и идеала, фильтрованное произведение) для построения боллеанов с экстремальными свойствами. В частности, построен неметризуемый групповой боллеан со свойством Фреше.

Following [6, 7], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for every $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius α around x*. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii.*

Given any $x \in X, A \subseteq X, \alpha \in P$ we put

$$B^*(x,\alpha) = \{y \in X : x \in B(y,\alpha)\},\ B(A,\alpha) = \bigcup_{a\in A} B(a,\alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,

$$B(x,\alpha) \subseteq B^*(x,\alpha'),\ B^*(x,\beta) \subseteq B(x,\beta');$$

- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma).$$

A ballean $\mathcal{B}$ on X can also be determined in terms of entourages of the diagonal Δ_X of $X \times X$, in this case it is called a *coarse structure* ([9]).

We suppose that all balleans under consideration are *connected*, i.e. for any $x, y \in X$ there exists $\alpha \in P$ such that $y \in B(x, \alpha)$.

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Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f\colon X_1 \to X_2$ is called a $\prec$-*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$, $f(B_1(x,\alpha)) \subseteq B_2(f(x),\beta)$. If there exists a bijection $f\colon X_1 \to X_2$ such that f and f^{-1} are $\prec$-mappings, $\mathcal{B}_1$ and $\mathcal{B}_2$ are called *asymorphic*. If $X_1 = X_2$ and the identity mapping $id\colon X_1 \to X_2$ is a $\prec$-mapping, we write $\mathcal{B}_1 \prec \mathcal{B}_2$. If $\mathcal{B}_1 \prec \mathcal{B}_2$ and $\mathcal{B}_2 \prec \mathcal{B}_1$, we identify $\mathcal{B}_1$ and $\mathcal{B}_2$, and write $\mathcal{B}_1 = \mathcal{B}_2$.

Let G be a group, $\mathcal{I}$ be an ideal in the Boolean algebra $\mathcal{P}_G$ of all subsets of G, i.e. $\varnothing \in \mathcal{I}$ and if $A, B \in \mathcal{I}$ and $A' \subseteq A$ then $A \cup B \in \mathcal{I}$ and $A' \in \mathcal{I}$. An ideal $\mathcal{I}$ is called a group ideal if, for all $A, B \in \mathcal{I}$, we have $AB \in \mathcal{I}$ and $A^{-1} \in \mathcal{I}$.

Now let X be a G-space with the action $G \times X \to X$, $(g,x) \mapsto gx$, and let $\mathcal{I}$ be a *group ideal* on G such that $\bigcup\{Ax_0\colon A \in \mathcal{I}\} = X$ for some $x_0 \in X$. We define a ballean $\mathcal{B}(G,X,\mathcal{I})$ as a triple $(X, \mathcal{I}, B)$ where $B(x,A) = Ax \cup \{x\}$ for all $x \in X$, $A \in \mathcal{I}$. By [5, Theorem 1], every ballean $\mathcal{B}$ with the support X is asymorphic to the ballean $\mathcal{B}(G,X,\mathcal{I})$ for some group G of permutations of X and some group ideal $\mathcal{I}$ on G. In the case $X = G$ and the left regular action G on X, we denote $\mathcal{B}(G,X,\mathcal{I})$ by $(G,\mathcal{I})$ and say that $(G,\mathcal{I})$ is a *group ballean*.

Given a ballean $\mathcal{B} = (X,P,B)$, a subset $A \subseteq X$ is called

- *large* if there exists $\alpha \in P$ such that $X = B(A,\alpha)$;
- *thick* if $X \setminus A$ is not large;
- *small* if $L \setminus A$ is large for each large subset L of X;
- *bounded* if $A \subseteq B(x,\alpha)$ for some $x \in X, \alpha \in P$;
- *thin* if, for each $\alpha \in P$, there exists a bounded subset V of X such that $B(a,\alpha) \cap A = \{a\}$ for each $a \in A \setminus V$.

By definition, two balleans $\mathcal{B} = (X,P,B)$ and $\mathcal{B}' = (X',P',B')$ are *coarsely equivalent* if there exist large subsets $A \subseteq X$, $A' \subseteq X'$ such that the subballeans $\mathcal{B}_A = (A,P,B_A)$, $\mathcal{B}'_A = (A',P',B'_{A'})$ are asymorphic, where $B_A(x,\alpha) = A \cap B(x,\alpha)$, $B'_{A'}(y,\beta) = A' \cap B'(y,\beta)$.

In this paper, we introduce three constructions (namely, a ballean-filter mix, a ballean-ideal mix and a filter product of directed sets) to give some examples of balleans with extremal properties. In particular, we construct a non-metrizable Fréchet group ballean.

1. Thin subsets and asymptotically isolated ball. Let $\mathcal{B} = (X,P,B)$ be a ballean, $\alpha \in P$. We use the natural preordering on P: $\alpha < \beta$ if and only if $B(x,\alpha) \subset B(x,\beta)$ for every $x \in X$.

We say that a subset $Y \subseteq X$ has *asymptotically isolated α-balls* if, for every $\beta > \alpha$, there exists $y \in Y$ such that $B(y,\beta) = B(y,\alpha)$, so $B(y,\beta) \setminus B(y,\alpha) = \varnothing$. If Y has asymptotically isolated α-balls for some $\alpha \in P$, we say that Y has an *asymptotically isolated balls*. For the case of metric balleans, this notion was introduced in [2].

In what follows, we assume that the set P of radii has a distinguished element 0 such that $B(x,0) = \{x\}$ for each $x \in X$, $B(x,\alpha) = B^*(x,\alpha)$ and write $\beta > \alpha$ if $B(x,\alpha) \subseteq B(x,\beta)$ for all $x \in X$.

If the support X of $\mathcal{B}$ is thin (unbounded), we say that $\mathcal{B}$ is thin (unbounded).

Theorem 1. *An unbounded ballean $\mathcal{B} = (X,P,B)$ is thin if and only if each unbounded subset $Y \subseteq X$ has asymptotically isolated 0-balls.*

Proof. We suppose that $\mathcal{B}$ is not thin. Then there is $\alpha \in P$ such that, for each bounded subset V of X, we can choose $x(V) \in X \setminus V$ such that $|B(x(V),\alpha)| > 1$. We put $Y =$

$\{x(V) \colon V$ is a bounded subset of $X\}$. Clearly, Y is unbounded and $|B(y,\alpha)| > 1$ for each $y \in Y$, so Y has no asymptotically isolated 0-balls.

If $\mathcal{B}$ is thin, for each $\alpha \in P$, there is a bounded subset V such that $|B(x,\alpha)| = 1$ for each $x \in X \setminus V$. If Y is unbounded then $Y \cap (X \setminus V) \neq \varnothing$ so Y has asymptotically isolated 0-balls. □

We say that $\mathcal{B}$ is *coarsely thin* if $\mathcal{B}$ is coarsely equivalent to some thin ballean.

Theorem 2. *For an unbounded ballean $\mathcal{B} = (X, P, B)$, the following statements are equivalent:*

(i) $\mathcal{B}$ is coarsely thin;

(ii) X contains a large thin subset;

(iii) there exists $\alpha \in P$ such that each unbounded subset $Y \subseteq X$ has asymptotically isolated α-balls.

Proof. $(i) \Rightarrow (ii)$. Evident.
$(ii) \Rightarrow (iii)$. Let T be a large thin subset of X, we take $\alpha \in P$ such that $X = B(T,\alpha)$ and choose $\delta \in P$ such that for each $x \in X$

$$B(B(x,\alpha),\alpha) \subseteq B(x,\delta).$$

We show that every unbounded subset Y of X has asymptotically isolated δ-balls. Let $\beta \in P$, $\beta > \delta$. Choose $\gamma \in P$ such that for all $x \in X$

$$B(B(x,\beta),\alpha) \subseteq B(x,\gamma).$$

Since T is thin, there exists a bounded $V \subseteq X$ such that for all $x \in X \setminus V$ $|B(x,\gamma) \cap T| \leqslant 1$. Since Y is an unbounded subset of X, there exists $y \in Y \setminus V$. So, $|B(y,\gamma) \cap T| \leqslant 1$.

Suppose that $z \in B(y,\beta) \setminus B(y,\delta)$. Since $z \in B(T,\alpha)$, $z \in B(t_1,\alpha)$ for some $t_1 \in T$. Then $t_1 \in B(z,\alpha) \subseteq B(B(y,\beta),\alpha) \subseteq B(y,\gamma)$. But $y \in B(t_2,\alpha)$ for some $t_2 \in T$ and, since $|B(y,\gamma) \cap T| \leqslant 1$, we have $t_1 = t_2$. Then $z \in B(t_1,\alpha) = B(t_2,\alpha) \subseteq B(B(y,\alpha),\alpha) \subseteq B(y,\delta)$ which contradicts the choice of z. Hence, $B(y,\beta) \setminus B(y,\delta) = \varnothing$ and Y has asymptotically isolated δ-balls
$(iii) \Rightarrow (i)$. We take a subset $A \subseteq X$ such that the family $\{B(x,\alpha) \colon x \in A\}$ is maximal disjoint. Since A is large, it suffices to show that A is thin. Otherwise, we can choose $\beta > \alpha$ such that, for every bounded subset V, there exists $x(V) \in A \setminus V$ satisfying $|B(x,\beta) \cap A| > 1$. Clearly, the subset $Y = \{x(V) \colon V$ is a bounded subset of $X\}$ is unbounded and $B(y,\beta) \setminus B(y,\alpha) \neq \varnothing$ for each $y \in Y$, so Y has no asymptotically isolated α-balls. □

Given a ballean $\mathcal{B} = (X, P, B)$, we say that a subset $Y \subseteq X$ is *isolated* if, for each $\alpha \in P$, there exists a bounded subset V such that $B(y,\alpha) \subseteq Y$ for each $y \in Y \setminus V$. A ballean $\mathcal{B}$ is *metrizable* if $\mathcal{B}$ is asymorphic to a ballean of some metric space. For the criterion of metrizability (see [7, Theorem 2.1.1]).

Theorem 3. *For an unbounded ballean $\mathcal{B} = (X, P, B)$, the following statements hold*

(i) if there is an unbounded isolated thin subset $Y \subseteq X$ then $\mathcal{B}$ has asymptotically isolated 0-balls;

(ii) if $\mathcal{B}$ is metrizable and has asymptotically isolated 0-balls then there exists an unbounded isolated thin subset Y of X.

Proof. (i) Evident.
(ii) Let (X, d) be an unbounded metric space with asymptotically isolated 0-balls. We choose inductively an injective sequence $(x_n)_{n\in\omega}$ in X such that $B(x_n, n) = \{x_n\}$ for each $n \in \omega$. Then $\{x_n : n \in \omega\}$ is an unbounded isolated thin subset of X. □

In the next section, we show that Theorem 3 (ii) fails to be true for non-metrizable balleans.

2. Ballean-filter mix. Let $\mathcal{B} = (X, P, B)$ be a ballean, φ be a filter on X such that $\cap\varphi = \varnothing$. We define a *ballean-filter mix* $\mathcal{B}_\varphi$ as a ball structure $(X, P \times \varphi, B_\varphi)$ where $P \times \varphi = \{(\alpha, \Phi) : \alpha \in P, \Phi \in \varphi\}$ and, for all $x \in X, \alpha \in P, \Phi \in \varphi$,

$$B_\varphi(x, (\alpha, \Phi)) = \begin{cases} \{x\} & \text{if } x \in \Phi, \\ B(x, \alpha) \cap (X \setminus \Phi) & \text{if } x \in X \setminus \Phi. \end{cases}$$

To verify that $\mathcal{B}_\varphi$ is a ballean, given $\alpha, \alpha' \in P$, we take $\beta \in P$ such that, for each $x \in X$, $B(B(x, \alpha), \alpha') \subseteq B(x, \beta)$, and note that

$$B_\varphi(B_\varphi(x, (\alpha, \Phi)), (\alpha', \Phi')) \subseteq B_\varphi(x, (\beta, \Phi \cap \Phi')),$$

$$B^*_\varphi(\alpha, \Phi) = \begin{cases} \{x\} & \text{if } x \in \Phi, \\ B^*(x, \alpha) \cap (X \setminus \Phi) & \text{if } x \in X \setminus \Phi, \end{cases}$$

$$\bigcup\{B(x, (\alpha, \Phi)) : (\alpha, \Phi) \in P \times \varphi)\} = X.$$

The third condition holds because $\mathcal{B}$ is connected.

By the definition, $\mathcal{B}_\varphi \prec \mathcal{B}$ and each subset $\Phi \in \varphi$ has asymptotically isolated 0-balls in $\mathcal{B}_\varphi$.

Let $\mathcal{B} = (X, P, B)$ be a ballean, $\alpha \in P$, $\Phi_\alpha = \{x \in X : B(x, \alpha) = \{x\}\}$. Clearly, $\mathcal{B}$ has asymptotically isolated 0-balls if and only if $\Phi_\alpha \neq \varnothing$ for each $\alpha \in P$. If $\mathcal{B}$ is unbounded with asymptotically isolated 0-balls then the family $\{\Phi_\alpha : \alpha \in P\}$ is a base of some filter φ_0 on X such that $\bigcap\varphi = \varnothing$.

An unbounded ballean $\mathcal{B}$ with the support X is called *maximal* ([8]) if each stronger ballean $\mathcal{B}$ on X is bounded. By the Zorn Lemma, each unbounded ballean can be strengthened to some maximal ballean. Under the CH, there is a maximal group ballean ([8, Example 4.3]), but it is unknown ([8, Question 4.4]) whether a maximal group ballean can be constructed in ZFC with no additional assumptions.

Theorem 4. *Let $\mathcal{B} = (X, P, B)$ be an unbounded ballean with asymptotically isolated balls. Then the following statements hold*

(i) $\mathcal{B} = \mathcal{B}_{\varphi_0}$;

(ii) *$\mathcal{B}$ is thin if and only if $\mathcal{B}$ is a mix of a bounded ballean on X and φ_0;*

(iii) *$\mathcal{B}$ is maximal if and only if φ_0 is an ultrafilter and $\mathcal{B}$ is thin.*

Proof. (i) It suffices to show that $\mathcal{B} \prec \mathcal{B}_{\varphi_0}$. Let $\alpha \in P$. Take $(\alpha, \Phi_\alpha) \in P \times \varphi_0$. If $x \in \Phi_\alpha$, $B(x, \alpha) = B_{\varphi_0}(x, (\alpha, \Phi_\alpha)) = \{x\}$. If $x \in X \setminus \Phi_\alpha$, $B(x, \alpha) \subseteq X \setminus \Phi_\alpha$, so for all $x \in X$,

$$B(x, \alpha) \subseteq B_{\varphi_0}(x, (\alpha, \Phi_\alpha)).$$

Then the identity mapping is a $\prec$-mapping, and $\mathcal{B} \prec \mathcal{B}_{\varphi_0}$. Hence, $\mathcal{B} = \mathcal{B}_{\varphi_0}$.

(ii) We take an arbitrary non-empty set P' and consider a bounded ballean $\mathcal{B}' = (X, P', B')$, $B'(x,\alpha) = X$ for all $x \in X$, $\alpha \in P'$. Consider a mix of $\mathcal{B}'$ and φ_0. For $\beta > \alpha$, we have

$$B'_{\varphi_0}(x, (\beta, \Phi)) = \begin{cases} \{x\} & \text{if } x \in \Phi, \\ X \setminus \Phi & \text{if } x \in X \setminus \Phi. \end{cases}$$

Suppose that $\mathcal{B}$ is thin and show that $\mathcal{B}_{\varphi_0} = \mathcal{B}'_{\varphi_0}$. Clearly, $\mathcal{B}_{\varphi_0} \prec \mathcal{B}'_{\varphi_0}$.

Let $(\beta, \Phi) \in P' \times \varphi_0$. Then $B'_{\varphi_0}(x, (\beta, \Phi)) \subseteq B'_{\varphi_0}(x, (\beta, \Phi_\gamma))$ for some $\gamma \in P$ such that $\Phi_\gamma \subseteq \Phi$. Since $\mathcal{B}$ is thin, $X \setminus \Phi_\gamma$ is bounded, so there exists $\delta \in P$ such that $X \setminus \Phi_\gamma \subseteq B(x, \delta)$ for every $x \in X$.

If $x \in \Phi_\gamma$, then $B'_{\varphi_0}(x, (\beta, \Phi_\gamma)) = \{x\} = B_{\varphi_0}(x, (\delta, \Phi_\gamma))$.

If $x \in X \setminus \Phi_\gamma$, then $B'_{\varphi_0}(x, (\beta, \Phi_\gamma)) = X \setminus \Phi_\gamma = B(x, \delta) \cap (X \setminus \Phi_\gamma) = B_{\varphi_0}(x, (\delta, \Phi_\gamma))$.

Hence, for all $(\beta, \Phi) \in P' \times \varphi_0$, there exists $(\delta, \Phi_\gamma) \in P \times \varphi_0$ such that, for all $x \in X$,

$$B'_{\varphi_0}(x, (\beta, \Phi)) \subseteq B_{\varphi_0}(x, (\delta, \Phi_\gamma)),$$

and $\mathcal{B}'_{\varphi_0} \prec \mathcal{B}_{\varphi_0}$. Then $\mathcal{B}_{\varphi_0} = \mathcal{B}'_{\varphi_0}$ and, by (i), $\mathcal{B} = \mathcal{B}'_{\varphi_0}$.

On the other hand, let $\mathcal{B}$ be a mix $\mathcal{B}'$ and φ_0. Then $X \setminus \Phi$ is bounded in $\mathcal{B}$ for each $\Phi \in \varphi_0$, so $\mathcal{B}$ is thin.

(iii) Apply (ii) and [7, Example 10.1.2] with $\varphi = \varphi_0$. □

The following two examples concern Theorem 3 (ii). We construct two unbounded balleans $\mathcal{B}_1$ and $\mathcal{B}_2$ with asymptotically isolated balls such that each thin subset in $\mathcal{B}_1$ is bounded, $\mathcal{B}_2$ has unbounded thin subsets but has no unbounded isolated thin subsets.

Example 1. Let G be the group of all permutations of ω, $\mathcal{B} = \mathcal{B}(G, \omega, \mathcal{F}_G)$, where $\mathcal{F}_G$ is the ideal of all finite subsets of G. We take a filter φ on ω such that $\cap\varphi = \varnothing$ and for every infinite subset Y of ω, there exists $\Phi \in \varphi$ such that $Y \setminus \Phi$ is infinite (each free ultrafilter has this property), and put $\mathcal{B}_1 = \mathcal{B}_\varphi$. Clearly, $\mathcal{B}_1$ has asymptotically isolated 0-balls. Let Y be an unbounded subset of $\mathcal{B}_1$. We take an infinite subset $Z \subset Y$ such that $X \setminus Z \in \varphi$, and choose a permutation $g \in G$ such that $g(Z) = Z$ and g has no fixed points on Z. Since $|B_\varphi(z, (\{g\} \times (X \setminus Z)))| = 2$ and each bounded subset in $\mathcal{B}_1$ is finite, Z is not thin.

Example 2. Let X be a binary tree with the root x_0 endowed with the path metric d, $\mathcal{B}$ be the metric ballean $(X, \mathbb{R}^+, B_d)$, $B_d(x, r) = \{y \in X \colon d(x,y) \leqslant r\}$. We take a filter φ on X with the base

$$\{X \setminus F \colon F \text{ is a finite union of infinite rays with the endpoint } x_0\}.$$

We put $\mathcal{B}_2 = \mathcal{B}_\varphi$. Since $\mathcal{B}$ and $\mathcal{B}_2$ have the same set of bounded subsets and $\mathcal{B}$ is a metric ballean, each unbounded subset of $\mathcal{B}_\varphi$ has a thin subset. By the König lemma, each infinite subset of X has an infinite intersection with some ray. It follows that $\mathcal{B}_2$ has no isolated thin subsets.

We recall that a metric space (X, ρ) is *uniformly locally finite* if, for any $n \in \mathbb{N}$, there exists $c(n) \in \mathbb{N}$ such that $|B_\rho(x)| < c(n)$ for each $x \in X$.

By [2], an unbounded uniformly locally finite ultrametric space with no asymptotically isolated balls is coarsely equivalent to the Cantor macro-cube $2^{<\mathbb{N}} = (\bigotimes_{\mathbb{N}} \mathbb{Z}_2, d)$ where $d(x,y) = \max\{n \in \mathbb{N} \colon x_i = y_i \text{ for every } i > n\}$.

By [7, Theorem 3.1.4], every countable ultrametric space is asymorphic to some subspace of the Baire space $\omega^{<\omega} = (\bigotimes_\omega \omega, d)$, $d(x,y) = \max\{n \colon x_i = y_i \text{ for every } i > n\}$.

In the first version of the paper we asked the following two questions.

- Let (X, ρ) be an unbounded *uniformly locally finite ultrametric* space with asymptotically isolated 0-balls. Does there exist a filter φ on $\bigotimes_\omega \mathbb{Z}_2$ with countable base such that (X, ρ) is coarsely equivalent to the ballean-filter mix of $2^{<\omega}$ and φ? If not then how to detect when this is so.
- Let (X, ρ) be a countable unbounded ultrametric space with asymptotically isolated 0-balls. Does there exists a filter φ on $\bigotimes_\omega \omega$ with a countable base such that (X, ρ) is coarsely equivalent to the ballean-filter mix of $\omega^{<\omega}$ and φ? If not then how to detect when this is so?

Taras Banakh has answered both questions affirmatively. With his kind permission, we rewrite corresponding arguments.

To answer the first question, take any unbouned uniformly locally finite ultrametric space X having asymptotically isolated balls. By Theorem 3, X contains an isolated subset $Y \subset X$ whose complement $Z = X \setminus Y$ is coarsely thin. By an Embedding Theorem 3.11 of Dranishnikov and Zarichnyi ([3]), the space X admits a coarse embedding into the Cantor macro-cube $2^{<\mathbb{N}}$. So, we can assume that X is a subset of $2^{<\mathbb{N}}$. Now consider the filter $\varphi = \{F \subset 2^{<\mathbb{N}} \colon 2^{<\mathbb{N}} \setminus (F \cup Y) \text{ is finite}\}$, which is the Fréchet filter on the set $2^{<\mathbb{N}} \setminus Y \supset Z$. It can be shown that the ballean-filter mix $2^{<\mathbb{N}}_\varphi$ is coarsely equivalent to X. A coarse equivalence $f \colon X \to 2^{<\mathbb{N}}_\varphi$ can be defined letting $f|Y = id$ and $f|Z \colon Z \to 2^{<\mathbb{N}} \setminus Y$ be any bijective map. In a similar fashion we can give an answer to the second question.

3. Ballean-ideal mix. Let $\mathcal{B} = (X, P, B)$ be a ballean, $\mathcal{I}$ be an ideal on X such that $B(A, \alpha) \in \mathcal{I}$ for all $A \in \mathcal{I}$, $\alpha \in P$. We define a *ballean-ideal mix* $\mathcal{B}_\mathcal{I}$ as a ball structure $(X, P \times \mathcal{I}, B_\mathcal{I})$ where $P \times \mathcal{I} = \{(\alpha, A) \colon \alpha \in P, A \in \mathcal{I}\}$ and, for each $x \in X$

$$B_\mathcal{I}(x, (\alpha, A)) = \begin{cases} B(A, \alpha) & \text{if } x \in A, \\ B(x, \alpha) & \text{if } x \in X \setminus A. \end{cases}$$

To see that $\mathcal{B}_\mathcal{I}$ is a ballean, we take $\alpha, \alpha' \in \mathcal{I}$ and choose $\beta \in \mathcal{I}$ such that $B(B(x, \alpha), \alpha') \subseteq B(x, \beta)$ for each $x \in X$. Then

$$B_\mathcal{I}(B_\mathcal{I}(x, (\alpha, A)), (\alpha', A')) \subseteq B_\mathcal{I}(x, (\beta, A \cup A')),$$

$$B^*_\mathcal{I}(x, (\alpha, A)) = \begin{cases} B^*(A, \alpha) & \text{if } x \in A, \\ B^*(x, \alpha) & \text{if } x \in X \setminus A. \end{cases}$$

Clearly, $\mathcal{B}_\mathcal{I} \succ \mathcal{B}$. We note that this construction was introduced unexplicitly in [1] for the case of some group balleans, so the following statement can be considered as a generalization of Theorem 2.2 from [1].

Theorem 5. *Let $\mathcal{B}$ be a ballean, $\mathcal{S}$ be the ideal of all small subsets of $\mathcal{B}$. Then the ideal of small subsets of $\mathcal{B}_\mathcal{S}$ coincides with $\mathcal{S}$ and each small subset of $\mathcal{B}_\mathcal{S}$ is bounded.*

Proof. We use the following observation: a subset S of X is small if and only if $X \setminus B(S, \alpha)$ is large for every $\alpha \in P$. Let S be a subset small in $\mathcal{B}$. Take arbitrary radius $(\alpha, A) \in P \times \mathcal{S}$. Then $X \setminus B_\mathcal{S}(S, (\alpha, A)) \supseteq X \setminus B(A \cup S, \alpha)$. Since $A \cup S$ is small in $\mathcal{B}$, $X \setminus B_\mathcal{S}(S, (\alpha, A))$ contains a subset large in $\mathcal{B}$, and hence is large in $\mathcal{B}_\mathcal{S}$. So, S is small in $\mathcal{B}_\mathcal{S}$.

Assume that S is small in $\mathcal{B}_\mathcal{S}$ but not small in $\mathcal{B}$. Take $\alpha \in P$ such that $B(S, \alpha)$ is thick in $\mathcal{B}$. Then $B(S, \alpha) \setminus A$ is thick in $\mathcal{B}$ for each $A \in \mathcal{S}$. By definition of balls in $\mathcal{B}_\mathcal{S}$, $B(S, \alpha)$ is thick in $\mathcal{B}_\mathcal{S}$. Since $B(S, \alpha) = B_\mathcal{S}(S, (\alpha, \varnothing))$, we conclude that S is not small in $B_\mathcal{S}$.

Let S be a small subset in $\mathcal{B}_{\mathcal{S}}$. Take any $x \in S$. Since S is also small in $\mathcal{B}$, $B_{\mathcal{S}}(x, (\alpha, S)) = B(S, \alpha) \supseteq S$, so S is bounded in $\mathcal{B}_{\mathcal{S}}$. □

Under the CH, there exists ([8, Example 4.2]) a group ideal $\mathcal{I}$ on a countable group of period 2 such that each unbounded set in the group ballean $(G, \mathcal{I})$ is large, so each small subset of $(G, \mathcal{I})$ is bounded. It is unknown ([8, Question 4.7]) if there is a ZFC-example of a group ballean in which every small subset is bounded. We note that every thin subset of a group ballean is small, and pose a weaker version of this question.

Question 1. *In ZFC, does there exist an unbounded group ballean in which each thin subset is bounded?*

Let X be an infinite set, $\mathcal{I}$ be an ideal of X such that $\bigcup \mathcal{I} = X$. We put $\varphi = \{X \setminus A \colon A \in \mathcal{I}\}$ and denote by $\mathcal{X}_\varphi$ the mix of the bounded ballean $\mathcal{X}$ on X and φ. Each small subset in $\mathcal{X}_\varphi$ is bounded and the ideal of bounded subsets coincides with $\mathcal{I}$.

Question 2. *Given an ideal $\mathcal{I}$ on an infinite group G, how to detect whether $\mathcal{I}$ is an ideal of small subsets of some group ballean on G? In particular, if G is amenable, does there exist a group ballean on G in which the ideal of small subsets coincides with the ideal of universal null sets? A subset $A \subseteq G$ is universal null if $\mu(A) = 0$ for each Banach measure μ on G.*

Question 3. *Let G be a countable group, A be a subset of G small in each group ballean on G. Is A finite?*

To answer Question 3 affirmatively, it suffices to give a positive answer to the following question.

Question 4. *Let G be a countable group, L be an infinite subset of G. Does there exist a group ballean on G in which L is large? For some groups (in particular, for $\mathbb{Z}$), this is so.*

4. Filter product. Let κ be an infinite cardinal, $\{X_\alpha \colon \alpha < \kappa\}$ be a family of sets with the distinguished elements $e_\alpha \in X_\alpha$. The direct product $X = \bigotimes_{\alpha<\kappa}(X_\alpha, e_\alpha)$ can be identified with the set of all κ-sequences $(x_\alpha)_{\alpha<\kappa}$ such that $x_\alpha \in X_\alpha$ and $x_\alpha = e_\alpha$ for all but finitely many $\alpha < \kappa$. We fix a filter φ on κ such that $\bigcap \varphi = \varnothing$ and consider the ball structure $\mathcal{B}_{X,\varphi} = (X, \varphi, B_\varphi)$ where

$$B_\varphi(x, \Phi) = \{y \in X \colon y_\alpha = x_\alpha \text{ for all } \alpha \in \kappa \setminus \Phi\}.$$

It is easy to see that $\mathcal{B}_{X,\varphi}$ is a ballean. If each X_α is a group with the identity e_α then X is a group and $\mathcal{B}_{X,\varphi}$ is a group ballean determined by the group ideal with the base

$$\{\bigotimes_{\alpha\in\kappa\setminus\Phi}(X_\alpha, e_\alpha) \colon \Phi \in \varphi\}.$$

Given a ballean $\mathcal{B} = (X, P, B)$, we say that a sequence $(x_n)_{n\in\omega}$ in X is *asymptotically convergent* if, for every bounded subset V of X, there exists $m \in \omega$ such that $x_n \in X \setminus V$ for each $n > m$. If each unbounded subset of X has an asymptotically convergent sequence, we say that $\mathcal{B}$ is Fréchet. Clearly, each metrizable ballean is Fréchet. Now we construct in ZFC a countable group ballean which is Fréchet but non-metrizable.

Example 3. A point s of a topological space S is called a *Fréchet point* if each subset $A \subseteq S \setminus \{x\}$ with $s \in \mathrm{cl}A$ contains a sequence convergent to s. It is not hard (see [4, Example 1.6.18]) to construct a countable topological space (S, τ) with a Fréchet point of uncountable character, i.e. the filter of neighbourhoods of s has a countable base. We identify S with $\omega \cup \{\infty\}$, s with ∞, and put

$$\varphi = \{U \setminus \{\infty\} \colon U \text{ is a neighbourhood of } \infty \text{ in } \tau\}.$$

Then we consider the filter product $\mathcal{B}_{X,\varphi}$ where $X = \bigotimes_{i<\omega}(X_i, e_i)$, X_i is a group, $|X_i| > 1$. Since φ has no countable base, by [7, Theorem 2.1.1], $\mathcal{B}_{X,\varphi}$ is not metrizable. Let Y be an unbounded subset of $\mathcal{B}_{X,\varphi}$. For each $\Phi \in \varphi$, we take $y_\Phi \in Y$ such that $|\text{supt } y_\Phi \cap \Phi \neq \varnothing$, where $\text{supt } y = \{i \in \omega \colon y_i \neq e_i\}$, choose $f(y_\Phi) \in \text{supt } y_\Phi \cap \Phi$ and put $Z = \{f(y_\Phi) \colon \Phi \in \varphi\}$. Since $\infty \in \mathrm{cl}_\tau\, Z$, there exists a sequence $(z_n)_{n\in\omega}$ in Z converging to ∞. For each $n \in \omega$, we take $y_n \in Y$ such that $f(y_n) = z_n$. Then the sequence $(y_n)_{n\in\omega}$ is asymptotically convergent.

We say that a ballean $\mathcal{B} = (X, P, B)$ is *thin Fréchet* if each unbounded subset Y of X contains an asymptotically convergent sequence $(y_n)_{n\in\omega}$ such that the subset $\{y_n \colon n \in \omega\}$ is thin. Again, each metrizable ballean is thin Fréchet. The ballean $\mathcal{B}$ from Example 1 is Fréchet but not thin Fréchet. The group ballean $B_{X,\varphi}$ from Example 3 is thin Fréchet. Indeed, passing to subsequences, we may suppose that $z_{n+1} \notin \bigcup_{i=0}^{n} \mathit{supt}\ y_i$ for each $n \in \omega$. Then the subset $\{y_n \colon n \in \omega\}$ is thin.

Question 5. *Is every Fréchet group ballean thin Fréchet?*

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be balleans with common support X such that $\mathcal{B}_1 \prec \mathcal{B}_2$. Clearly, every large subset of $\mathcal{B}_1$ is large in $\mathcal{B}_2$. However, in this case the ideals $\mathcal{S}_1, \mathcal{S}_2$ of small subsets of $\mathcal{B}_1, \mathcal{B}_2$ could be non-incident: $\mathcal{S}_1 \setminus \mathcal{S}_2 \neq \varnothing$ and $\mathcal{S}_2 \setminus \mathcal{S}_1 \neq \varnothing$.

Example 4. Let X be a direct product $\bigotimes_{i<\omega} X_i$ of countable groups. We denote by $\mathcal{B}_1$ the group ballean $(X, \mathcal{F}_X)$. Then we put $G_n = \bigotimes_{i\leqslant n} X_n$, denote by $\mathcal{I}$ the group ideal with the base $\{G_n \colon n \in \omega\}$ and put $\mathcal{B}_2 = (X, \mathcal{I})$. To show that $\mathcal{S}_1 \setminus \mathcal{S}_2 \neq \varnothing$, we note that $\bigotimes_{0<i<\omega} X_i$ is small in $\mathcal{B}_1$ but large in $\mathcal{B}_2$. To show that $\mathcal{S}_2 \setminus \mathcal{S}_1 \neq \varnothing$, we enumerate $\{K_n \colon n \in \omega\}$ all finite subsets of X and choose inductively a sequence $(x_n)_{n\in\omega}$ in X such that, for each $n \in \omega$,

$$G_{n+1}(K_0x_0 \cup \ldots \cup K_nx_n) \cap G_{n+1}K_{n+1}x_{n+1} = \varnothing.$$

We put $A = \bigcup_{n\in\omega} K_nx_n$. By the construction, A is thick in $\mathcal{B}_1$ so $A \notin \mathcal{S}_1$. On the other hand, $G_{n+1}g \nsubseteq G_nA$ for all $g \in G$, $n \in \omega$. It follows that G_nA is not thick in $\mathcal{B}_2$ so $A \in \mathcal{S}_2$.

The following example was suggested by Sergiy Slobodianiuk.

Example 5. We construct two balleans $\mathcal{B}_1, \mathcal{B}_2$ with common support X such that $\mathcal{B}_1 \prec \mathcal{B}_2$ but the families $\mathcal{T}_1$ and $\mathcal{T}_2$ of thin subsets of $\mathcal{B}_1$ and $\mathcal{B}_2$ are not incident. Let X be a direct product $X = \bigotimes_{i\in\omega} X_i$ of countable groups with the identities e_i. We consider the group ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ on X with the bases

$$\{F \times X_1 \times \ldots \times X_n \colon F \in \mathcal{F}_{X_0}, n \in \omega\},\ \{X_1 \times \ldots \times X_n \colon n \in \omega\},$$

put $\mathcal{B}_1 = (X, \mathcal{I}_1)$, $\mathcal{B}_2 = (X, \mathcal{I}_2)$ and note that $\mathcal{B}_1 \prec \mathcal{B}_2$. Clearly, $X_0 \notin \mathcal{T}_1$ but X_0 is bounded in $\mathcal{B}_2$ so $X_0 \in \mathcal{T}_2 \setminus \mathcal{T}_1$. Then we take an infinite thin subset T in the ballean $(X, \mathcal{F}_{X_0})$, enumerate $T = \{g_m \colon m \in \omega\}$, pick $x_n \in X_n \setminus \{e_n\}$ and put $A = \{g_mx_n \colon 1 < n \leqslant m\}$. It is easy to see that $A \in \mathcal{T}_1 \setminus \mathcal{T}_2$.

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