УДК 519.6

Yu. A. Muzychuk, R. S. Chapko

ON VARIATIONAL FORMULATIONS OF INNER BOUNDARY VALUE PROBLEMS FOR INFINITE SYSTEMS OF ELLIPTIC EQUATIONS OF SPECIAL KIND

Yu. A. Muzychuk, R. S. Chapko. *On variational formulations of inner boundary value problems for infinite systems of elliptic equations of special kind*, Mat. Stud. **38** (2012), 12–34.

We consider boundary value problems for infinite triangular systems of elliptic equations with variable coefficients in 3d Lipschitz domains. Variational formulations of Dirichlet, Neumann and Robin problems are received and their well posedness in corresponding Sobolev spaces is established. With the help of introduced q-convolution the integral representations of generalized solutions of formulated problems in the case of constant coefficients are built. We investigate the properties of integral operators and well posedness of received systems of boundary integral equations.

Ю. А. Музычук, Р. С. Хапко. *О вариационных формулировках внутренних граничных задач для бесконечных систем эллиптических уравнений специального вида* // Мат. Студії. – 2012. – Т.38, №1. – С.12–34.

В трехмерных ограниченных областях с липшицевой границей рассматриваются граничные задачи для бесконечных систем эллиптических уравнений специального треугольного вида с переменными коэффициентами. Сформулированы вариационные постановки задач Дирихле, Неймана и Робина и установлено их корректность в соответствующих пространствах Соболева. С помощью введенного понятия q-свертки для поставленных задач в случае постоянных коэффициентов построены интегральные представления решений. Исследованы свойства интегральных операторов и корректность полученных систем граничных интегральных уравнений.

1. Introduction. The variational approach is being widely used for the investigation of the well-posedness of boundary value problems for elliptic equations and for their subsequent solution. Since it provides the ability to transfer the ellipticity property of differential operators on the corresponding boundary integral operators, it has been proved to be very effective when reducing boundary value problems for elliptic equations with constant coefficients to boundary integral equations. This allows to use suitable Green formulae and the integral representation of the solution through its trace and the co-normal derivative on the boundary. Substantial results on the variational approach in the context of the reduction of boundary value problems for elliptic equations and their finite systems to boundary integral equations are given in [13–16].

Various semi-discretization approaches for time-dependent boundary value problems often lead to necessity of investigation and numerical solution of boundary value problems

2010 *Mathematics Subject Classification:* 31B10, 35D30, 35J50, 35J57, 45B05.
Keywords: elliptic equation, infinite system, Sobolev spaces, Lipschitz boundary, variational formulation, generalized solution, Green formulae, layer potential.

for infinite elliptic systems ([2, 3, 6, 7, 9–11]). Variational formulations of the Dirichlet and Neumann boundary value problems for an infinite triangular system of elliptic equations with constant coefficients have been studied in [5]. Our goal is to generalize these results for the case of a triangular system with variable coefficients and to investigate the Robin problem besides the Dirichlet and Neumann boundary value problems.

The paper is organized as follows. In the first chapter we define the Sobolev spaces necessary for our research and provide information about the objects we are dealing with. In the second chapter we introduce variational formulations of boundary value problems for the infinite triangular system of elliptic equations with measurable and bounded coefficients and establish their well-posedness. In chapter 3 we provide equivalent variational formulations of boundary value problems that lead to an effective integral model. For this purpose we use a special operation of q-convolution ([5]) and derive the analogues of the Green formulae. Then in chapter 4 we assume that the coefficients of elliptic equations are constant. It allows us to build the integral representation of the solution of boundary value problems based on the fundamental solution of the system of differential equations and to investigate the properties of corresponding boundary integral operators. Finally, in the fifth chapter boundary integral equations equivalent to the boundary value problems are obtained and we show the well-posedness of these problems.

Thus we start with definitions of necessary functional spaces and the elliptic operator. Let $\Omega \subset \mathbb{R}^3$ be a bounded and simply connected domain with Lipschitz boundary Γ and $\bar{\nu}(x) = (\nu_1(x), \nu_2(x), \nu_3(x))$ in a unit normal to Γ at the point x. $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are the spaces of all test functions and distributions on them correspondingly ([15]). We will use the Lebesgue space $L_2(\Omega)$ and the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ of real-valued scalar functions and duals to them $\widetilde{H}^{-1}(\Omega) := (H^1(\Omega))'$ and $H^{-1}(\Omega) := (H_0^1(\Omega))'$ correspondingly ([13]). We will have in mind that

$$H_0^1(\Omega) \subset L_2(\Omega) \subset H^{-1}(\Omega) := \left(H_0^1(\Omega)\right)', \tag{1}$$

$$H^1(\Omega) \subset L_2(\Omega) \subset \widetilde{H}^{-1}(\Omega). \tag{2}$$

It is known ([13]), that the space $\widetilde{H}^{-1}(\Omega)$ can be given in the form of a direct sum

$$\widetilde{H}^{-1}(\Omega) = \widetilde{H}_0^{-1}(\Omega) \oplus \widetilde{H}_\Gamma^{-1}(\Omega), \tag{3}$$

where the subspace $\widetilde{H}_\Gamma^{-1}(\Omega) := \{f \in \widetilde{H}^{-1}(\Omega) | \langle f, v\rangle_{H^1(\Omega)} = 0 \ \forall v \in H_0^1(\Omega)\}$ consists of functionals with support only on Γ and $\widetilde{H}_0^{-1}(\Omega) := (\widetilde{H}_\Gamma^{-1}(\Omega))^\perp$. Note that $\widetilde{H}_0^{-1}(\Omega)$ can be identified with a subspace of $H^{-1}(\Omega)$. Hereinafter forms $\langle \cdot, \cdot\rangle_{H^1(\Omega)}$ and $\langle \cdot, \cdot\rangle_{H_0^1(\Omega)}$ denote duality pairings on spaces $H^1(\Omega)$ and $\widetilde{H}^{-1}(\Omega)$ and on $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ respectively. $(\cdot, \cdot)_{L_2(\Omega)}$ is the inner product in $L_2(\Omega)$.

Let $a_{i,j}$ $(i, j \in \{1, 2, 3\})$ and a_0 be measurable and bounded functions that satisfy the conditions

$$a_{i,j}(x) = a_{j,i}(x) \quad (i, j \in \{1, 2, 3\}) \text{ for almost all } x \in \Omega, \tag{4}$$

$$\sum_{i,j=1}^{3} a_{i,j}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{3} \xi_i^2 \text{ for arbitrary } \xi_1, \xi_2, \xi_3 \in \mathbb{R} \text{ and almost all } x \in \Omega, \tag{5}$$

where α is some positive constant and

$$a_0(x) > 0 \text{ for almost all } x \in \Omega. \tag{6}$$

Consider the formal second order differential operator

$$(Pu)(x):=-\sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}\Big[a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\Big]+a_0(x)u(x),\quad x\in\Omega, \tag{7}$$

and the bilinear form associated with it

$$a_\Omega(u,v):=\int_\Omega\Big[\sum_{i,j=1}^{3}a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}+a_0(x)u(x)v(x)\Big]dx,\quad u,v\in H^1(\Omega).$$

Here the derivatives are interpreted in sense of $\mathcal{D}'(\Omega)$.

Let us introduce the space $H^1(\Omega,P):=\{u\in H^1(\Omega)|Pu\in\widetilde{H}_0^{-1}(\Omega)\}$. Let $\gamma_0\colon H^1(\Omega)\to H^{1/2}(\Gamma)$ be a trace operator and $\gamma_1\colon H^1(\Omega,P)\to\big(H^{1/2}(\Gamma)\big)'=:H^{-1/2}(\Gamma)$ — a co-normal derivative operator ([12]) that in case of functions from $H^2(\Omega)$, a sufficiently smooth boundary Γ and continuous on $\bar\Omega$ coefficients $a_{i,j}(\in\{1,2,3\})$ coincides with the co-normal derivative

$$\partial_{\bar\nu}u(x):=\sum_{i,j=1}^{3}a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\nu_j(x),\ x\in\Gamma.$$

It is known ([12]) that for functions $u\in H^1(\Omega)$ and $v\in H_0^1(\Omega)$ the following equality holds

$$\langle Pu,v\rangle_{H_0^1(\Omega)}=a_\Omega(u,v), \tag{8}$$

and for $u\in H^1(\Omega,P)$ and $v\in H^1(\Omega)$ — the equality

$$\langle Pu,v\rangle_{H^1(\Omega)}=a_\Omega(u,v)-\langle\gamma_1u,\gamma_0v\rangle_\Gamma. \tag{9}$$

Equality (9) is also called the *first Green formula*.

2. Variational formulations of boundary value problems. We consider an infinite system of elliptic equations for the unknown functions $u_0,u_1,...,u_k,...$ on Ω

$$\begin{cases}Pu_0=f_0,\\ c_{1,0}u_0+Pu_1=f_1,\\ c_{2,0}u_0+c_{2,1}u_1+Pu_2=f_2,\\ \quad\cdots\cdots\cdots\cdots\\ c_{k,0}u_0+c_{k,1}u_1+...+c_{k,k-1}u_{k-1}+Pu_k=f_k,\\ \quad\cdots\cdots\cdots\cdots\cdots\end{cases} \tag{10}$$

where $c_{i,j}(i,j\in\mathbb{N}_0,\mathbb{N}_0:=\mathbb{N}\cup\{0\})$ are some given measurable and bounded on Ω functions with $c_{i,j}=0$ when $j\geq i$ and $f_i(i\in\mathbb{N}_0)$ are given functions (functionals) on Ω.

We investigate boundary value problems for system (10) that consists in finding the solutions of (10) which satisfy one of the following boundary value conditions

- Dirichlet condition:
$$u_k|_\Gamma=\tilde h_k,\ k\in\mathbb{N}_0, \tag{11}$$
- Neumann condition:
$$\partial_{\bar\nu}u_k|_\Gamma=\tilde g_k,\ k\in\mathbb{N}_0, \tag{12}$$
- Robin condition:
$$(\partial_{\bar\nu}u_k+b_{k,0}u_0+b_{k,1}u_1+...+b_{k,k-1}u_{k-1}+b_{k,k}u_k)|_\Gamma=\tilde g_k,\ k\in\mathbb{N}_0, \tag{13}$$

where $\tilde{h}_i, \tilde{g}_i$ $(i \in \mathbb{N}_0)$ are given functions (functionals) on Γ, $b_{i,j} \in L_\infty(\Gamma)$ $(i, j \in \mathbb{N}_0)$ — given functions on Γ with $b_{i,j} = 0$ when $j > i \geq 0$, $b_{i,i} \geq \tilde{b}_i > 0$, $\tilde{b}_i$ — given constants. In other words, we will consider the Dirichlet problem (10), (11), the Neumann problem (10), (12) and the Robin problem (10), (13).

We build variational formulations for each of the formulated boundary value problems. At first we conduct some auxiliary observations. Let $u_0, u_1, ...$ be an infinite system of functions from $H^1(\Omega)$. Then we denote

$$G_k(u_0, u_1, ...) := c_{k,0}u_0 + c_{k,1}u_1 + ... + c_{k,k-1}u_{k-1} + Pu_k, \ \ k \in \mathbb{N}_0. \tag{14}$$

It is obvious that $G_k(u_0, u_1, ...) \in H^{-1}(\Omega), k \in \mathbb{N}_0$. Considering for each $k \in \mathbb{N}_0$ the action of such functional $G_k(u_0, u_1, ...)$ on an arbitrary function $v_k \in H^1_0(\Omega)$ and using the relation (8), we arrive at such system of the variational equalities

$$\begin{cases} \langle G_0(u_0, u_1, ...), v_0\rangle_{H^1_0(\Omega)} = a_\Omega(u_0, v_0), \\ \langle G_1(u_0, u_1, ...), v_1\rangle_{H^1_0(\Omega)} = (c_{1,0}u_0, v_1)_{L_2(\Omega)} + a_\Omega(u_1, v_1), \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \langle G_k(u_0, u_1, ...), v_k\rangle_{H^1_0(\Omega)} = \sum_{i=0}^{k-1} (c_{k,i}u_i, v_k)_{L_2(\Omega)} + a_\Omega(u_k, v_k), \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases} \tag{15}$$

Similarly if functions $u_0, u_1, ...$ are elements of the space $H^1(\Omega, P)$ then taking into account that $G_k(u_0, u_1, ...) \in \widetilde{H}^{-1}_0(\Omega)$ for each $k \in \mathbb{N}_0$ and using the first Green formula (9), we obtain the following relations

$$\begin{cases} \langle G_0(u_0, u_1, ...), v_0\rangle_{H^1(\Omega)} = a_\Omega(u_0, v_0) - \langle \gamma_1 u_0, \gamma_0 v_0\rangle_\Gamma, \\ \langle G_1(u_0, u_1, ...), v_1\rangle_{H^1(\Omega)} = (c_{1,0}u_0, v_1)_{L_2(\Omega)} + a_\Omega(u_1, v_1) - \langle \gamma_1 u_1, \gamma_0 v_1\rangle_\Gamma, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \langle G_k(u_0, u_1, ...), v_k\rangle_{H^1(\Omega)} = \sum_{i=0}^{k-1} (c_{k,i}u_i, v_k)_{L_2(\Omega)} + a_\Omega(u_k, v_k) - \langle \gamma_1 u_k, \gamma_0 v_k\rangle_\Gamma, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases} \tag{16}$$

for an arbitrary infinite system of functions $v_0, v_1, ... \in H^1(\Omega)$.

Relations (15) and (16) for system (10) are analogues to relations (8) and (9) correspondingly. Based on the equalities (15) and (16) we can analogously to elliptic equations introduce definitions of generalized solutions of boundary value problems for system (10).

Definition 1. Let $f_0, f_1, ...$ be a set of elements of $H^{-1}(\Omega)$ and $\tilde{h}_0, \tilde{h}_1, ...$ — elements of $H^{1/2}(\Gamma)$. The sequence of functions $u_0, u_1, ...$ from $H^1(\Omega)$ is called a *generalized solution of the Dirichlet problem* (10), (11) if it satisfies the (infinite) system of equalities

$$\begin{cases} a_\Omega(u_0, v_0) = \langle f_0, v_0\rangle_{H^1_0(\Omega)}, \\ (c_{1,0}u_0, v_1)_{L_2(\Omega)} + a_\Omega(u_1, v_1) = \langle f_1, v_1\rangle_{H^1_0(\Omega)}, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \sum_{i=0}^{k-1} (c_{k,i}u_i, v_k)_{L_2(\Omega)} + a_\Omega(u_k, v_k) = \langle f_k, v_k\rangle_{H^1_0(\Omega)}, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases} \tag{17}$$

for an arbitrary (infinite) set of functions $v_0, v_1, ... \in H^1_0(\Omega)$ and boundary conditions

$$\gamma_0 u_k = \tilde{h}_k, \ \ k \in \mathbb{N}_0. \tag{18}$$

Definition 2. Let $f_0, f_1, ...$ be a sequence of elements of $\widetilde{H}_0^{-1}(\Omega)$ and $\tilde{g}_0, \tilde{g}_1, ...$ — of $H^{-1/2}(\Gamma)$. A sequence of functions $u_0, u_1, ...$ з $H^1(\Omega)$ is called a *generalized solution of the Neumann problem* (10), (12) if it satisfies the (infinite) system of equalities

$$\begin{cases} a_\Omega(u_0, v_0) = \langle f_0, v_0\rangle_{H^1(\Omega)} + \langle \tilde{g}_0, \gamma_0 v_0\rangle_\Gamma, \\ a_\Omega(u_1, v_1) + (c_{1,0}u_0, v_1)_{L_2(\Omega)} = \langle f_1, v_1\rangle_{H^1(\Omega)} + \langle \tilde{g}_1, \gamma_0 v_1\rangle_\Gamma, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ a_\Omega(u_k, v_k) + \sum_{i=0}^{k-1} (c_{k,i}u_i, v_k)_{L_2(\Omega)} = \langle f_k, v_k\rangle_{H^1(\Omega)} + \langle \tilde{g}_k, \gamma_0 v_k\rangle_\Gamma, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases} \tag{19}$$

for an arbitrary (infinite) sequence of functions $v_0, v_1, ... \in H^1(\Omega)$.

Definition 3. Let $f_0, f_1, ...$ be a sequence of elements of $\widetilde{H}_0^{-1}(\Omega)$ and $\tilde{g}_0, \tilde{g}_1, ...$ — elements of $H^{-1/2}(\Gamma)$. The sequence of functions $u_0, u_1, ...$ from $H^1(\Omega)$ is called a *generalized solution of the Robin problem* (10), (13) if it satisfies the (infinite) system of equalities

$$\begin{cases} a_\Omega(u_0, v_0) + \langle b_{k,0}\gamma_0 u_0, \gamma_0 v_0\rangle_\Gamma = \langle f_0, v_0\rangle_{H^1(\Omega)} + \langle \tilde{g}_0, \gamma_0 v_0\rangle_\Gamma, \\ a_\Omega(u_1, v_1) + (c_{1,0}u_0, v_1)_{L_2(\Omega)} + \sum_{i=0}^{1} \langle b_{k,i}\gamma_0 u_i, \gamma_0 v_1\rangle_\Gamma = \langle f_1, v_1\rangle_{H^1(\Omega)} + \langle \tilde{g}_1, \gamma_0 v_1\rangle_\Gamma, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ a_\Omega(u_k, v_k) + \sum_{i=0}^{k-1} (c_{k,i}u_i, v_k)_{L_2(\Omega)} + \sum_{i=0}^{k} \langle b_{k,i}\gamma_0 u_i, \gamma_0 v_k\rangle_\Gamma = \langle f_k, v_k\rangle_{H^1(\Omega)} + \langle \tilde{g}_k, \gamma_0 v_k\rangle_\Gamma, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases} \tag{20}$$

for an arbitrary (infinite) sequence of functions $v_0, v_1, ... \in H^1(\Omega)$.

Concerning the well-posedness of the specified problems the following results have been obtained.

Theorem 1. *The Dirichlet boundary value problem* (10), (11) *has a unique generalized solution.*

Proof. The triangular manner of the system (17) gives us an opportunity to consider equations of the system one by one and apply the same procedure on each step of the proof.

We start with the Dirichlet boundary value problem for the first equation $a_\Omega(u_0, v) = \langle f_0, v\rangle_{H_0^1(\Omega)}$ for each $v \in H_0^1(\Omega)$. Since each function $\tilde{h}_k \in H^{1/2}(\Gamma)$ due to the trace theorem ([12]) can be extended in Ω with some (non-unique) element $\tilde{u}_k \in H^1(\Omega)$, we can obtain the following variational equation for the difference $u_0 - \tilde{u}_0 =: w \in H_0^1(\Omega)$

$$a_\Omega(w, v) = \langle \tilde{f}_0, v\rangle_{H_0^1(\Omega)} := \langle f_0, v\rangle_{H_0^1(\Omega)} - a_\Omega(\tilde{u}_0, v) \text{ for each } v \in H_0^1(\Omega). \tag{21}$$

Due to the $H^1(\Omega)$-ellipticity of the bilinear form and the boundedness of the functional $\tilde{f}_0$ on $H_0^1(\Omega)$ according to the Lax-Milgram theorem this equation has a unique solution in $H_0^1(\Omega)$. This proves the existence of the unique function $u_0 \in H^1(\Omega)$ that is a generalized solution of the first problem.

When considering the second problem we move the function u_0 into the right hand side of the corresponding equation and for the difference $u_1 - \tilde{u}_1 =: w \in H_0^1(\Omega)$ we arrive at the variational equation that differs from (21) only in the right hand side. That is why using the previous considerations we prove the assertion of the theorem for the solution u_1. Obviously, acting this way on each succeeding step we will obtain the variational equation (21) with the following right hand side

$$\langle \tilde{f}_k, v\rangle_{H_0^1(\Omega)} := \langle f_k, v\rangle_{H_0^1(\Omega)} - \sum_{i=0}^{k-1} (c_{k,i}\tilde{u}_i, v)_{L_2(\Omega)} - a_\Omega(\tilde{u}_k, v) \text{ for each } v \in H_0^1(\Omega), k \in \mathbb{N}.$$

Here u_i ($i \in \{0, 1, \ldots, k-1\}$) are generalized solutions of the problems considered on the previous steps. Since $\tilde{f}_k \in H^{-1}(\Omega)$, for each boundary value problem with an arbitrary index $k \in \mathbb{N}$ the generalized solution $u_k \in H^1(\Omega)$ exists and is unique. □

Theorem 2. *The Robin boundary value problem* (10), (13) *has a unique generalized solution.*

Proof. Consider the first equation of system (20)

$$a_\Omega(u_0, v) + b_{\Gamma,0}(u_0, v) = \langle f_0, v\rangle_{H^1(\Omega)} + \langle \tilde{g}_0, \gamma_0 v\rangle_\Gamma \text{ for each } v \in H^1(\Omega). \tag{22}$$

Here the bilinear form $b_{\Gamma,k}(\cdot,\cdot)$ ($k \in \mathbb{N}_0$) is expressed through traces of elements of space $H^1(\Omega)$ on the boundary Γ

$$b_{\Gamma,k}(u, v) = \int_\Gamma b_{k,k}(x)\gamma_0 u(x)\gamma_0 v(x) dS_x, u, v \in H^1(\Omega).$$

As long as $b_{k,k} \in L_\infty(\Gamma)$ and $\gamma_0 u, \gamma_0 v \in H^{1/2}(\Gamma) \subset L_2(\Gamma)$, such integral exists. The expression $a_\Omega(u, v) + b_{\Gamma,0}(u, v) =: \tilde{a}_\Omega(u, v)$ can be treated as some bilinear form for $u, v \in H^1(\Omega)$. Obviously, it is coercive. On the other hand, according to (3) the functionals $f_0 \in \widetilde{H}_0^{-1}(\Omega)$ and $\tilde{g}_0 \in H^{-1/2}(\Gamma) \subset \widetilde{H}_\Gamma^{-1}(\Omega)$ give in addition some element of the space $\widetilde{H}^{-1}(\Omega)$. Then, taking into account the Lax-Milgram theorem there exists a unique solution $u_0 \in H^1(\Omega)$ of the equation (22).

Next we follow the scheme, used in the proof of the previous theorem. Consider the equation with an arbitrary index $k \in \mathbb{N}$. After shifting all expressions in it that contain functions u_i ($i \in \{0, 1, \ldots, k-1\}$) into the right hand side, this equation takes the form

$$a_\Omega(u_k, v) + b_{\Gamma,k}(u_k, v) = \langle \tilde{f}_k, v\rangle_{H^1(\Omega)} \text{ for each } v \in H^1(\Omega),\ k \in \mathbb{N}, \tag{23}$$

where

$$\langle \tilde{f}_k, v\rangle_{H^1(\Omega)} := \langle f_k, v\rangle_{H^1(\Omega)} - \sum_{i=0}^{k-1} (c_{k,i}u_i, v)_{L_2(\Omega)} + \sum_{i=0}^{k-1} \langle b_{k,i}\gamma_0 u_i, \gamma_0 v\rangle_\Gamma + \langle \tilde{g}_k, \gamma_0 v\rangle_\Gamma.$$

Obviously, $\tilde{f}_k \in \widetilde{H}^{-1}(\Omega)$. Since the obtained variational equation differs from (22) in only the right hand side, we arrive at the conclusion that there exists its unique solution $u_k \in H^1(\Omega)$.

Thus, we have shown the existence and the uniqueness of any component of the variational solution of system (20). □

Note that coerciveness of the bilinear form $\tilde{a}_\Omega(u, v)$ in the previous proof also persists when $b_{i,j} = 0$ ($i, j \in \mathbb{N}_0$). In this case the Robin boundary value problem is transformed into the Neuman boundary value problem.

Numerical solutions of the considered problems can be found by Galerkin methods. As it could be seen from the proofs of the above theorems, the structure of the system allows consequent solution of the corresponding variational equations. Herewith the k-th component of the generalized solution $u_0, u_1, ..., u_k, ...$ found on the k-th step is transferred into the right hand side of the $k+1$-th variational equation as a given function and the operator of the left side of the numerical scheme will remain the same for all components. This condition is crucial for an effective implementation of the numerical method.

In the case of constant coefficients of system (10) variational relations (16) can be used to build an integral representation of the solution of boundary value problems through simple- and double-layer potentials. It will allow us to apply boundary integral equations method for the numerical solution ([13–15]). However, consequent transferring of the components of the generalized solution found on the previous steps into the right hand side of the current equation will lead to necessity to recalculate the volume potential over the whole domain Ω on each step. Therefore we introduce other variational formulations, equivalent to the given above, but without such a flaw.

3. Variational formulations with usage of q-convolution. To make notations of the specified generalized solutions more convenient we give some new definitions.

Let $\mathbf{X}$ be an arbitrary linear space over the field of real numbers, $\mathbb{Z}$ — a set of integers. By $\mathbf{X}^\infty$ we denote the linear space of reflections $\mathbf{u}\colon \mathbb{Z} \to \mathbf{X}$ having $u(k) = 0$ when $k < 0$. For any element $\mathbf{u} \in \mathbf{X}^\infty$ we have $u_k \equiv (\mathbf{u})_k{:=}\mathbf{u}(k)$, $k \in \mathbb{Z}$, and will write it as $\mathbf{u}{:=}(u_0, u_1, ..., u_k, ...)^\top$. Henceforth we will call elements of $\mathbf{X}^\infty$ sequences.

Let us introduce triangular matrix operators $\mathbf{C}\colon (L_2(\Omega))^\infty \to (L_2(\Omega))^\infty$ and $\mathbf{B}\colon (L_2(\Gamma))^\infty \to (L_2(\Gamma))^\infty$, that act in the following way

$$(\mathbf{Cu})_k = \sum_{l=0}^{k} c_{k,l}(\mathbf{u})_l,\ \ k \in \mathbb{N}_0, \tag{24}$$

$$(\mathbf{Bu})_k = \sum_{l=0}^{k} b_{k,l}(\mathbf{u})_l,\ \ k \in \mathbb{N}_0, \tag{25}$$

where $c_{k,l}$ and $b_{k,l}$ are coefficients of system (10) and Robin boundary condition correspondingly. Let's denote a sequence of bilinear forms by

$$\mathbf{a}_\Omega(\mathbf{u}, \mathbf{v}){:=}(a_\Omega(u_0, v_0), a_\Omega(u_1, v_1), ...)^\top, \mathbf{u}, \mathbf{v} \in (H^1(\Omega))^\infty.$$

Under notations $(\mathbf{u}, \mathbf{v})_\mathbf{X}$ and $\langle \mathbf{u}, \mathbf{v}\rangle_\mathbf{X}$ we will understand sequences $((u_0, v_0)_\mathbf{X}, (u_1, v_1)_\mathbf{X}, ...)^\top$ and $(\langle u_0, v_0\rangle_\mathbf{X}, \langle u_1, v_1\rangle_\mathbf{X}, ...)^\top$ correspondingly. Analogously, the corresponding linear functionals will be treated as component-wise. For a sequence $\mathbf{u} \in (H^1(\Omega))^\infty$ we introduce the definition of a trace as a sequence of traces of its components i.e. $\gamma_0\mathbf{u}{:=}(\gamma_0 u_0, \gamma_0 u_1, ...)^\top$ will be called a trace of sequence $\mathbf{u}$ on the boundary Γ. In the same manner, sequence $\gamma_1\mathbf{u}{:=}(\gamma_1 u_0, \gamma_1 u_1, ...)^\top$ will be called a co-normal derivative of sequence $\mathbf{u}$ on the boundary Γ.

Taking into account previous definitions, generalized solutions of the Dirichlet, Neumann and Robin boundary value problems for system (10) can be defined in the following way.

- Let $\mathbf{f} \in (H^{-1}(\Omega))^\infty$ and $\tilde{\mathbf{h}} \in (H^{1/2}(\Gamma))^\infty$. Sequence $\mathbf{u} \in (H^1(\Omega))^\infty$ is called a *generalized solution of the Dirichlet problem* (10), (11) if it satisfies the variational equality

$$\mathbf{a}_\Omega(\mathbf{u}, \mathbf{v}) + (\mathbf{Cu}, \mathbf{v})_{L_2(\Omega)} = \langle \mathbf{f}, \mathbf{v}\rangle_{H_0^1(\Omega)} \text{ for each } \mathbf{v} \in (H_0^1(\Omega))^\infty \tag{26}$$

  and the boundary condition

$$\gamma_0\mathbf{u} = \tilde{\mathbf{h}} \ \text{ on } \Gamma. \tag{27}$$

- Let $\mathbf{f} \in (\widetilde{H}_0^{-1}(\Omega))^\infty$ and $\tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$. A sequence $\mathbf{u} \in (H^1(\Omega))^\infty$ is called a *generalized solution of the Neumann problem* (10), (12) if it satisfies the variational equality

$$\mathbf{a}_\Omega(\mathbf{u},\mathbf{v}) + (\mathbf{Cu},\mathbf{v})_{L_2(\Omega)} = \langle \mathbf{f},\mathbf{v}\rangle_{H^1(\Omega)} + \langle \tilde{\mathbf{g}}, \gamma_0\mathbf{v}\rangle_{H^{1/2}(\Gamma)} \text{ for each } \mathbf{v} \in (H^1(\Omega))^\infty. \quad (28)$$

- Let $\mathbf{f} \in (\widetilde{H}_0^{-1}(\Omega))^\infty$ and $\tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$. A sequence $\mathbf{u} \in (H^1(\Omega))^\infty$ is called a *generalized solution of the Robin problem* (10), (13) if it satisfies the variational equality

$$\begin{aligned}&\mathbf{a}_\Omega(\mathbf{u},\mathbf{v}) + (\mathbf{Cu},\mathbf{v})_{L_2(\Omega)} + \langle \mathbf{B}\gamma_0\mathbf{u}, \gamma_0\mathbf{v}\rangle_{H^{1/2}(\Gamma)} = \\ &= \langle \mathbf{f},\mathbf{v}\rangle_{H^1(\Omega)} + \langle \tilde{\mathbf{g}}, \gamma_0\mathbf{v}\rangle_{H^{1/2}(\Gamma)} \text{ for each } \mathbf{v} \in (H^1(\Omega))^\infty. \end{aligned} \quad (29)$$

Consider the variational equation (26) for a sequence $\mathbf{u} \in (H^1(\Omega))^\infty$. Bearing in mind (15), we can rewrite it in the following way

$$\langle \mathbf{Pu},\mathbf{v}\rangle_{H_0^1(\Omega)} + (\mathbf{Cu},\mathbf{v})_{L_2(\Omega)} = \langle \mathbf{f},\mathbf{v}\rangle_{H_0^1(\Omega)} \text{ for each } \mathbf{v} \in (H_0^1(\Omega))^\infty, \quad (30)$$

where the matrix operator $\mathbf{P}\colon (H_0^1(\Omega))^\infty \to (H^{-1}(\Omega))^\infty$ acts by the rule $(\mathbf{Pu})_k = Pu_k$, $k \in \mathbb{N}_0$. Taking into account the embedding of spaces (1), the equality (30) can be presented as follows $\langle \mathbf{Pu},\mathbf{v}\rangle_{H_0^1(\Omega)} + \langle \mathbf{Cu},\mathbf{v}\rangle_{H_0^1(\Omega)} = \langle \mathbf{f},\mathbf{v}\rangle_{H_0^1(\Omega)}$ for each $\mathbf{v} \in (H_0^1(\Omega))^\infty$. Introducing the notation

$$\mathbf{G}{:=}\mathbf{P} + \mathbf{C}, \quad (31)$$

the previous equality can be given in the form of an operator equation

$$\mathbf{Gu} = \mathbf{f} \text{ in } (H^{-1}(\Omega))^\infty. \quad (32)$$

Thus, the generalized solution of the Dirichlet problem (10), (11) is the solution of the operator equation (32) and satisfies the boundary condition (27), and vice versa, the solution of (32), (27) is a generalized solution of the Dirichlet problem (10), (11).

Now we show that the generalized solution $\mathbf{u} \in (H^1(\Omega, P))^\infty$ of the Robin problem (10), (13) will also satisfy a similar to (32) operator equation and the Robin boundary condition. After application of the Green formula analogue (16) to the variational equality (29) we get

$$\begin{aligned}&\langle \mathbf{Pu},\mathbf{v}\rangle_{H^1(\Omega)} + \langle \gamma_1\mathbf{u}, \gamma_0\mathbf{v}\rangle_{H^{1/2}(\Gamma)} + \langle \mathbf{Cu},\mathbf{v}\rangle_{H^1(\Omega)} + \\ &+\langle \mathbf{B}\gamma_0\mathbf{u}, \gamma_0\mathbf{v}\rangle_{H^{1/2}(\Gamma)} = \langle \mathbf{f},\mathbf{v}\rangle_{H^1(\Omega)} + \langle \tilde{\mathbf{g}}, \gamma_0\mathbf{v}\rangle_{H^{1/2}(\Gamma)},\end{aligned} \quad (33)$$

or

$$\langle \mathbf{Gu} - \mathbf{f},\mathbf{v}\rangle_{H^1(\Omega)} + \langle \mathbf{B}\gamma_0\mathbf{u} + \gamma_1\mathbf{u} - \tilde{\mathbf{g}}, \gamma_0\mathbf{v}\rangle_{H^{1/2}(\Gamma)} = 0 \text{ for each } \mathbf{v} \in (H^1(\Omega))^\infty. \quad (34)$$

Since $\mathbf{u} \in (H^1(\Omega, P))^\infty$ and $\mathbf{f} \in (\widetilde{H}_0^{-1}(\Omega))^\infty$, as a result we get $\mathbf{Gu} - \mathbf{f} \in (\widetilde{H}_0^{-1}(\Omega))^\infty$. Moreover, if we take in (34) an arbitrary element $\mathbf{v} \in (H_0^1(\Omega))^\infty$ then we come to the equality $\langle \mathbf{Gu}-\mathbf{f},\mathbf{v}\rangle_{H_0^1(\Omega)} = 0$, that means that the element $\mathbf{Gu}-\mathbf{f}$ belongs to space $(\widetilde{H}_\Gamma^{-1}(\Omega))^\infty$ as well. Hence, due to the decomposition (3) $\mathbf{Gu}-\mathbf{f}$ coincides with zero element of space $(\widetilde{H}^{-1}(\Omega))^\infty$ i.e. the following operator equation

$$\mathbf{Gu} - \mathbf{f} = 0 \text{ in } (\widetilde{H}^{-1}(\Omega))^\infty \quad (35)$$

or

$$\mathbf{G}\mathbf{u} = \mathbf{f} \text{ in } (\widetilde{H}^{-1}(\Omega))^{\infty} \tag{36}$$

is well defined.

Therefore, substituting an arbitrary sequence $\mathbf{v} \in (H^1(\Omega))^{\infty}$ into (34), we arrive at the relation $\langle \mathbf{B}\gamma_0\mathbf{u} + \gamma_1\mathbf{u} - \tilde{\mathbf{g}}, \gamma_0\mathbf{v}\rangle_{\Gamma} = 0$, that taking into account that values of the trace operator $\gamma_0 \colon H^1(\Omega) \to H^{1/2}(\Gamma)$ fill in the whole space $H^{1/2}(\Gamma)$, is a variational formulation of the Robin boundary condition

$$\gamma_1\mathbf{u} + \mathbf{B}\gamma_0\mathbf{u} = \tilde{\mathbf{g}}\text{in } (H^{-1/2}(\Gamma))^{\infty}. \tag{37}$$

Thus, we have shown that the generalized solution of the Robin boundary value problem can be characterized by the operator equation (36) and the boundary condition (37). Analogously it can be shown that the generalized solution of the Neumann problem can be characterized by the same operator equation (36) and the Neumann boundary condition

$$\gamma_1\mathbf{u} = \tilde{\mathbf{g}} \text{ in } (H^{-1/2}(\Gamma))^{\infty}. \tag{38}$$

Boundary conditions (37) and (38), as in the theory of elliptic equations, will be referred to as the natural boundary conditions.

We introduce according to [5] the operation of the q-convolution of sequences, that would give us ability to write down equivalent variational formulations of the problems specified above.

Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be arbitrary linear spaces and $q \colon \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ — some reflection.

Definition 4. By q-convolution of sequences $\mathbf{u} \in \mathbf{X}^{\infty}$ and $\mathbf{v} \in \mathbf{Y}^{\infty}$ we will understand a sequence $\mathbf{w} \in \mathbf{Z}^{\infty}$, that is defined according to the following rule

$$\mathbf{w} = \mathbf{u} \underset{q}{\circ} \mathbf{v}, \tag{39}$$

where $w_n \equiv (\mathbf{u} \underset{q}{\circ} \mathbf{v})_n \equiv \mathbf{u} \overset{n}{\underset{q}{\circ}} \mathbf{v} := \sum_{i=0}^{n} q\,(u_{n-i}, v_i)$, when $n \geq 0$, and $w_n = 0$ when $n < 0$.

The most important properties of q-convolutions and their examples are given in [5]. We will simplify the notation of the convolution for some reflections. For example, in case of $q(u,v) := \langle u, v\rangle_{H_0^1(\Omega)}$ we will write $\mathbf{u} \underset{H^{-1}(\Omega)}{\circ} \mathbf{v} := \mathbf{u} \underset{q}{\circ} \mathbf{v}$.

We give a definition of a generalized solution of the Dirichlet problem (10), (11) by means of q-convolution. Consider a sequence $\mathbf{u} \in \left(H^1(\Omega)\right)^{\infty}$, that satisfies equation (32). Let us substitute it into this equation and, treating the obtained equality as equality of elements from $(H^{-1}(\Omega))^{\infty}$ and taking $q(w,v) = \langle w, v\rangle_{H_0^1(\Omega)}, v \in H_0^1(\Omega), w \in H^{-1}(\Omega)$, we apply the q-convolution with an arbitrary sequence $\mathbf{v} \in \left(H_0^1(\Omega)\right)^{\infty}$ to both sides of this equality. Taking into account lemma 3.1 ([5]), we arrive at the following variational equation

$$(\mathbf{G}\mathbf{u}) \underset{H^{-1}(\Omega)}{\circ} \mathbf{v} = \mathbf{f} \underset{H^{-1}(\Omega)}{\circ} \mathbf{v} \quad \text{for each } \mathbf{v} \in \left(H_0^1(\Omega)\right)^{\infty}. \tag{40}$$

Thus, the generalized solution of the Dirichlet problem (10), (11) can be characterized by the variational equality (40) and the boundary condition (27).

Now we assume that sequence $\mathbf{u} \in \left(H^1(\Omega, P)\right)^{\infty}$ satisfies operator equation (36). We apply the q-convolution with some arbitrary sequence $\mathbf{v} \in \left(H^1(\Omega)\right)^{\infty}$ to both of its sides as elements of $(\widetilde{H}^{-1}(\Omega))^\infty$, taking $q(w,v) = \langle w,v\rangle_{H^1(\Omega)}, v \in H^1(\Omega), w \in \widetilde{H}^{-1}(\Omega)$. As a result we get

$$(\mathbf{Gu}) \underset{\widetilde{H}^{-1}(\Omega)}{\circ} \mathbf{v} = \mathbf{f} \underset{\widetilde{H}^{-1}(\Omega)}{\circ} \mathbf{v} \quad \text{for each } \mathbf{v} \in \left(H^1(\Omega)\right)^\infty. \tag{41}$$

Thus, the generalized solution of the Robin boundary value problem can be characterized by variational equality (41) and the boundary condition (37). Obviously, this property also holds for the generalized solution of the Neumann boundary value problem.

Let us obtain an analogue of the first Green formula using the q-convolution of sequences. At first note that the component of the q-convolution in the left hand side of (41) with an arbitrary index $k \in \mathbb{N}_0$ after application of the first Green formula (9) can be written as

$$(\mathbf{Gu}) \overset{k}{\underset{\widetilde{H}^{-1}(\Omega)}{\circ}} \mathbf{v} = \sum_{i=0}^{k} a_\Omega\left(u_i, v_{k-i}\right) - \sum_{i=0}^{k} \langle \gamma_1 u_i, \gamma_0 v_{k-i}\rangle_\Gamma + \sum_{i=1}^{k} \left(\sum_{j=0}^{i-1} c_{i,j} u_j, v_{k-i}\right)_{L_2(\Omega)}. \tag{42}$$

Henceforth we assume that the sum expressions equal zero if their last index is less than the first one i.e. in the case of $k=0$ the last item in the previous formula is absent.

Consider a sequence $(\Phi_0(\mathbf{u},\mathbf{v}), \Phi_1(\mathbf{u},\mathbf{v}), ..., \Phi_k(\mathbf{u},\mathbf{v}), ...)^\top$, components of which are the following expressions

$$\Phi_0(\mathbf{u},\mathbf{v}) := a_\Omega\left(u_0, v_0\right), \tag{43}$$

$$\Phi_k(\mathbf{u},\mathbf{v}) := \sum_{i=0}^{k} a_\Omega\left(u_i, v_{k-i}\right) + \sum_{i=1}^{k} \left(\sum_{j=0}^{i-1} c_{i,j} u_j, v_{k-i}\right)_{L_2(\Omega)}, \quad k \in \mathbb{N}_0. \tag{44}$$

Definition 5. A sequence $\mathbf{\Phi}(\mathbf{u},\mathbf{v}) = (\Phi_0(\mathbf{u},\mathbf{v}), \Phi_1(\mathbf{u},\mathbf{v}), ..., \Phi_k(\mathbf{u},\mathbf{v}), ...)^\top$, $\mathbf{u},\mathbf{v} \in (H^1(\Omega))^\infty$ defined by formula (43) is called a *bilinear form associated with the operator* $\mathbf{G}$.

Such notation of the bilinear form gives us ability to present relation (42) in the following way

$$(\mathbf{Gu}) \underset{\widetilde{H}^{-1}(\Omega)}{\circ} \mathbf{v} = \mathbf{\Phi}(\mathbf{u},\mathbf{v}) - \gamma_1 \mathbf{u} \underset{H^{-1/2}(\Gamma)}{\circ} \gamma_0 \mathbf{v} \quad \text{for each } \mathbf{u} \in \left(H^1(\Omega,P)\right)^\infty, \mathbf{v} \in \left(H^1(\Omega)\right)^\infty, \tag{45}$$

and treat it as the the first Green formula for operator $\mathbf{G}$. Note that for the left part of the variational equality (40) we can analogously obtain the expression

$$(\mathbf{Gu}) \underset{H^{-1}(\Omega)}{\circ} \mathbf{v} = \mathbf{\Phi}(\mathbf{u},\mathbf{v}) \quad \text{for each } \mathbf{u} \in \left(H^1(\Omega)\right)^\infty, \ \mathbf{v} \in \left(H^1_0(\Omega)\right)^\infty, \tag{46}$$

when using equality (8).

Due to the triangular structure of operator $\mathbf{C}$, definition of the second Green formula may be complicated. Because of that we assume that the $\mathbf{C}$ part of the differential operator $\mathbf{G}$ satisfies the equality

$$(\mathbf{Cu}) \underset{L_2(\Omega)}{\circ} \mathbf{v} = (\mathbf{Cv}) \underset{L_2(\Omega)}{\circ} \mathbf{u} \text{ for each} \mathbf{u},\mathbf{v} \in (L_2(\Omega))^\infty. \tag{47}$$

This fact grants the symmetry of the operator $\mathbf{G}$ with regard to the operation of q-convolution of sequences and gives ability, analogously to [5], to present the second Green formula for operator $\mathbf{G}$ in the case of variable coefficients.

Theorem 3. *For sequences $\boldsymbol{u}, \boldsymbol{v} \in (H^1(\Omega, P))^\infty$ the following equality holds*

$$(\boldsymbol{G}\boldsymbol{u}) \underset{\widetilde{H}^{-1}(\Omega)}{\circ} \boldsymbol{v} - (\boldsymbol{G}\boldsymbol{v}) \underset{\widetilde{H}^{-1}(\Omega)}{\circ} \boldsymbol{u} = \gamma_1 \boldsymbol{v} \underset{H^{-1/2}(\Gamma)}{\circ} \gamma_0 \boldsymbol{u} - \gamma_1 \boldsymbol{u} \underset{H^{-1/2}(\Gamma)}{\circ} \gamma_0 \boldsymbol{v}. \tag{48}$$

Formula (48) is used to obtain an integral representation of the sequence $\mathbf{u} \in (H^1(\Omega, P))^\infty$.

4. Integral representation of the solutions. Henceforth we assume all coefficients of the operator $\mathbf{G}$ to be constant. Then property (47) will hold if the matrix $\mathbf{C}$ satisfies for arbitrary sequences $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^\infty$ the following condition

$$\sum_{k=1}^{n} \sum_{i=0}^{k-1} c_{k,i} \xi_i \eta_{n-k} = \sum_{k=1}^{n} \sum_{i=0}^{k-1} c_{k,i} \eta_i \xi_{n-k} \text{ for each } n \in \mathbb{N}. \tag{49}$$

In systems (10) that appear during the solution of non-stationary problems with first and second order partial derivatives over the time variable the matrix $\mathbf{C}$ contains equal elements on the diagonals

$$c_{k,i} = c_{k-1,i-1}, \ i, k \in \mathbb{N}, \tag{50}$$

which can be treated as a partial case of property (49). We will use it to simplify proofs of some statements.

Let $\widetilde{\mathbf{E}}(x) = (\widetilde{E}_0(x), \widetilde{E}_1(x), ...)^\top, x \in \mathbb{R}^3$, be a fundamental solution of the operator $\mathbf{G}$, i.e. a solution of the equation

$$\mathbf{G}\widetilde{\mathbf{E}} = \tilde{\boldsymbol{\delta}} \text{ in } \left(\mathcal{D}'(\mathbb{R}^3)\right)^\infty, \tag{51}$$

where $\tilde{\boldsymbol{\delta}}(\theta) = (\delta(\theta), \delta(\theta), ...)^\top$ is a functional sequence which elements are Dirac's δ-function ([1]). It is known (see for example [13–15]), that in the case of systems with finite number of equations and constant coefficients such a solution exists.

In [2, 3, 6, 7, 9–11] fundamental solutions for some infinite systems that are partial cases of (10) have been obtained. These solutions can be characterized by the fact that their first component is a function $\widetilde{E}_0(x) = \alpha_1 e^{-\alpha_2 |x|} |x|^{-1}$ and all other components have asymptotes $\alpha_1 e^{-\alpha_3 |x|} |x|^{-1}$ when $|x| \to \infty$ and $\alpha_4 |x|^{-1}$ when $|x| \to 0$, where $\alpha_i (i \in \{1, 2, 3, 4\})$ are some given positive constants. Moreover, each difference

$$E_i(x) := \widetilde{E}_i(x) - \widetilde{E}_{i-1}(x), \ i \in \mathbb{N}, \ x \in \mathbb{R}^3, \tag{52}$$

can be continued to some continuous function that has a continuous partial derivative $\frac{\partial}{\partial r} E_i(x), \ r := |x|$. Henceforth we assume that a fundamental solution with suggested properties can be built for the operator $\mathbf{G}$.

Consider a sequence $\mathbf{E}(x) = (E_0(x), E_1(x), ...)^\top$, where $E_0(x) = \widetilde{E}_0(x)$, and the rest of the components are defined according to (52). The following statement holds.

Lemma 1. *Sequence $\boldsymbol{E}(x)$ is a solution of*

$$\boldsymbol{G}\boldsymbol{E} = \bar{\boldsymbol{\delta}} \text{ in } \left(\mathcal{D}'(\mathbb{R}^3)\right)^\infty, \tag{53}$$

where $\bar{\boldsymbol{\delta}}(\theta) = (\delta(\theta), 0, 0, ...)^\top$.

Proof. Consider the first equation of (53). Since $E_0(x) = \widetilde{E}_0(x), x \in \mathbb{R}^3$, then taking into account the definition of a fundamental solution $\widetilde{\mathbf{E}}$ we get $PE_0 = P\widetilde{E}_0 = \delta$ in $\mathcal{D}'(\mathbb{R}^3)$. If $k = 1$ then the left part of the corresponding equation (53) can be written as a difference of the expressions in the left parts of equations with indices $k = 0$ and $k = 1$. As a result we obtain $c_{1,0}E_0 + PE_1 = c_{1,0}\widetilde{E}_0 + P\widetilde{E}_1 - P\widetilde{E}_0 = \delta - \delta = 0$ in $\mathcal{D}'(\mathbb{R}^3)$. Let's consider now an arbitrary k-th equation of (53).

$$\sum_{i=0}^{k-1} c_{k,i}E_i + PE_k = c_{k,0}\widetilde{E}_0 + \sum_{i=1}^{k-1} c_{k,i}(\widetilde{E}_i - \widetilde{E}_{i-1}) + P(\widetilde{E}_k - P\widetilde{E}_{k-1}) =$$
$$= \sum_{i=0}^{k-1} c_{k,i}\widetilde{E}_i + P\widetilde{E}_k - \sum_{i=1}^{k-1} c_{k,i}\widetilde{E}_{i-1} - P\widetilde{E}_{k-1} \text{ in } \mathcal{D}'(\mathbb{R}^3).$$

Then, taking into account the k-th and the $(k-1)$-st equations of (51) and the property (50) of the components of the operator $\mathbf{C}$, we get

$$\sum_{i=0}^{k-1} c_{k,i}E_i + PE_k = \delta - \left(\sum_{i=1}^{k-1} c_{k-1,i-1}\widetilde{E}_{i-1} + P\widetilde{E}_{k-1}\right) = \delta - \delta = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3). \qquad \square$$

By means of the q-convolution and the sequence $\mathbf{E}$ we build sequences that, by analogy with the theory of elliptic equations, will also be called potentials.

Definition 6. Let $\mathbf{f} \in (\widetilde{H}_0^{-1}(\Omega))^\infty$ be a given sequence. The sequence $\mathbf{u}{:=}\mathbf{Uf}$, where

$$\mathbf{Uf}(x){:=}(\mathbf{Uf})(x) \equiv \mathbf{f}(\cdot) \underset{\widetilde{H}^{-1}(\Omega)}{\circ} \mathbf{E}(x - \cdot),\ x \in \Omega, \tag{54}$$

is called a *volume potential of operator* $\mathbf{G}$.

Theorem 4. *The operator* $\boldsymbol{U}$: $(\widetilde{H}_0^{-1}(\Omega))^\infty \to (H^1(\Omega, P))^\infty$ *is linear and separate continuous. Moreover, for an arbitrary sequence* $\boldsymbol{f} \in (\widetilde{H}_0^{-1}(\Omega))^\infty$ *the volume potential* $\boldsymbol{u} = \boldsymbol{Uf}$ *belongs to the space* $(H^1(\Omega, P))^\infty$ *and satisfies the equation* $\boldsymbol{Gu} = \boldsymbol{f}$ *in terms of* $(\mathcal{D}'(\Omega))^\infty$.

Proof. The operator $\mathbf{U}$ is component-wise linear since this property holds for each expression $U_j f_i(x){:=}\langle f_i(\cdot), E_j(x - \cdot)\rangle_{H^1(\Omega)},\ x \in \Omega,\ (i, j \in \mathbb{N}_0)$, of which every component of the potential consists.

In the case of $j = 0$ function E_0 is a fundamental solution of the operator P and $U_0 f_i$ is a volume potential with density f_i. This potential can be given as a sum $U_0 = U^* + U^+$, where $U^* f_i(x){:=}\langle f_i(\cdot), E^*(x - \cdot)\rangle_{H^1(\Omega)}$, $U^+ f_i(x){:=}\langle f_i(\cdot), E^+(x - \cdot)\rangle_{H^1(\Omega)}$, $E^*(x){:=}\alpha_1|x|^{-1}$ is a fundamental solution of the Laplacian and $E^+(x) \sim (c_1 + c_2|x|)$ when $|x| \to 0$, c_1 and c_2 are some constants. According to Theorem 6.1 ([15]) for the reflection $U^*\colon \widetilde{H}^{-1}(\Omega) \to H^1(\Omega)$ we have $\|U^* f_i\|_{H^1(\Omega)} \le c^*\|f_i\|_{\widetilde{H}^{-1}(\Omega)},\ i \in \mathbb{N}_0$, where c^* is some constant. The function E^+ can be extended to $x = 0$ so that it and its derivative over $r{:=}|x|$ are continuous when $|x| \to 0$. That is why for the reflection $U^+\colon \widetilde{H}^{-1}(\Omega) \to H^1(\Omega)$ we also have $\|U^+ f_i\|_{H^1(\Omega)} \le c^+\|f_i\|_{\widetilde{H}^{-1}(\Omega)},\ i \in \mathbb{N}_0$, where c^+ is some constant. Thus, the operator $U_0\colon \widetilde{H}^{-1}(\Omega) \to H^1(\Omega)$ is continuous and $\|U_0 f_i\|_{H^1(\Omega)} \le c_0\|f_i\|_{\widetilde{H}^{-1}(\Omega)},\ i \in \mathbb{N}_0$, where c_0 is some constant.

If $j \in \mathbb{N}$, then the kernels E_j in all the expressions $U_j f_i$, $i \in \mathbb{N}_0$, behave themselves as the function E^+ when $|x| \to 0$. Hence, each of these expressions defines a continuous map from $\widetilde{H}^{-1}(\Omega)$ to $H^1(\Omega)$ and for each $k \in \mathbb{N}$ we have the estimate

$$\|(\mathbf{Uf})_k\|_{H^1(\Omega)} = \left\| \sum_{i=0}^{k} \langle f_i(\cdot), E_{k-i}(\bullet - \cdot) \rangle_{H^1(\Omega)} \right\|_{H^1(\Omega)} \le \tilde{c}_k \sum_{i=0}^{k} \|(\mathbf{f})_i\|_{\widetilde{H}^{-1}(\Omega)}, \tag{55}$$

where $\tilde{c}_k$ is some constant.

Consider the component-wise action of the volume potential on some test functions for the case of $\mathbf{f} \in (L_2(\Omega))^\infty$

$$\langle (\mathbf{Uf})_k(x), \phi(x) \rangle_{\mathcal{D}(\Omega)} = \left(\sum_{i=0}^{k} \left(f_i(y), E_{k-i}(x-y) \right)_{L_2(\Omega)}, \phi(x) \right)_{L_2(\Omega)} =$$
$$= \sum_{i=0}^{k} \left(f_i(y), U_{k-i}\phi(y) \right)_{L_2(\Omega)}, \ k \in \mathbb{N}_0, \ \phi \in \mathcal{D}(\Omega). \tag{56}$$

Now if $\mathbf{f} \in (\widetilde{H}^{-1}(\Omega))^\infty$, then there exists some sequence $\mathbf{f}_0, \mathbf{f}_1, ..., \mathbf{f}_n, ...,$ where $\mathbf{f}_n \in (L_2(\Omega))^\infty$, that converges to $\mathbf{f}$ component-wisely by the norm of the space $\widetilde{H}^{-1}(\Omega)$. Since for each component of this sequence as for an element of space $(L_2(\Omega))^\infty$ property (56) holds, it can be extended to the elements of the space $(\widetilde{H}_0^{-1}(\Omega))^\infty$

$$\langle (\mathbf{Uf})_k(x), \phi(x) \rangle_{\mathcal{D}(\Omega)} = \sum_{i=0}^{k} \langle f_i(y), U_{k-i}\phi(y) \rangle_{H^1(\Omega)}, \ k \in \mathbb{N}_0, \ \phi \in \mathcal{D}(\Omega). \tag{57}$$

Let us substitute $\mathbf{u} = \mathbf{Uf}$ into the equation $\mathbf{Gu} = \mathbf{f}$, treating it as a system. Then, taking into account (57), the equation with an arbitrary index $k \in \mathbb{N}_0$ can be written as

$$\langle (\mathbf{Gu})_k(x), \phi(x) \rangle_{\mathcal{D}(\Omega)} = \sum_{i=0}^{k-1} c_{k,i} \langle u_i(x), \phi(x) \rangle_{\mathcal{D}(\Omega)} + \langle u_k(x), P\phi(x) \rangle_{\mathcal{D}(\Omega)} = \tag{58}$$
$$= \sum_{i=0}^{k-1} c_{k,i} \sum_{j=0}^{i} \langle f_j(y), U_{i-j}\phi(y) \rangle_{H^1(\Omega)} + \sum_{i=0}^{k} \langle f_i(y), U_{k-i}P\phi(y) \rangle_{H^1(\Omega)}.$$

After that, using the formula (see proof of lemma 5.2 in [5])

$$\sum_{i=0}^{k-1} c_{k,i} \sum_{j=0}^{i} \langle f_j(x), U_{i-j}\phi(x) \rangle_{H^1(\Omega)} = \sum_{i=0}^{k-1} \left\langle f_i(y), \left(\sum_{j=0}^{k-i-1} c_{k-i,j} E_j(x-y), \phi(x) \right)_{L_2(\Omega)} \right\rangle_{H^1(\Omega)},$$

and shifting the operator P on the kernels of the potentials in the second part of the expression (58), we get

$$\langle (\mathbf{Gu})_k(x), \phi(x)\rangle_{\mathcal{D}(\Omega)} = \sum_{i=0}^{k-1} \left\langle f_i(y), \left(\sum_{j=0}^{k-i-1} c_{k-i,j} E_j(x-y), \phi(x) \right)_{L_2(\Omega)} \right\rangle_{H^1(\Omega)} +$$
$$+ \sum_{i=0}^{k} \left\langle f_i(y), \langle PE_{k-i}(x-y), \phi(x)\rangle_{\mathcal{D}(\Omega)} \right\rangle_{H^1(\Omega)} =$$
$$= \sum_{i=0}^{k-1} \left\langle f_i(y), \left\langle \sum_{j=0}^{k-i-1} c_{k-i,j} E_j(x-y) + PE_{k-i}(x-y), \phi(x) \right\rangle_{\mathcal{D}(\Omega)} \right\rangle_{H^1(\Omega)} +$$
$$+ \left\langle f_k(y), \langle PE_0(x-y), \phi(x)\rangle_{\mathcal{D}(\Omega)} \right\rangle_{H^1(\Omega)} = \sum_{i=0}^{k} \left\langle f_i(y), \langle (\mathbf{GE})_{k-i}(x-y), \phi(x)\rangle_{\mathcal{D}(\Omega)} \right\rangle_{H^1(\Omega)} =$$
$$= \langle f_k(y), \phi(y)\rangle_{\mathcal{D}(\Omega)}, \ k \in \mathbb{N}_0.$$

Hence, $\mathbf{Gu} = \mathbf{f}$ in $(\widetilde{H}^{-1}(\Omega))^\infty$. In particular, taking into account (55), in the case of $\mathbf{f} \in (\widetilde{H}_0^{-1}(\Omega))^\infty$ we will obtain $\mathbf{Uf} \in (H^1(\Omega, P))^\infty$ according to the definition of the space $(H^1(\Omega, P))^\infty$. □

Now analogously to the volume potential we give the definitions of the layer potentials.

Definition 7. Let $\boldsymbol{\lambda} \in \left(H^{1/2}(\Gamma)\right)^\infty$ and $\boldsymbol{\mu} \in \left(H^{-1/2}(\Gamma)\right)^\infty$. Sequences

$$\mathbf{V}\boldsymbol{\mu}(x) := (\mathbf{V}\boldsymbol{\mu})(x) \equiv \boldsymbol{\mu}(\cdot) \underset{H^{-1/2}(\Gamma)}{\circ} \mathbf{E}(x-\cdot), \ x \in \Omega, \tag{59}$$

$$\mathbf{W}\boldsymbol{\lambda}(x) := (\mathbf{W}\boldsymbol{\lambda})(x) \equiv \partial_{\bar{\nu}(\cdot)} \mathbf{E}(x-\cdot) \underset{H^{-1/2}(\Gamma)}{\circ} \boldsymbol{\lambda}(\cdot), \ x \in \Omega, \tag{60}$$

are called the *simple and the double layer potentials of the operator* $\mathbf{G}$ on the surface Γ correspondingly.

Lemma 2. *For arbitrary sequences* $\boldsymbol{\lambda} \in \left(H^{1/2}(\Gamma)\right)^\infty$ *and* $\boldsymbol{\mu} \in \left(H^{-1/2}(\Gamma)\right)^\infty$ *the layer potentials* $\mathbf{u}(x) = \boldsymbol{V\mu}(x), \ x \in \Omega$, *and* $\mathbf{u}(x) = \boldsymbol{W\lambda}(x), \ x \in \Omega$, *are the solutions of the homogeneous equation*

$$\boldsymbol{Gu} = \boldsymbol{0} \ (in\ \Omega). \tag{61}$$

According to the definitions the components of the layer potentials consist of the following expressions $V_j\mu_i(x) := \langle \mu_i(\cdot), E_j(x-\cdot)\rangle_{H^{1/2}(\Gamma)}$, $W_j\lambda_i(x) := \left\langle \partial_{\bar{\nu}(\cdot)} E_j(x-\cdot), \lambda_i(\cdot)\right\rangle_{H^{1/2}(\Gamma)}$, $i, j \in \mathbb{N}_0, \ x \in \Omega$.

We consider them more accurately. Expressions $V_0\mu_i$ and $W_0\lambda_i (i \in \mathbb{N}_0)$ are the simple and the double layer potentials of the operator P respectively. In the case of the Lipschitz boundary these potentials as long as their traces and co-normal derivatives define continuous boundary operators on spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ respectively (see for example Theorem 1 [12] and Theorem 5.6.2 [13]). Therefore, taking into account the continuity in the domain Ω of functions E_j and their derivatives $\frac{\partial}{\partial r} E_j$ when $j \in \mathbb{N}$, we obtain the following statement.

Theorem 5. *Operators*

$$\begin{aligned} \boldsymbol{V}\colon (H^{-1/2}(\Gamma))^\infty &\to (H^1(\Omega,P))^\infty, & \boldsymbol{W}\colon (H^{1/2}(\Gamma))^\infty &\to (H^1(\Omega,P))^\infty, \\ \gamma_0\boldsymbol{V}\colon (H^{-1/2}(\Gamma))^\infty &\to (H^{1/2}(\Gamma))^\infty, & \gamma_1\boldsymbol{V}\colon (H^{-1/2}(\Gamma))^\infty &\to (H^{-1/2}(\Gamma))^\infty, \\ \gamma_0\boldsymbol{W}\colon (H^{1/2}(\Gamma))^\infty &\to (H^{1/2}(\Gamma))^\infty, & \gamma_1\boldsymbol{W}\colon (H^{1/2}(\Gamma))^\infty &\to (H^{-1/2}(\Gamma))^\infty \end{aligned}$$

are linear and separate continuous.

Using the potentials (54), (59)–(60) and the second Green formula we arrive at the integral representation of the components of an arbitrary sequence $\mathbf{u} \in (H^1(\Omega,P))^\infty$.

Theorem 6 ([5], Theorem 1)**.** *For sequences $\boldsymbol{u} \in (H^1(\Omega,P))^\infty$ the following representation holds*

$$\boldsymbol{u}(x) = \boldsymbol{U}\boldsymbol{f}(x) + \boldsymbol{V}\boldsymbol{\mu}(x) - \boldsymbol{W}\boldsymbol{\lambda}(x),\ x \in \Omega, \tag{62}$$

where $\boldsymbol{f}{:=}\boldsymbol{G}\boldsymbol{u}$, $\boldsymbol{\lambda}{:=}\gamma_0\boldsymbol{u}$ and $\boldsymbol{\mu}{:=}\gamma_1\boldsymbol{u}$.

Note, that according to Theorem 4 by means of the volume potential (54) we can always build a partial solution of system (10) and reduce it to a homogeneous system. So, from now on we will consider homogeneous systems only, that can be presented in the form of (61). Then by Theorem 6, their generalized solution can be given by its boundary value and the co-normal derivative on the boundary — the Cauchy data. As it can be seen from the boundary conditions (27) and (37), in each of the boundary problems these data are incomplete. To get a complete Cauchy data we need to consider corresponding boundary integral equations that can be obtained by means of presentation (62). Note that this is the so-called direct approach ([13]) to replacement of the boundary value problems by integral equations. To apply it we need to consider some properties of the layer potentials first.

Let $[v]$ denote a jump of some function v over the boundary Γ from outside into inside of the domain Ω.

Theorem 7 ([5], Theorem 9)**.** *Layer potentials* (59) *and* (60) *can be characterized by the relations*

$$\begin{aligned} [\gamma_0\boldsymbol{V}\boldsymbol{\mu}] = 0,\ [\gamma_1\boldsymbol{V}\boldsymbol{\mu}] = -\boldsymbol{\mu} \quad &\text{for each } \boldsymbol{\mu} \in (H^{-1/2}(\Gamma))^\infty; \\ [\gamma_0\boldsymbol{W}\boldsymbol{\lambda}] = \boldsymbol{\lambda},\ [\gamma_1\boldsymbol{W}\boldsymbol{\lambda}] = 0 \quad &\text{for each } \boldsymbol{\lambda} \in (H^{1/2}(\Gamma))^\infty. \end{aligned}$$

Henceforth we consider the boundary operators

$$\begin{aligned} \mathbf{V}\colon (H^{-1/2}(\Gamma))^\infty &\to (H^{1/2}(\Gamma))^\infty, & \mathbf{K}'\colon (H^{-1/2}(\Gamma))^\infty &\to (H^{-1/2}(\Gamma))^\infty, \\ \mathbf{K}\colon (H^{1/2}(\Gamma))^\infty &\to (H^{1/2}(\Gamma))^\infty, & \mathbf{D}\colon (H^{1/2}(\Gamma))^\infty &\to (H^{-1/2}(\Gamma))^\infty, \end{aligned}$$

that are defined by means of q-convolution with corresponding reflection q in the following way

$$(\mathbf{V}\boldsymbol{\mu})_i := \sum_{j=0}^{i} V_j\mu_{i-j},\ (\mathbf{K}\boldsymbol{\lambda})_i := \sum_{j=0}^{i} K_j\lambda_{i-j},$$

$$(\mathbf{K}'\boldsymbol{\mu})_i := \sum_{j=0}^{i} K'_j\mu_{i-j},\ (\mathbf{D}\boldsymbol{\lambda})_i := \sum_{j=0}^{i} D_j\lambda_{i-j},\ i \in \mathbb{N}_0,$$

for arbitrary sequences $\boldsymbol{\lambda} \in \left(H^{1/2}(\Gamma)\right)^\infty$ and $\boldsymbol{\mu} \in \left(H^{-1/2}(\Gamma)\right)^\infty$.

Here we use the notation $V_j\mu{:=}\gamma_0 V_j\mu, D_j\lambda{:=}-\gamma_1 W_j\lambda, j \in \mathbb{N}_0$. The boundary operators K_j, K_j' $(j \in \mathbb{N}_0)$ act by the following rules

$$K_0'\mu{:=}\gamma_1 V_0\mu - \mu/2,\ K_j'\mu{:=}\gamma_1 V_j\mu,\ K_0\lambda{:=}\gamma_0 W_0\lambda + \lambda/2,\ K_j\lambda{:=}\gamma_0 W_j\lambda,\ j \in \mathbb{N}. \tag{63}$$

According to Theorem 5 and the continuity of operators γ_0 and γ_1 the operators $\mathbf{V}$, $\mathbf{K}'$, $\mathbf{K}$ and $\mathbf{D}$ are also separate continuous. Thus, the following relations hold

$$\mathbf{V}\boldsymbol{\mu} = \gamma_0\mathbf{V}\boldsymbol{\mu},\ \mathbf{K}\boldsymbol{\lambda} = (\gamma_0\mathbf{W} + \mathbf{I}/2)\boldsymbol{\lambda},\ \mathbf{K}'\boldsymbol{\mu} = (\gamma_1\mathbf{V} - \mathbf{I}/2)\boldsymbol{\mu},\ \mathbf{D}\boldsymbol{\lambda} = -\gamma_1\mathbf{W}\boldsymbol{\lambda}. \tag{64}$$

Lemma 3. *Operators*

$$\boldsymbol{V}\colon (H^{-1/2}(\Gamma))^\infty \to (H^{1/2}(\Gamma))^\infty,\ (\boldsymbol{I}/2 \pm \boldsymbol{K}')\colon (H^{-1/2}(\Gamma))^\infty \to (H^{-1/2}(\Gamma))^\infty, \tag{65}$$

$$\boldsymbol{D}\colon (H^{1/2}(\Gamma))^\infty \to (H^{-1/2}(\Gamma))^\infty,\ (-\boldsymbol{I}/2 \pm \boldsymbol{K})\colon (H^{1/2}(\Gamma))^\infty \to (H^{1/2}(\Gamma))^\infty \tag{66}$$

are bijective.

Proof. Let's consider a homogeneous boundary integral equation of the first kind

$$\mathbf{V}\boldsymbol{\mu} = 0, \quad \boldsymbol{\mu} \in \left(H^{-1/2}(\Gamma)\right)^\infty. \tag{67}$$

If we come to the component-wise notation, then for the component μ_0 we will have the equation $V_0\mu_0 = 0,\ \mu_0 \in H^{-1/2}(\Gamma)$. It is known ([12, 13]), that the operator V_0 is $H^{-1/2}(\Gamma)$-elliptic. Hence, the previous equation has only trivial solution $\mu_0 = 0$. For the component μ_1 we obtain the same equation $V_0\mu_1 = 0, \quad \mu_1 \in H^{-1/2}(\Gamma)$, whereof follows that $\mu_1 = 0$. If we continue these steps then each time we will have the same integral equation with trivial solution. Thus, the equation (67) has only trivial solution $\boldsymbol{\mu} = 0$, hence the operator $\mathbf{V}$ is injective.

Now we prove the surjectivity of the operator $\mathbf{V}$. Let $\mathbf{h} \in \left(H^{1/2}(\Gamma)\right)^\infty$ be a given sequence. Consider the boundary equation

$$\mathbf{V}\boldsymbol{\mu} = \mathbf{h} \text{ in } \left(H^{1/2}(\Gamma)\right)^\infty. \tag{68}$$

In particular, for the component μ_0 we have the following boundary integral equation of the first kind

$$V_0\mu_0 = h_0 \text{ in } H^{1/2}(\Gamma). \tag{69}$$

When considering its weak formulation, in the right hand side we obtain a continuous on $H^{-1/2}(\Gamma)$ operator $l_0(h_0){:=}\langle\phi, h_0\rangle_{H^{1/2}(\Gamma)}$. Moreover, the operator V_0 is $H^{-1/2}(\Gamma)$-elliptic ([12, 13]). Hence from the Lax-Milgram lemma the existence and the uniqueness of the solution $\mu_0 \in H^{-1/2}(\Gamma)$ of equation (69) follows.

After finding μ_0 it can be moved to the right hand side of the boundary equation obtained from (68) for μ_1. After this transformation only the component μ_1 will remain unknown in this equation. Assuming that for an arbitrary index $k \in \mathbb{N}$ equations with indices $i \in \{0, 1, \ldots, k-1\}$ have been already solved, then in the k-th equation we can separate the elements and leave in the left hand side only the component with the unknown function μ_k and the rest of the elements we can shift into the right hand side

$$V_0\mu_k = h_k - \sum_{i=0}^{k-1} V_{k-i}\mu_i \text{ in } H^{1/2}(\Gamma). \tag{70}$$

The obtained equation will differ from (69) only in the right hand side

$$\tilde{h}_k := h_k - \sum_{i=0}^{k-1} V_{k-i}\mu_i, \tag{71}$$

which contains the components $\mu_i, i \in \{0, 1, \ldots, k-1\}$, — solutions of the previous equations. It can be easily seen that $\tilde{h}_k \in H^{1/2}(\Gamma)$. Hence, by analogy to equation (69), on each step there exists a unique solution $\mu_k \in H^{-1/2}(\Gamma)$ of equation (70). Thus, for an arbitrary sequence $\mathbf{h} \in \left(H^{1/2}(\Gamma)\right)^\infty$ there exists a solution $\boldsymbol{\mu} \in (H^{-1/2}(\Gamma))^\infty$ of equation (68) i.e. the operator $\mathbf{V}$ is surjective.

Statement of the lemma regarding the other operators can be proved using the same scheme. For instance, in the case of the operator $\mathbf{D}$ the homogeneous boundary integral equation $\mathbf{D}\boldsymbol{\lambda} = 0$, $\boldsymbol{\lambda} \in \left(H^{1/2}(\Gamma)\right)^\infty$, can be split into a sequence of equations $D_0\lambda_k = 0$, $\lambda_k \in H^{1/2}(\Gamma)$, $k \in \mathbb{N}_0$, that due to the $H^{1/2}(\Gamma)$-ellipticity of the operator D_0 ([12, 13]) have only trivial solution.

Consider the sequence $\mathbf{g} \in \left(H^{-1/2}(\Gamma)\right)^\infty$. Then the boundary equation $\mathbf{D}\boldsymbol{\lambda} = \mathbf{g}$ in $\left(H^{-1/2}(\Gamma)\right)^\infty$ will lead to the following sequence of boundary integral equations of the first kind

$$D_0\lambda_k = \tilde{g}_k \text{ in } H^{-1/2}(\Gamma),\ k \in \mathbb{N}_0, \tag{72}$$

that differ only in their right hand sides

$$\tilde{g}_k = g_k - \sum_{i=0}^{k-1} D_{k-i}\lambda_i,\ k \in \mathbb{N}_0. \tag{73}$$

Since $\tilde{g}_k \in H^{-1/2}(\Gamma)$, the existence of a unique solution of each of these equations follows from the Lax-Milgram lemma due to the $H^{1/2}(\Gamma)$-ellipticity of the operator D_0 ([12, 13]). Thus, the bijectivity of the operator $\mathbf{D}$ follows from the bijectivity of the operator D_0 as well.

We will demonstrate the proof of the statement of the lemma regarding the rest of the operators based on the example of the operator $(\mathbf{I}/2 - \mathbf{K}')$. Consider the equation

$$(\mathbf{I}/2 - \mathbf{K}')\boldsymbol{\mu} = 0 \text{ in } \left(H^{-1/2}(\Gamma)\right)^\infty. \tag{74}$$

Firstly, we will show that the equation

$$\mu_0/2 - K_0'\mu_0 = 0 \text{ in } H^{-1/2}(\Gamma) \tag{75}$$

has only trivial solution. We assume that there exists another solution of this equation $\tilde{\mu} \in H^{-1/2}(\Gamma)$ and construct a function $\tilde{u}(x) = V_0\tilde{\mu}(x)$. This is a simple layer potential, so, firstly, it satisfies the equation $P\tilde{u}(x) = 0$ in $\widetilde{H}_0^{-1}(\Omega \cup \Omega^e)$, where $\Omega^e := \mathbb{R}^3 \setminus \bar{\Omega}$ ([13]). Secondly, the left hand side of equation (75) reflects the co-normal derivative γ_1^e of the simple layer potential in the exterior domain Ω^e. Hence we have $\gamma_1^e\tilde{u} = 0$ on Γ. Then from the first Green formula in the domain Ω^e (see for example lemma 5.1.2 [13]) we come to the variational equation $a_{\Omega^e}(\tilde{u}, v) = 0$ for each $v \in H^1(\Omega^e)$. Due to the ellipticity of the bilinear form $a_{\Omega^e}(\cdot,\cdot)$ this equation has only trivial solution $\tilde{u} = 0$ in Ω^e. Hence, we have $\gamma_0^e\tilde{u} = 0$ on Γ. After that, based on the property of traces of a simple layer potential (see Theorem 7) we obtain another boundary equality for the density $\tilde{\mu}$ $\gamma_0V_0\tilde{\mu} = \gamma_0^e\tilde{u} = 0$ in $H^{-1/2}(\Gamma)$. Taking into account the injectivity of the operator V_0 from this follows that $\tilde{\mu} = 0$ i.e. the kernel of the operator $(I/2 - K_0')$ contains only zero element.

Since $\mu_0 \equiv 0$, for the component μ_1 in equation (74) we will again obtain equation (75) and as a result will have $\mu_1 \equiv 0$. If we continue this process, on each step we will get trivial solution. Thus, $\boldsymbol{\mu} \equiv \mathbf{0}$ i.e. the operator $(\mathbf{I}/2 - \mathbf{K}')$ is injective.

Consider a sequence $\mathbf{g} \in \left(H^{-1/2}(\Gamma)\right)^{\infty}$. Let us apply to the equation

$$(\mathbf{I}/2 - \mathbf{K}')\boldsymbol{\mu} = \mathbf{g} \text{ in } \left(H^{-1/2}(\Gamma)\right)^{\infty}. \tag{76}$$

The same approach as we did for the equations of the first kind. As a result, we will obtain a sequence of the boundary integral equations of the second kind $\mu_k/2 - K_0'\mu_k = \check{g}_k$ in $H^{-1/2}(\Gamma),\ k \in \mathbb{N}_0$, where

$$\check{g}_k := g_k + \sum_{i=0}^{k-1} K_{k-i}'\mu_i. \tag{77}$$

Since the operator $(I/2 - K_0')$ is injective and $\check{g}_k \in H^{-1/2}(\Gamma)$, according to Theorem 5.6.12 ([13]), that reflects the Fredholm alternative for such boundary equations, each of the equations of the second kind (76) will have a unique solution. Thus we get the existence of the solution of the equation (76) for an arbitrary sequence $\mathbf{g} \in \left(H^{-1/2}(\Gamma)\right)^{\infty}$ i.e. the surjectivity of the operator $(\mathbf{I}/2 - \mathbf{K}')$. □

Theorem 8. ***(i)*** *If a pair of sequences* $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \left(H^{1/2}(\Gamma)\right)^{\infty} \times \left(H^{-1/2}(\Gamma)\right)^{\infty}$ *are the Cauchy data of some generalized solution of the equation* (61)*, then they satisfy both equations*

$$-\left(\boldsymbol{I}/2 + \boldsymbol{K}\right)\boldsymbol{\lambda} + \boldsymbol{V}\boldsymbol{\mu} = 0 \text{ in } (H^{1/2}(\Gamma))^{\infty}, \tag{78}$$

$$\boldsymbol{D}\boldsymbol{\lambda} + \left(-\boldsymbol{I}/2 + \boldsymbol{K}'\right)\boldsymbol{\mu} = 0 \text{ in } (H^{-1/2}(\Gamma))^{\infty}. \tag{79}$$

(ii) *If a pair of sequences* $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \left(H^{1/2}(\Gamma)\right)^{\infty} \times \left(H^{-1/2}(\Gamma)\right)^{\infty}$ *satisfy one of equations* (78) *or* (79)*, then they satisfy the second one and are the Cauchy data of some generalized solution of equation* (61)*.*

Proof. Let $\mathbf{u} \in (H^1(\Omega, P))^{\infty}$ be a generalized solution of equation (61). Then there exists a pair of sequences $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \left(H^{1/2}(\Gamma)\right)^{\infty} \times \left(H^{-1/2}(\Gamma)\right)^{\infty}$ that $\gamma_0\mathbf{u} = \boldsymbol{\lambda}$ and $\gamma_1\mathbf{u} = \boldsymbol{\mu}$. Theorem 6 yields that

$$\mathbf{u}(x) = \mathbf{V}\boldsymbol{\mu}(x) - \mathbf{W}\boldsymbol{\lambda}(x),\ x \in \Omega. \tag{80}$$

Let us apply the trace operator γ_0 to both parts of (80). Then, taking into account the expressions for the traces of potentials (64), we obtain

$$\gamma_0\mathbf{u} = \mathbf{V}\boldsymbol{\mu} + (\mathbf{I}/2 - \mathbf{K})\boldsymbol{\lambda}. \tag{81}$$

Hence, after substitution of $\gamma_0\mathbf{u}$ with its value $\boldsymbol{\lambda}$ we come to the relation (78). After application of the co-normal derivative operator γ_1 to both parts of (80) we obtain the following equality

$$\gamma_1\mathbf{u} = (\mathbf{I}/2 + \mathbf{K}')\boldsymbol{\mu} + \mathbf{D}\boldsymbol{\lambda}, \tag{82}$$

that after substitution of the co-normal derivative with its value $\boldsymbol{\mu}$ can be reduced to (79).

Consider an arbitrary pair of sequences $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \left(H^{1/2}(\Gamma)\right)^{\infty} \times \left(H^{-1/2}(\Gamma)\right)^{\infty}$. Then the sequence $\mathbf{u}$ built by the formula (80) belongs to the space $(H^1(\Omega, P))^{\infty}$ (Theorem 5), satisfies equation (61) (lemma 2) and its Cauchy data can be found by formulae (81) and (82). If we assume now that the sequence $\mathbf{u}$ is built of such pair $(\boldsymbol{\lambda}, \boldsymbol{\mu})$, that satisfies relation (78), then we will come to the equality $-\mathbf{K}\boldsymbol{\lambda} + \mathbf{V}\boldsymbol{\mu} = \boldsymbol{\lambda}/2$. After substituting the obtained expression into (81) we finally get $\gamma_0\mathbf{u} = \boldsymbol{\lambda}$.

Taking into account that for a sequence $\mathbf{u}$ as for an element of $(H^1(\Omega, P))^\infty$ the co-normal is defined i.e. there exists some element $\boldsymbol{\mu}^* \in \left(H^{-1/2}(\Gamma)\right)^\infty$ that has $\gamma_1\mathbf{u} = \boldsymbol{\mu}^*$, then besides (80) we will also have the following integral representation $\mathbf{u}(x) = \mathbf{V}\boldsymbol{\mu}^*(x) - \mathbf{W}\boldsymbol{\lambda}(x)$. After that we are able to write down the equality $\mathbf{V}\boldsymbol{\mu}(x) - \mathbf{W}\boldsymbol{\lambda}(x) = \mathbf{V}\boldsymbol{\mu}^*(x) - \mathbf{W}\boldsymbol{\lambda}(x),\ x \in \Omega$, or $\mathbf{V}(\boldsymbol{\mu}(x) - \boldsymbol{\mu}^*(x)) = 0,\ x \in \Omega$. Hence, applying the trace operator, we come to the boundary integral equation $\mathbf{V}(\boldsymbol{\mu}(x) - \boldsymbol{\mu}^*(x)) = 0,\ x \in \Gamma$, that according to Lemma 3 has only trivial solution. As a result we get $\boldsymbol{\mu}^* = \boldsymbol{\mu}$, whereof follows that $\gamma_1\mathbf{u} = \boldsymbol{\mu}$.

Now let $\mathbf{u}$ be a sequence, built of the pair $(\boldsymbol{\lambda}, \boldsymbol{\mu})$, that satisfies equality (79). After finding the expression $\mathbf{D}\boldsymbol{\lambda} + \mathbf{K}'\boldsymbol{\mu} = \frac{1}{2}\boldsymbol{\mu}$ from this equality and substituting it into (82) we get $\gamma_1\mathbf{u} = \boldsymbol{\mu}$. The sequence $\mathbf{u}$ has a trace i.e. there exists some element $\boldsymbol{\lambda}^* \in \left(H^{1/2}(\Gamma)\right)^\infty$ that $\gamma_0\mathbf{u} = \boldsymbol{\lambda}^*$. Then the sequence $\mathbf{u}$ can be represented as $\mathbf{u}(x) = \mathbf{V}\boldsymbol{\mu}(x) - \mathbf{W}\boldsymbol{\lambda}^*(x)$. Similarly to the previous case we have the equality $\mathbf{V}\boldsymbol{\mu}(x) - \mathbf{W}\boldsymbol{\lambda}(x) = \mathbf{V}\boldsymbol{\mu}(x) - \mathbf{W}\boldsymbol{\lambda}^*(x),\ x \in \Omega$, or $\mathbf{W}(\boldsymbol{\lambda}(x) - \boldsymbol{\lambda}^*(x)) = 0,\ x \in \Omega$. Hence, applying the co-normal derivative operator, we obtain the following boundary integral equation $\mathbf{D}(\boldsymbol{\lambda}(x) - \boldsymbol{\lambda}^*(x)) = 0,\ x \in \Gamma$, that according to Lemma 3 admits only trivial solution. As a result $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$ and $\gamma_0\mathbf{u} = \boldsymbol{\lambda}$. □

5. Boundary integral equations. Theorem 8 is a basis for the replacement of boundary value problems with the corresponding boundary integral equations in regards to the Cauchy datum that is not given explicitly in the formulation of the problem. Let us demonstrate this procedure for the Dirichlet problem (10), (11) first. In this case the boundary condition contains the given sequence $\boldsymbol{\lambda} = \tilde{\mathbf{h}} \in (H^{1/2}(\Gamma))^\infty$. Then, taking into account equation (78), after substitution of the given trace into it we will obtain the following boundary integral equation of the first kind in regards to the sequence $\boldsymbol{\mu}$

$$\mathbf{V}\boldsymbol{\mu} = (\mathbf{I}/2 + \mathbf{K})\,\tilde{\mathbf{h}} \text{ in } (H^{1/2}(\Gamma))^\infty. \tag{83}$$

If we substitute the known trace into the equation (79), we will come to the following boundary integral equation

$$(\mathbf{I}/2 - \mathbf{K}')\,\boldsymbol{\mu} = \mathbf{D}\tilde{\mathbf{h}} \text{ in } (H^{-1/2}(\Gamma))^\infty. \tag{84}$$

Theorem 9. *The co-normal derivative of the generalized solution* $\boldsymbol{u} \in (H^1(\Omega, P))^\infty$ *of the Dirichlet problem* (10), (11) *satisfies both boundary integral equations* (83) *and* (84). *Conversely, if a sequence* $\boldsymbol{\mu} \in (H^{-1/2}(\Gamma))^\infty$ *is a solution of one of boundary integral equations* (83) *or* (84) *then it will satisfy the other one and the function built by formula* (80) *with* $\boldsymbol{\lambda} = \tilde{\boldsymbol{h}}$ *will be a generalized solution of the Dirichlet problem* (10), (11).

Proof. Since the boundary integral equations (83) and (84) are in fact modified relations (78) and (79), the correctness of both the direct and the inverse statements is granted by the Theorem 8. □

Now we consider the Neumann boundary value problem (10), (12). In this case the co-normal derivative $\boldsymbol{\mu} = \tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$ is given and we can obtain the following boundary integral equation for the unknown trace $\boldsymbol{\lambda}$ from equation (78)

$$(\mathbf{I}/2 + \mathbf{K})\,\boldsymbol{\lambda} = \mathbf{V}\tilde{\mathbf{g}} \text{ in } (H^{1/2}(\Gamma))^\infty. \tag{85}$$

On the other hand, if we start with equation (79) and substitute the given co-normal derivative into it we will arrive at the boundary integral equation

$$\mathbf{D}\boldsymbol{\lambda} = (\mathbf{I}/2 - \mathbf{K}')\, \tilde{\mathbf{g}} \text{ in } (H^{-1/2}(\Gamma))^\infty. \tag{86}$$

Analogously to the previous case, Theorem 8 implies the following statement.

Theorem 10. *The trace of a generalized solution* $\boldsymbol{u} \in (H^1(\Omega, P))^\infty$ *of the Neumann problem* (10), (12) *satisfies both boundary integral equations* (85) *and* (86). *Conversely, if a sequence* $\boldsymbol{\lambda} \in (H^{1/2}(\Gamma))^\infty$ *is a solution of one of the boundary integral equations* (85) *or* (86) *then it will satisfy the other one and the function built by formula* (80) *with* $\boldsymbol{\mu} = \tilde{\mathbf{g}}$ *will is a generalized solution of the Neumann problem* (10), (12).

Unlike the previous problems, in case of the Robin boundary value problem (10), (13) we use both integral equalities (78) and (79). Since the Cauchy data of the solution are bound by the boundary condition (13), we will find the co-normal derivative $\boldsymbol{\mu}$ from it and substitute it into (79). Then we will come to the following system of boundary integral equations

$$\begin{cases} \mathbf{V}\boldsymbol{\mu} - (\mathbf{I}/2 + \mathbf{K})\boldsymbol{\lambda} = 0 \text{ in } (H^{1/2}(\Gamma))^\infty, \\ (\mathbf{I}/2 + \mathbf{K}')\boldsymbol{\mu} + (\mathbf{B} + \mathbf{D})\boldsymbol{\lambda} = \tilde{\mathbf{g}} \text{ in } (H^{-1/2}(\Gamma))^\infty. \end{cases} \tag{87}$$

Theorem 11. *The trace and the co-normal derivative of a generalized solution* $\mathbf{u} \in (H^1(\Omega, P))^\infty$ *of the Robin problem* (10), (13) *satisfy the system of boundary integral equations* (87). *Conversely, if a pair of sequences* $\boldsymbol{\lambda} \in (H^{1/2}(\Gamma))^\infty$ *and* $\boldsymbol{\mu} \in (H^{-1/2}(\Gamma))^\infty$ *is a solution of the system of boundary integral equations* (87) *then the function built by formula* (80) *will be the generalized solution of the Robin problem* (10), (13).

Proof. Let a sequence $\mathbf{u}$ be a generalized solution of the boundary value problem (10), (13). Since $\mathbf{u} \in (H^1(\Omega, P))^\infty$, according to the trace theorem and Lemma 3.2 ([12]) there exists the trace and the co-normal derivative of each component of this sequence i.e. there exists a pair of sequences $\gamma_0\mathbf{u}{=:}\boldsymbol{\lambda} \in (H^{1/2}(\Gamma))^\infty$ and $\gamma_1\mathbf{u}{=:}\boldsymbol{\mu} \in (H^{-1/2}(\Gamma))^\infty$. By Theorem 8 this pair of sequences satisfies equality (78) (also the first equation of the system (87)) and the equality (79). Moreover, this pair satisfies the boundary condition (13). Hence we can define the co-normal derivative through its trace. After substituting it into equality (79) we obtain the second equation of system (87).

Now let a pair of sequences $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in (H^{1/2}(\Gamma))^\infty \times (H^{-1/2}(\Gamma))^\infty$ be solutions of system (87). Since these sequences satisfy equality (78), by Theorem 8 they also satisfy (79) and are the Cauchy data of the generalized solution of equation (61) (i.e. the system (10)). From equality (79) we get $(\mathbf{I}/2 + \mathbf{K}')\boldsymbol{\mu} + \mathbf{D}\boldsymbol{\lambda} = \boldsymbol{\mu}$ and substitute this expression into the second equation of system (87). Finally we obtain the equality $\boldsymbol{\mu} + \mathbf{B}\boldsymbol{\lambda} = \tilde{\mathbf{g}}$ in $(H^{-1/2}(\Gamma))^\infty$, that is a notation of boundary condition (13) through the pair of sequences $(\boldsymbol{\lambda}, \boldsymbol{\mu})$. □

Let us investigate the well-posedness of the obtained boundary integral equations.

Theorem 12. ***(i)*** *For an arbitrary sequence* $\tilde{\boldsymbol{h}} \in (H^{1/2}(\Gamma))^\infty$ *boundary integral equations* (83) *and* (84) *have a unique solution* $\boldsymbol{\mu} \in (H^{-1/2}(\Gamma))^\infty$.
(ii) *For an arbitrary sequence* $\tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$ *boundary integral equations* (85) *and* (86) *have a unique solution* $\boldsymbol{\lambda} \in (H^{1/2}(\Gamma))^\infty$.
(iii) *For an arbitrary sequence* $\tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$ *the system of boundary integral equations* (87) *has a unique solution* $(\boldsymbol{\mu}, \boldsymbol{\lambda}) \in (H^{-1/2}(\Gamma))^\infty \times (H^{1/2}(\Gamma))^\infty$.

Proof. Consider the sequences $\tilde{\mathbf{h}} \in (H^{1/2}(\Gamma))^\infty$ and $\tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$. Then according to Theorem 5 and relation (64) we obtain $(\mathbf{I}/2+\mathbf{K})\tilde{\mathbf{h}} \in (H^{1/2}(\Gamma))^\infty$, $\mathbf{D}\tilde{\mathbf{h}} \in (H^{-1/2}(\Gamma))^\infty$, $\mathbf{V}\tilde{\mathbf{g}} \in (H^{1/2}(\Gamma))^\infty$, $(\mathbf{I}/2-\mathbf{K}')\tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$. Thus the first two statements of the theorem follow from lemma 3.

Now consider the system (87). Due to the bijectivity of the operator $\mathbf{V}$ there exists the inverse operator $\mathbf{V}^{-1}\colon (H^{1/2}(\Gamma))^\infty \to (H^{-1/2}(\Gamma))^\infty$. From the first equation of the system we get $\boldsymbol{\mu} = \mathbf{V}^{-1}(\mathbf{I}/2+\mathbf{K})\boldsymbol{\lambda}$. Note that in the general case it is complicated to find this operator explicitly but the components of $\boldsymbol{\mu}$ can be calculated in a numerical way.

Let us examine the structure of system (87). Thus, regarding the components λ_0 and μ_0 we obtain the following system

$$\begin{cases} V_0\mu_0 - (I/2+K_0)\lambda_0 = 0, \\ (I/2+K_0')\mu_0 + (b_{0,0}I + D_0)\,\lambda_0 = \tilde{g}_0. \end{cases}$$

After solving this system, in the following pair of the boundary integral equations regarding (λ_1, μ_1) the solutions found on the previous step can be moved to the right hand side. Continuing this process for an arbitrary index $k \in \mathbb{N}$ we obtain the following system of two boundary integral equations

$$\begin{cases} V_0\mu_k - (I/2+K_0)\lambda_k = -\sum_{i=0}^{k-1} V_{k-i}\mu_i + \sum_{i=0}^{k-1} K_{k-i}\lambda_i, \\ (I/2+K_0')\mu_k + (b_{k,k}I + D_0)\lambda_k = \tilde{g}_k - \sum_{i=0}^{k-1} K'_{k-i}\mu_i - \sum_{i=0}^{k-1}(b_{k,i}I + D_{k-i})\lambda_i. \end{cases} \tag{88}$$

Hereafter we will treat it as a system, enclosed into infinite system (87).

Now we investigate the solvability of system (88). Let us rewrite its first equation for an arbitrary fixed $k \in \mathbb{N}_0$ in the following form

$$V_0\mu_k = \frac{1}{2}\lambda_k + \sum_{i=0}^{k} K_{k-i}\lambda_i - \sum_{i=0}^{k-1} V_{k-i}\mu_i \text{ in } H^{1/2}(\Gamma),\ k \in \mathbb{N}_0, \tag{89}$$

and substitute (formally) $\mu_k = V_0^{-1}(\lambda_k/2 + \sum_{i=0}^{k} K_{k-i}\lambda_i - \sum_{i=0}^{k-1} V_{k-i}\mu_i)$ into the second equation of the system (88). After moving all of the components with indices $i \in \{0, 1, \dots, k-1\}$ into the right hand side we will get the following boundary integral equation

$$(S_0 + b_{k,k}I)\lambda_k = \tilde{g}_k^{\,*} \text{ in } H^{-1/2}(\Gamma),\ k \in \mathbb{N}_0, \tag{90}$$

where

$$\tilde{g}_k^{\,*} := \tilde{g}_k - \sum_{i=0}^{k-1} K'_{k-i}\mu_i - \sum_{i=0}^{k-1}(b_{k,i}I + D_{k-i})\lambda_i + \left(\frac{1}{2}I + K_0'\right) V_0^{-1}\left(\sum_{i=0}^{k-1} K_{k-i}\lambda_i - \sum_{i=0}^{k-1} V_{k-i}\mu_i\right), \tag{91}$$

and the operator $S_0\colon H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, that acts by the following rule $S_0 := D_0 + (I/2 + K_0')V_0^{-1}(I/2 + K_0)$, is called a *symmetric Steklov-Poincare operator* ([13, 15]). It is known (see for example 5.6.7 in [13]), that the following equality holds for it $\langle S_0\eta, \eta\rangle_\Gamma = \langle D_0\eta, \eta\rangle_\Gamma + \|(I/2 + K_0)\eta\|^2_{V^{-1}}$ for each $\eta \in H^{1/2}(\Gamma)$, where $\|\eta\|^2_{V^{-1}} := \langle V^{-1}\eta, \eta\rangle_\Gamma$ — is another norm in $H^{1/2}(\Gamma)$.

Then, taking into account the ellipticity of the operator D_0, we can write the following relation for each $\eta \in H^{1/2}(\Gamma)$ $\langle (S_0 + b_{k,k}I)\eta, \eta\rangle_\Gamma = \langle D_0\eta, \eta\rangle_\Gamma + \|(I/2+K_0)\eta\|^2_{V^{-1}} + \langle b_{k,k}\eta, \eta\rangle_\Gamma \geq$

$\tilde{b}_k\|\eta\|^2_{H^{1/2}(\Gamma)}$. It means that the operator of the left hand side of the equation (90) is $H^{1/2}(\Gamma)$-elliptic and according to the Lax-Milgram theorem this equation has a unique solution $\lambda_k \in H^{1/2}(\Gamma)$. After the inverse substitution of λ_k into the equation (89) we can find its unique solution $\mu_k \in H^{-1/2}(\Gamma)$.

Thus, for an arbitrary sequence $\tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$ there exists a unique pair of sequences $(\boldsymbol{\mu}, \boldsymbol{\lambda}) \in (H^{-1/2}(\Gamma))^\infty \times (H^{1/2}(\Gamma))^\infty$, that satisfy system (87). □

Note the structure of boundary integral equations that have been obtained during the reduction of the boundary value problems. When proving Theorem 12 for the Robin problem we obtained a sequence of systems (88). If all of the coefficients $b_{k,k}$ in the Robin boundary condition are equal then the operator of the left hand side in (88) will remain the same for each $k \in \mathbb{N}_0$ and the right hand side will contain the solutions found on previous steps besides given sequences.

Let's show that the sequences of boundary integral equations for the Dirichlet and Neumann boundary value problems will have the same property. Consider boundary integral equation (83). It can be reduced to a sequence of boundary integral equations

$$V_0\mu_k = \frac{1}{2}\tilde{h}_k + \sum_{i=0}^{k} K_{k-i}\tilde{h}_i - \sum_{i=0}^{k-1} V_{k-i}\mu_i \text{ in } H^{1/2}(\Gamma),\ k \in \mathbb{N}_0. \tag{92}$$

As we can see for different values of k, equations (92) will differ only in their right hand sides. The sequence of boundary integral equations of the first kind

$$D_0\lambda_k = \frac{1}{2}\tilde{g}_k - \sum_{i=0}^{k} K'_{k-i}\tilde{g}_i - \sum_{i=0}^{k-1} D_{k-i}\lambda_i \quad \text{in} \quad H^{-1/2}(\Gamma),\ k \in \mathbb{N}_0,$$

that are obtained from the equation (86), have the same property.

Applying the same approach for equations (84) and (85) we get the following sequences of boundary integral equations of the second kind

$$\frac{1}{2}\mu_k - K'_0\mu_k = \sum_{i=0}^{k} D_{k-i}\tilde{h}_i + \sum_{i=0}^{k-1} K'_{k-i}\mu_i \text{ in } H^{-1/2}(\Gamma),\ k \in \mathbb{N}_0,$$

$$\frac{1}{2}\lambda_k + K_0\lambda_k = \sum_{i=0}^{k} V_{k-i}\tilde{g}_i - \sum_{i=0}^{k-1} K_{k-i}\lambda_i \text{ in } H^{1/2}(\Gamma),\ k \in \mathbb{N}_0,$$

respectively.

As follows, after application of q-convolution of sequences to variational formulations of boundary value problems all of the obtained sequences of boundary integral equations will have a specified property. Since the boundary operators in the left hand sides of these equations remain the same for each $k \in \mathbb{N}_0$, we can build efficient algorithms for their numerical solution.

Thus, variational problems for infinite triangular systems, which consist of elliptic equations with variable coefficients, have been formulated and their well-posedness has been shown. In the case of constant coefficients a representation of generalized solutions in the form of potentials has been obtained, with which variational problems have been reduced to triangular systems of boundary integral equations. By using the q-convolution of sequences, components of the solution of the systems of boundary integral equations can consistently be found from the relevant equations which differ only in the right hand side. In this case the right hand side consists of the components of the solutions, found on previous steps, besides of the given Cauchy data. Solvability of such systems in the appropriate Sobolev spaces has been established. Further, we plan to build efficient numerical methods, based on boundary elements, for the solution of such systems.

Acknowledgements. Authors would like to thank the referee for valuable suggestions that led to an improvement in the paper's presentation.

REFERENCES

1. Vladimirov V.S. Equations of mathematical physics. – Moscow: Fizmatlit, 2000. – 398p. (in Russian)
2. Galazyuk V.A., Lyudkevych J.V., Muzychuk A.O. An integral equation method for time-depended diffraction problems. Lviv. univ., 1984. – Dep. v UkrNIINTI, №601Uk-85Dep. – 16p. (in Russian)
3. Galazyuk V.A., Chapko R.S. *Methods of the Tschebyschov–Laguerre transformation and integral equations for initial boundary value problems for the telegraph equation*// Dokl. AN USSR. – 1990. – Ser.A, V.8. – P. 11–14. (in Russian)
4. Ladyzhenskaya O.A. The boundary value problems of mathematical physics. – Moscow: Nauka, 1973. – 407p. (in Russian)
5. Litynskyy S., Muzychuk Yu., Muzychuk A. *On week solutions of boundary value problems for an infinite system of elliptic equations*// Visn. Lviv. univ. Ser. prykl. matem. inform. – 2009. – V.15. – P. 52–70. (in Ukrainian)
6. Lyudkevych J.V., Muzychuk A.O. Numerical solution of boundary value problems for a wave equation.– Lviv: Lviv. university, 1990. – 90p. (in Russian)
7. Lyudkevych J.V., Skaskiv R.B. *Numerical solution of the Dirichlet boundary value problem for a heat equation by integral transformations and integral equations methods in the case of open axial-symmetric surfaces*// Visn. Lviv. univ. Ser. mekh.-mat. – 1989.– V.31. – P. 2–8. (in Ukrainian)
8. Mikhlin S.G. Linear equations in partial derivatives. – Moscow: Vysshaya shkola, 1977. – 432p. (in Russian)
9. Chapko R. *On the combination of some semi-discretization methods and boundary integral equations for the numerical solution of initial boundary value problems*// PAMM, Proc. Appl. Math. Mech. – 2002. – V.1. – P. 424–425.
10. Chapko R., Kress R. *Rothe's method for the heat equation and boundary integral equations*// Journal of Integral Equations and Applications. – 1997. – V.9. – P. 47–69.
11. Chapko R., Kress R. *On the numerical solution of initial boundary value problems by the Laguerre transformation and boundary integral equations*// In Agarwal, O'Regan, eds. *Integral and Integrodifferential Equations: Theory, Methods and Applications. Series in Mathematical Analysis and Applications.* – V.2, Cordon and Breach Science Publishers, Amsterdam, 2000. – P. 55–69.
12. Costabel M. *Boundary integral operators on Lipschitz domains: elementary results*// SIAM J. Math. Anal. – 1988. – V.19. – P. 613–626.
13. Hsiao G.C., Wendland W.L. Boundary Integral Equations. – Springer-Verlag, Berlin, 2008. – 640p.
14. McLean W. Strongly Elliptic Systems and Boundary Integral Equations. – Cambridge University Press, 2000. – 357p.
15. Steinbach O. Numerical Approximation Methods for Elliptic Boundary Value Problems. – Springer Science, 2008. – 396p.
16. Sybil Yu. *Three-dimensional elliptic boundary value problems for an open Lipschitz surface*// Mat. Stud. – 1997. – V.8, №1. – P. 79–96.

Department of Numerical Mathematics
Faculty of Applied Mathematics and Informatics
Ivan Franko National University of Lviv
yuriy.muzychuk@gmail.com
roman.chapko@gmail.com

Received 28.04.2011
Revised 7.06.2012