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TWO OPEN PROBLEMS FOR ABSOLUTELY CONVERGENT DIRICHLET SERIES

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For the absolutely convergent in a half-plane Dirichlet series we establish upper estimates without exceptional sets.

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Для абсолютно сходящихся в полуплоскости рядов Дирихле устанавливаются оценки сверху без исключительного множества.

1. Introduction. Let $\Lambda=(\lambda_n)_{n=0}^{\infty}$ be a sequence of nonnegative numbers such that $0=\lambda_0<\lambda_n<\lambda_{n+1}$ $(1\le n\uparrow+\infty)$ and $S^a(\Lambda)$ be the class of absolutely convergent in the complex half-plane $\Pi_a=\{z=\sigma+it\in\mathbb{C}\colon \sigma<a, t\in\mathbb{R}\}$, $-\infty<a\le+\infty$, Dirichlet series of the form

$$F(z)=\sum_{n=0}^{+\infty}a_ne^{z\lambda_n},\quad z=\sigma+it. \tag{1}$$

For $F\in S^a(\Lambda)$ and $\sigma<a$ we define by $M(\sigma,F)=\sup\{|F(\sigma+it)|\colon t\in\mathbb{R}\}$ the maximum modulus of the function F and by $\mu(\sigma,F)=\max\{|a_n|\exp\{\sigma\lambda_n\}\colon n\ge0\}$ the maximal term.

The class of nonnegative continuous on $[0;+\infty)$ functions $l(x)$ such that $l(x)\to+\infty$ $(x\to+\infty)$ is denoted by L_0, the subclass of functions $l\in L_0$ such that $l(x)\nearrow+\infty$ as $0\le x\uparrow+\infty$ is denoted by L. The subclass of functions $l\in L_0$ such that $\frac{x}{l(x)}\nearrow+\infty$ $(x\to+\infty)$ is denoted by L_1.

We introduce the following classes of Dirichlet series. For $\psi\in L$ let $S_\psi(\Lambda)$ denote the class of entire Dirichlet series (1) (i.e. $F\in S(\Lambda)\overset{def}{=}S^{+\infty}(\Lambda)$) such that $\sharp\{n\colon a_n\ne0\}=+\infty$ and

$$|a_n|\le\exp\{-\lambda_n\psi(\lambda_n)\}\quad(n\ge n_0). \tag{2}$$

For $\psi\in L_1$ let $S_\psi^0(\Lambda)$ denote the class of Dirichlet series $F\in S^0(\Lambda)$ such that

$$|a_n|\le\exp\Big\{\frac{\lambda_n}{\psi(\lambda_n)}\Big\},\quad n\ge n_0. \tag{3}$$

We remark that condition (2) is equivalent to the inequality $\ln\mu(x,F)\le x\Phi(x)$ $(x\in(x_0,+\infty))$ and condition (3) is equivalent to $\ln\mu(x,F)\le\Phi_0(1/|x|)$ $(x\in(x_0,0))$, $x_0<0$, where $\Phi,\ \Phi_0$ are some functions from the class L.

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In 2000 P. V. Filevych ([1]) proved the following theorem.

Theorem A ([1]). *Let $\psi \in L$, condition*

$$\overline{\lim_{n\to\infty}} \frac{\ln\ n}{\lambda_n \psi(\lambda_n)} = q < 1 \tag{4}$$

holds and $h \in L_0$. Then

$$(\forall F \in S_\psi(\Lambda))(\exists x_0)(\forall x \geq x_0)\colon\ M(x,F) < \mu(x,F)h(\ln \mu(x,F))$$

holds if and only if

$$(\forall l_1, l_2 \in L)(\exists n_0)(\forall n \geq n_0)\colon\ n < l_1(n) + h\big(l_2(n)\psi(\lambda_n)\big). \tag{5}$$

Using the ideas of the proof of Theorem A (Theorem 2 in [1]) one can prove the following assertion.

Proposition 1. *If $\psi \in L$, $h \in L_0$ and conditions* (4), (5) *hold, then*

$$(\forall d \in L_0)(\forall F \in S_\psi(\Lambda))(\exists x_0)(\forall x \geq x_0)\colon\ M(x,F) < \mu(x,F)h(xd(x)).$$

Note that since the function $\ln \mu(x,F)$ is convex (i.e. $\ln \mu(x,F)/x \to +\infty$ $(x \to +\infty)$), sufficiency in Theorem A follows from Proposition 1. Proposition 1 implies also that by conditions (4), (5)

$$(\forall F \in S_\psi(\Lambda))(\exists c > 0(\exists x_0)(\forall x \geq x_0)\colon\ M(x,F) < \mu(x,F)h(cx).$$

Question 1. *What is the exact value of the constant c in the previous inequality?*

Conjecture 1. *Sharp values are $c = \tau + \varepsilon$, where $\varepsilon > 0$ is arbitrary,*

$$\tau = \overline{\lim_{t\to+\infty}} \frac{h^{-1}(n(t))}{\psi(t)}, \quad n(t) = \sum\nolimits_{\lambda_n \leq t} 1, \tag{6}$$

and h^{-1} is the inverse function to function $h \in L$.

Consider first the following problem.

Problem 1. *Set an analogue of Theorem A or Proposition 1 for the class $S^0(\Lambda)$.*

So, we consider Dirichlet series $F \in S^0(\Lambda)$ of the form (1). Next, we assume that conditions

$$q := \overline{\lim_{n\to+\infty}} \frac{\ln n}{\lambda_n}\psi(\lambda_n) < \theta < +\infty \tag{7}$$

and

$$\sup\{|a_n|\colon n \geq 0\} = +\infty \tag{8}$$

hold.

Remark 1. It is easy to see from (7) that $\ln n = o(\lambda_n)$ $(n \to +\infty)$ and thus every Dirichlet series of form (1) has the abscissa of absolutely convergence $\sigma_a \geq 0$ as (3) and (7) hold. In the case of $\sup\{|a_n|\colon n \geq 0\} = +\infty$ from conditions (3) and (7) we obtain that $\sigma_a = 0$.

The following theorem shows that direct analogues of Theorem A for the class $S^0(\Lambda)$ can not be obtained.

Theorem 1. *Let $h \in L$, $\Phi \in L$ be arbitrary functions. For all sequences $\Lambda = (\lambda_n)_{n=0}^{\infty}$ such that $0 = \lambda_0 < \lambda_n < \lambda_{n+1}$ $(1 \le n \uparrow +\infty)$ there exist some function $F \in S^0(\Lambda)$ such that condition* (8) *is satisfied and*

$$\frac{M(x,F)}{\mu(x,F)h(\ln \mu(x,F))} \to +\infty \quad (x \to -0), \tag{9}$$

$$(\exists x_0 < 0)(\forall x \in (x_0, 0)) \colon \ \ln \mu(x,F) \le \Phi(1/|x|). \tag{10}$$

The following theorem is a certain analogue of Proposition 1 for the class $S^0(\Lambda)$.

Theorem 2. *If $\psi \in L_1$, $h \in L_0$ and conditions* (7), (5) *hold, then*

$$(\forall d \in L_0)(\forall F \in S^0_{\psi}(\Lambda))(\exists x_0 < 0)(\forall x \in (x_0, 0)) \colon \ M(x,F) < \mu(x,F) h\left(\frac{1}{|x|} d\left(\frac{1}{|x|}\right)\right). \tag{11}$$

Theorem 2 implies the following corollary.

Corollary 1. *If $\psi \in L_1$, $h \in L_0$ and conditions* (7), (5) *hold, then*

$$(\forall F \in S^0_{\psi}(\Lambda))(\exists c > 0)(\exists x_0 < 0)(\forall x \in [x_0, 0)) \colon \ M(x,F) < \mu(x,F)h(c/|x|).$$

Question 2. *What is the exact value of constant c in the previous inequality?*

Conjecture 2. *Sharp values are $c = \tau + \varepsilon$, where $\varepsilon > 0$ is arbitrary and τ is defined by* (6).

In the case of $\psi(x) = x^{\alpha}$, $\alpha \in (0,1)$, $h(x) = x^{\beta}$, $\beta > 0$, Corollary 1 implies the following statement.

Corollary 2. *If $F \in S^0(\Lambda))$, $|a_n| \le \exp\{\lambda_n^{1-\alpha}\}$ $(n \ge n_0)$ and $\overline{\lim\limits_{n \to +\infty}} \dfrac{n^{1/(\alpha\beta)}}{\lambda_n} < +\infty$, then*

$$(\exists c > 0)(\exists x_0 < 0)(\forall x \in [x_0, 0)) \colon \ M(x,F) < c\mu(x,F)(1/|x|)^{\beta}.$$

The degree of β in the last inequality, in general, cannot be improved. This follows from the example of functions below

Example 1. Let $\lambda_n = n^q$ $(n \ge 0)$, $\beta > 0$, $\alpha \in (0,1)$, $\alpha \cdot \beta \ge 1$, $p = 1 - \alpha$ and

$$F_0(z) = \sum_{n=1}^{+\infty} e^{n^{pq} + zn^q}. \tag{12}$$

It is obvious that $F_0 \in S^0(\Lambda)$. For the series (12) we have

$$F_0(x) = (1 + o(1)) \frac{\sqrt{2\pi}}{q\sqrt{1-p}} p^{\frac{2-q}{2q(1-p)}} \mu(x, F_0) |x|^{\frac{pq-2}{2q(1-p)}} \ (x \to -0) \tag{13}$$

and

$$\mu(x, F_0) = (1 + (o(1))) e^{|x|^{-\frac{p}{1-p}} p^{\frac{p}{1-p}} (1-p)} \ (x \to -0).$$

i) Now for fixed $\beta > 0$ and $\varepsilon \in (0,1)$ we choose $\alpha \in (0,1)$, $q > 0$ such that $\alpha = \frac{1}{1+\varepsilon}$, $\alpha\beta q = 1$. Then $\frac{2-pq}{2(1-p)q} = \beta - \frac{\varepsilon}{2}$, therefore from relation (13) we have

$$|x|^{\beta - \varepsilon} F_0(x)/\mu(x, F_0) \to +\infty \quad (x \to -0).$$

ii) We can rewrite relation (13) in the form

$$F_0(x)=\frac{\sqrt{2\pi}\big(1+o(1)\big)}{q}(1-p)^{\frac{pq-2-q}{2q}}p^{\frac{2-q-pq}{2q}}\mu(x,F_0)\Big(\frac{\ln\mu(x,F_0)}{|x|}\Big)^{\frac{2-pq}{2q}},\ (x\to-0). \qquad (14)$$

Now for fixed $\beta>0$ and $\varepsilon\in(0,\beta)$ we choose $q=\frac{2\beta+1}{(2\beta+1-\varepsilon)\beta}$ and $\alpha\in(0,1)$ such that $\alpha\beta q=1$. Then $\frac{2-pq}{2q}=\frac{1}{q}\cdot\frac{2\beta+1}{2\beta}-\frac{1}{2}=\beta-\frac{\varepsilon}{2}$ and from relation (14) we obtain

$$F_0(x)/\mu(x,F_0)\Big(\frac{\ln\mu(x,F_0)}{|x|}\Big)^{-(\beta-\varepsilon)}\to+\infty\quad(x\to-0).$$

2. Proof of the theorems.

Proof of Theorem 1. The idea of the proof is the same as the idea of the proof of Theorem 1 from [2] (see, also [3, 4]). Without loss of generality we assume that $h(1)=1$, $t+\ln h(t)<0$ $(0\le t\le 1/2)$. Let (E_n) be some sequence such that $1=E_1\le E_n<E_{n+1}$ and $1\le n/E_n<(n+1)/E_{n+1}$ $(n\ge1)$. For every $n\ge1$ there exists L_n such that $L_n+\ln h(L_n)=\ln(n/E_n)$. Then $L_1=1$, $L_n<L_{n+1}$ $(n\ge1)$, $L_n\to+\infty$ $(n\to+\infty)$.

Let $c_n=L_{n-1}-L_{n-2}$ $(n\ge3)$ and (λ_j^*) be a subsequence of the sequence (λ_n) such that

$$\ln n=o(\lambda_n^*)\quad(n\to+\infty),\qquad\sum_{j=3}^{+\infty}\frac{c_j}{\lambda_j^*}\Phi^{-1}(L_{j-1})\le1, \qquad (15)$$

where Φ^{-1} is the inverse function to the function Φ. We define

$$\ln a_n^*\stackrel{def}{=}L_{n-1}+\lambda_n^*\sum_{j=n+1}^{+\infty}\frac{c_j}{\lambda_j^*}\quad\text{and}\quad a_n=\begin{cases}a_j^*, & \text{if}\ \ \lambda_n=\lambda_j^*,\\ 0, & \text{if}\ \ \lambda_n\notin\{\lambda_j^*\colon j\ge1\}.\end{cases}$$

Then

$$F(z)=\sum_{n=0}^{+\infty}a_ne^{z\lambda_n}=\sum_{n=1}^{+\infty}a_n^*e^{z\lambda_n^*}.$$

Conditions (15) immediately yield that $F\in S^0(\Lambda)$. Indeed, from (15) by the definition of L_n we obtain

$$\ln a_n^*\le\ln n+\lambda_n^*\sum_{j=n+1}^{+\infty}\frac{c_j}{\lambda_j^*}=o(\lambda_n^*)\quad(n\to+\infty).$$

Hence, for $x<0$

$$a_n^*\exp\{x\lambda_n^*\}=\exp\Big\{-\frac{|x|}{2}\lambda_n^*\Big\}\le\exp\{-2\ln n\}\quad(n\ge n_0),$$

where $n_0=n_0(x)$. Therefore $F\in S^0(\Lambda)$.

We need following lemma (see [5, p.19]).

Lemma 1. *Let $g\in S^0(\Lambda)$ be a Dirichlet series of the form $g(z)=\sum_{n=0}^{+\infty}g_ne^{z\lambda_n}$. If*

$$\varkappa_n\stackrel{def}{=}\frac{\ln|g_{n-1}|-\ln|g_n|}{\lambda_n-\lambda_{n-1}}\nearrow0\quad(1\le n\uparrow+\infty),$$

then

$$(\forall n\ge1)(\forall x\in[\varkappa_n\varkappa_{n+1}])\colon\quad\mu(x,g)=|g_n|e^{x\lambda_n}.$$

We note that for the function F

$$\varkappa_n = \frac{\ln a^*_{n-1} - \ln a^*_n}{\lambda^*_n - \lambda^*_{n-1}} = -\sum_{j=n}^{+\infty} \frac{c_j}{\lambda^*_j} \uparrow 0 \quad (n \to +\infty).$$

By Lemma 1 for all $x \in [\varkappa_n, \varkappa_{n+1}]$

$$\mu(x, F) = a^*_n \exp\{x\lambda^*_n\}.$$

Hence, for all $x \in [\varkappa_n, \varkappa_{n+1}]$ and $n \geq 2$

$$\ln \mu(x, F) = \ln a^*_n + x\lambda^*_n \leq L_{n-1} + \lambda^*_n \sum_{j=n+1}^{+\infty} \frac{c_j}{\lambda^*_j} + \varkappa_{n+1}\lambda^*_n = L_{n-1} \tag{16}$$

and for $m,\ 2 \leq m \leq n-1$,

$$\ln a^*_m + x\lambda^*_m \geq L_{m-1} + \lambda^*_m \sum_{j=m+1}^{+\infty} \frac{c_j}{\lambda^*_j} + \varkappa_n\lambda^*_m \geq L_{m-1} \geq 1.$$

From the previous inequality for all $x \in [\varkappa_n, \varkappa_{n+1}]$ and $n \geq 2$ we obtain

$$M(x, F) = F(x) = \sum_{k=0}^{+\infty} a_k e^{z\lambda_k} \geq \sum_{m=1}^{n} a^*_m e^{z\lambda^*_m} \geq n =$$
$$= E_n h(L_n) \exp\{L_n\} > E_n \mu(x, F) h(\ln \mu(x, F))$$

and (9) follows. From (15) we have $L_{n-1} \leq \Phi(1/|\varkappa_n|)$ $(n \geq 2)$. Therefore, from (16) for all $x \in [\varkappa_n, \varkappa_{n+1}]$ and $n \geq 2$

$$\ln \mu(x, F) \leq L_{n-1} \leq \Phi(1/|\varkappa_n|) \leq \Phi(1/|x|).$$

This complete the proof of Theorem 1. □

Proof of Theorem 2. Assume that condition (5) is fulfilled but there exist a function $d \in L_0$ and a sequence $x_j \uparrow 0$ $(j \to +\infty)$ such that

$$(\forall j \geq 1): \quad M(x_j, F) \geq \mu(x_j, F) h\Big(\frac{1}{|x_j|} d\Big(\frac{1}{|x_j|}\Big)\Big). \tag{17}$$

For $x < 0$ we set $n_1(x) = \min\{n\colon \psi(\lambda_n) \geq (1+\theta+\delta)\frac{1}{|x|}\},\ \delta > 0$. Then

$$n_1(x) \nearrow +\infty \ (x \uparrow 0) \quad \text{and} \quad \psi(\lambda_{n_1(x)-1}) < (1+\theta+\delta)\frac{1}{|x|}. \tag{18}$$

Moreover, using (3) and definition of $n_1(x)$ we have

$$\sum_{n=n_1(x)}^{+\infty} |a_n| \exp\{x\lambda_n\} \leq \sum_{n=n_1(x)}^{+\infty} \exp\Big\{-(\theta+\delta)\frac{\lambda_n}{\psi(\lambda_n)}\Big\} = \int\limits_{\lambda_{n_1(x)-1}}^{+\infty} \exp\Big\{-(\theta+\delta)\frac{t}{\psi(t)}\Big\} dn(t).$$

Condition (7) implies

$$\int_{t_0}^{+\infty} \exp\Big\{-(\theta+\delta)\frac{t}{\psi(t)}\Big\} dn(t) < +\infty,$$

hence by (18) we have

$$\sum_{n=n_1(x)}^{+\infty} |a_n| \exp\{x\lambda_n\} = o(1) \ \ (x \uparrow 0)\,. \tag{19}$$

The condition $\sup\{|a_n|\colon n \geq 0\} = +\infty$ implies $\mu(x, F) \uparrow +\infty \quad (x \uparrow 0)$. Thus from (19) we obtain

$$\sum_{n=n_1(x)}^{+\infty} |a_n| \exp\{x\lambda_n\} < \frac{1}{2}\mu(x, F) \quad (x \in [\sigma_1, 0)). \tag{20}$$

Since for all fixed m and $x < 0$

$$\sum_{n=0}^{m} |a_n| \exp\{x\lambda_n\} \leq \sum_{n=0}^{m} |a_n|,$$

there exists a continuous function $n_2 = n_2(x)$ such that $n_2(x) \nearrow +\infty$ $(x \uparrow 0)$ and

$$\sum_{n=0}^{[n_2(x)]} |a_n| \exp\{x\lambda_n\} \leq \frac{1}{2}\mu(x, F) \quad (x \in [\sigma_2, 0)),$$

where $[a]$ means the most integer such that $[a] \leq a$. Let $\sigma_3 = \max\{\sigma_1, \sigma_2\}$. Then from the previous inequality and (20) for $x \in [\sigma_3, 0)$ we have

$$\begin{aligned} M(x, F) < \mu(x, F) + \sum_{n=[n_2(x)]+1}^{n_1(x)-1} |a_n| \exp\{x\lambda_n\} \leq (n_1(x) - [n_2(x)])\mu(x, F) \leq \\ \leq (n_1(x) - n_2(x) + 1)\mu(x, F). \end{aligned}$$

Now choose a function $l_1 \in L$ such that

$$l_1(n_1(x) - 1) \leq n_2(x) - 2 \quad \big(x \in [\sigma_3, 0)\big).$$

Then $n_1(x) - n_2(x) + 1 = n_1(x) - 1 - (n_2(x) - 2) \leq n_1(x) - 1 - l_1(n_1(x) - 1)$, and for all function $l_2 \in L$ by condition (5)

$$\begin{aligned} M(x, F) < (n_1(x) - 1 - l_1(n_1(x) - 1))\mu(x, F) \leq h(l_2(n_1(x) - 1)\psi(\lambda_{n_1(x)-1}))\mu(x, F) \leq \\ \leq h(l_2(n_1(x) - 1)(1 + \theta + \delta)/|x|)\mu(x, F), \quad \big(x \in [\sigma_4, 0)\big). \end{aligned} \tag{21}$$

Without loss of generality, we may assume that $d(1/|x_j|) \uparrow +\infty$ $(j \to +\infty)$. Let $n_1^*(x)$ be a continuous function such that $n_1(x) - 1 \leq n_1^*(x) \leq n_1(x)$ $(x < 0)$. At last, we choose a function $l_2 \in L$ such that for $x = x_j$ $(j \geq 1)$

$$l_2(n_1^*(x))(1 + \theta + \delta) = d(1/|x|).$$

Then from inequality (21) for $x = x_j$ $(j \geq 1)$ we obtain

$$M(x, F) < \mu(x, F) h\left(d(1/|x|)/|x|\right),$$

and this contradicts (17). □

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