
For the absolutely convergent in a half-plane Dirichlet series we establish upper estimates without exceptional sets.


Для абсолютно сходящихся в полуплоскости рядов Дирихле устанавливаются оценки сверху без исключительного множества.

1. Introduction. Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be a sequence of nonnegative numbers such that $0 = \lambda_0 < \lambda_n < \lambda_{n+1}$ ($1 \leq n \uparrow +\infty$) and $S^\omega(\Lambda)$ be the class of absolutely convergent in the complex half-plane $\Pi_a = \{z = \sigma + it \in \mathbb{C} : \sigma < a, t \in \mathbb{R} \}, -\infty < a \leq +\infty$, Dirichlet series of the form

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}, \quad z = \sigma + it. \quad (1)$$

For $F \in S^\omega(\Lambda)$ and $\sigma < a$ we define by $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ the maximum modulus of the function $F$ and by $\mu(\sigma, F) = \max\{|a_n| \exp\{\sigma \lambda_n\} : n \geq 0\}$ the maximal term.

The class of nonnegative continuous on $[0; +\infty)$ functions $l(x)$ such that $l(x) \rightarrow +\infty$ ($x \rightarrow +\infty$) is denoted by $L_0$, the subclass of functions $l \in L_0$ such that $l(x) \nearrow +\infty$ as $0 \leq x \uparrow +\infty$ is denoted by $L$. The subclass of functions $l \in L_0$ such that $\frac{x}{l(x)} \nearrow +\infty$ ($x \rightarrow +\infty$) is denoted by $L_1$.

We introduce the following classes of Dirichlet series. For $\psi \in L$ let $S_{\psi}(\Lambda)$ denote the class of entire Dirichlet series (1) (i.e. $F \in S(\Lambda) \overset{\text{def}}{=} S^{+\infty}(\Lambda)$) such that $\sharp\{n : a_n \neq 0\} = +\infty$ and

$$|a_n| \leq \exp\{-\lambda_n \psi(\lambda_n)\} \quad (n \geq n_0). \quad (2)$$

For $\psi \in L_1$ let $S_{\psi}^0(\Lambda)$ denote the class of Dirichlet series $F \in S^0(\Lambda)$ such that

$$|a_n| \leq \exp\left\{\frac{\lambda_n}{\psi(\lambda_n)}\right\}, \quad n \geq n_0. \quad (3)$$

We remark that condition (2) is equivalent to the inequality $\ln \mu(x, F) \leq x \Phi(x)$ ($x \in (x_0, +\infty)$) and condition (3) is equivalent to $\ln \mu(x, F) \leq \Phi_0(1/|x|)$ ($x \in (x_0, 0)$), $x_0 < 0$, where $\Phi, \Phi_0$ are some functions from the class $L$.

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In 2000 P. V. Filevych ([1]) proved the following theorem.

**Theorem A ([1]).** Let \( \psi \in L \), condition

\[
\lim_{n \to \infty} \frac{\ln n}{\lambda_n \psi(\lambda_n)} = q < 1 \tag{4}
\]

holds and \( h \in L_0 \). Then

\[
(\forall F \in S_\psi(\Lambda))(\exists x_0)(\forall x \geq x_0): \ M(x, F) < \mu(x, F)h(\ln \mu(x, F))
\]

holds if and only if

\[
(\forall l_1, l_2 \in L)(\exists n_0)(\forall n \geq n_0): \ n < l_1(n) + h(l_2(n)\psi(\lambda_n)). \tag{5}
\]

Using the ideas of the proof of Theorem A (Theorem 2 in [1]) one can prove the following assertion.

**Proposition 1.** If \( \psi \in L \), \( h \in L_0 \) and conditions (4), (5) hold, then

\[
(\forall d \in L_0)(\forall F \in S_\psi(\Lambda))(\exists x_0)(\forall x \geq x_0): \ M(x, F) < \mu(x, F)h(xd(x)).
\]

Note that since the function \( \ln \mu(x, F) \) is convex (i.e. \( \ln \mu(x, F)/x \to +\infty \) \( x \to +\infty \)), sufficiency in Theorem A follows from Proposition 1. Proposition 1 implies also that by conditions (4), (5)

\[
(\forall F \in S_\psi(\Lambda))(\exists c > 0)(\exists x_0)(\forall x \geq x_0): \ M(x, F) < \mu(x, F)h(cx).
\]

**Question 1.** What is the exact value of the constant \( c \) in the previous inequality?

**Conjecture 1.** Sharp values are \( c = \tau + \varepsilon \), where \( \varepsilon > 0 \) is arbitrary,

\[
\tau = \lim_{t \to +\infty} \frac{h^{-1}(n(t))}{\psi(t)}, \quad n(t) = \sum_{\lambda_n \leq t} 1, \tag{6}
\]

and \( h^{-1} \) is the inverse function to function \( h \in L \).

Consider first the following problem.

**Problem 1.** Set an analogue of Theorem A or Proposition 1 for the class \( S^0(\Lambda) \).

So, we consider Dirichlet series \( F \in S^0(\Lambda) \) of the form (1). Next, we assume that conditions

\[
q := \lim_{n \to +\infty} \frac{\ln n}{\lambda_n \psi(\lambda_n)} < \theta < +\infty \tag{7}
\]

and

\[
\sup\{|a_n|: n \geq 0\} = +\infty \tag{8}
\]

hold.

**Remark 1.** It is easy to see from (7) that \( \ln n = o(\lambda_n) \) \( n \to +\infty \) and thus every Dirichlet series of form (1) has the abscissa of absolutely convergence \( \sigma_a \geq 0 \) as (3) and (7) hold. In the case of \( \sup\{|a_n|: n \geq 0\} = +\infty \) from conditions (3) and (7) we obtain that \( \sigma_a = 0 \).

The following theorem shows that direct analogues of Theorem A for the class \( S^0(\Lambda) \) can not be obtained.
**Theorem 1.** Let \( h \in L \), \( \Phi \in L \) be arbitrary functions. For all sequences \( \Lambda = (\lambda_n)_{n=0}^{\infty} \) such that \( 0 = \lambda_0 < \lambda_n < \lambda_{n+1} \) \((1 \leq n \uparrow +\infty)\) there exist some function \( F \in S^0(\Lambda) \) such that condition (8) is satisfied and

\[
\frac{M(x, F)}{\mu(x, F) h(\ln \mu(x, F))} \to +\infty \quad (x \to -0), \tag{9}
\]

\[
(\exists x_0 < 0)(\forall x \in (x_0, 0)):\quad \ln \mu(x, F) \leq \Phi(1/|x|). \tag{10}
\]

The following theorem is a certain analogue of Proposition 1 for the class \( S^0(\Lambda) \).

**Theorem 2.** If \( \psi \in L_1 \), \( h \in L_0 \) and conditions (7), (5) hold, then

\[
(\forall d \in L_0)(\forall F \in S^0_\psi(\Lambda))(\exists x_0 < 0)(\forall x \in (x_0, 0)):\quad M(x, F) < \mu(x, F) h \left( \frac{1}{|x|} \right) \left( \frac{1}{|x|} \right). \tag{11}
\]

Theorem 2 implies the following corollary.

**Corollary 1.** If \( \psi \in L_1 \), \( h \in L_0 \) and conditions (7), (5) hold, then

\[
(\forall F \in S^0_\psi(\Lambda))(\exists c > 0)(\exists x_0 < 0)(\forall x \in [x_0, 0)):\quad M(x, F) < \mu(x, F) h(c/|x|). \tag{12}
\]

**Question 2.** What is the exact value of constant \( c \) in the previous inequality?

**Conjecture 2.** Sharp values are \( c = \tau + \varepsilon \), where \( \varepsilon > 0 \) is arbitrary and \( \tau \) is defined by (6).

In the case of \( \psi(x) = x^\alpha \), \( \alpha \in (0, 1) \), \( h(x) = x^\beta \), \( \beta > 0 \), Corollary 1 implies the following statement.

**Corollary 2.** If \( F \in S^0(\Lambda) \), \( |a_n| \leq \exp\{\lambda_n^{1-\alpha}\} \) \((n \geq n_0)\) and \( \lim_{n \to +\infty} n^{1/(\alpha \beta)} \lambda_n^{-1} < +\infty \), then

\[
(\exists c > 0)(\exists x_0 < 0)(\forall x \in [x_0, 0)):\quad M(x, F) < c \mu(x, F)(1/|x|)^\beta. \tag{13}
\]

The degree of \( \beta \) in the last inequality, in general, cannot be improved. This follows from the example of functions below

**Example 1.** Let \( \lambda_n = n^q \) \((n \geq 0)\), \( \beta > 0 \), \( \alpha \in (0, 1) \), \( \alpha \cdot \beta > 1 \), \( p = 1 - \alpha \) and

\[
F_0(z) = \sum_{n=1}^{+\infty} e^{np+znq}. \tag{14}
\]

It is obvious that \( F_0 \in S^0(\Lambda) \). For the series (14) we have

\[
F_0(x) = (1 + o(1)) \sqrt{2\pi} q \sqrt{1-p} \mu(x, F_0) x^{(p-2)/2} \log^{p-2} \mu(x, F_0)|x|^{(p-2)/2} \mu(x, F_0) \mu(x, F_0) x \to -0 \tag{15}
\]

and

\[
\mu(x, F_0) = (1 + o(1)) e^{-x} \mu(x, F_0) x \to -0. \tag{16}
\]

\( i) \) Now for fixed \( \beta > 0 \) and \( \varepsilon \in (0, 1) \) we choose \( \alpha \in (0, 1) \), \( q > 0 \) such that \( \alpha = \frac{1}{1+\varepsilon} \), \( \alpha \beta q = 1 \). Then \( \frac{2-pq}{2(1-p)q} = \beta - \frac{\varepsilon}{2} \), therefore from relation (13) we have

\[
|x|^{\beta-\varepsilon} F_0(x)/\mu(x, F_0) \to +\infty \quad (x \to -0). \tag{17}
\]
We can rewrite relation (13) in the form
\[ F_0(x) = \sqrt{\frac{2\pi}{q}} (1 + o(1)) (1 - p)^{\frac{a_n - 2a - q}{2q}} e^{\frac{2a - a q}{2q}} \mu(x, F_0) \left( \frac{\log \mu(x, F_0)}{|x|} \right)^{\frac{2 - pq}{2q}}, \quad (x \to -0). \tag{14} \]

Now for fixed \( \beta > 0 \) and \( \varepsilon \in (0, \beta) \) we choose \( q = \frac{2\beta + 1}{(2\beta + 1 - \varepsilon)\beta} \) and \( \alpha \in (0, 1) \) such that \( \alpha \beta q = 1 \). Then \( \frac{2 - pq}{2q} = \frac{1}{q} \cdot \frac{2\beta + 1}{2\beta} - \frac{1}{2} = \beta - \frac{\varepsilon}{2} \) and from relation (14) we obtain
\[ F_0(x)/\mu(x, F_0) \left( \frac{\log \mu(x, F_0)}{|x|} \right)^{(\beta - \varepsilon)} \to +\infty \quad (x \to -0). \]

2. Proof of the theorems.

**Proof of Theorem 1.** The idea of the proof is the same as the idea of the proof of Theorem 1 from [2] (see, also [3, 4]). Without loss of generality we assume that \( h(1) = 1, t + \ln h(t) < 0 \) \((0 \leq t \leq 1/2)\). Let \((E_n)\) be some sequence such that \( 1 = E_1 \leq E_n < E_{n+1} \) and \( 1 \leq n/E_n < (n + 1)/E_{n+1} \) \((n \geq 1)\). For every \( n \geq 1 \) there exists \( L_n \) such that \( L_n + h(L_n) = \ln(n/E_n) \). Then \( L_1 = 1, L_n < L_{n+1} \) \((n \geq 1)\), \( L_n \to +\infty \) \((n \to +\infty)\).

Let \( c_n = L_{n-1} - L_{n-2} \) \((n \geq 3)\) and \((\lambda^*_j)\) be a subsequence of the sequence \((\lambda_n)\) such that
\[ \ln n = o(\lambda^*_n) \quad (n \to +\infty), \quad \sum_{j=3}^{+\infty} c_j \Phi^{-1}(L_{j-1}) \leq 1, \tag{15} \]
where \( \Phi^{-1} \) is the inverse function to the function \( \Phi \). We define
\[ \ln a^*_n \overset{def}{=} L_{n-1} + \lambda^*_n \sum_{j=n+1}^{+\infty} \frac{c_j}{\lambda^*_j} \quad \text{and} \quad a_n = \begin{cases} a^*_j, & \text{if} \quad \lambda_n = \lambda^*_j, \\ 0, & \text{if} \quad \lambda_n \notin \{\lambda^*_j: j \geq 1\}. \end{cases} \]

Then
\[ F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n} = \sum_{n=1}^{+\infty} a^*_n e^{z\lambda^*_n}. \]

Conditions (15) immediately yield that \( F \in S^0(\Lambda) \). Indeed, from (15) by the definition of \( L_n \)
we obtain
\[ \ln a^*_n \leq \ln n + \lambda^*_n \sum_{j=n+1}^{+\infty} \frac{c_j}{\lambda^*_j} = o(\lambda^*_n) \quad (n \to +\infty). \]

Hence, for \( x < 0 \)
\[ a^*_n \exp\{x\lambda^*_n\} = \exp\{-\frac{|x|}{2}\lambda^*_n\} \leq \exp\{-2\ln n\} \quad (n \geq n_0), \]
where \( n_0 = n_0(x) \). Therefore \( F \in S^0(\Lambda) \).

We need following lemma (see [5, p.19]).

**Lemma 1.** Let \( g \in S^0(\Lambda) \) be a Dirichlet series of the form \( g(z) = \sum_{n=0}^{+\infty} g_n e^{z\lambda_n} \). If
\[ \varkappa_n \overset{def}{=} \frac{\ln |g_{n-1}| - \ln |g_n|}{\lambda_n - \lambda_{n-1}} \overset{\nearrow}{\not\rightarrow} 0 \quad (1 \leq n \uparrow +\infty), \]
then
\[ (\forall n \geq 1)(\forall x \in [\varkappa_n, \varkappa_{n+1}]): \quad \mu(x, g) = |g_n| e^{x\lambda_n}. \]
We note that for the function $F$

$$\varphi_n = \frac{\ln a_{n-1}^* - \ln a_n^*}{\lambda_n^* - \lambda_{n-1}^*} = -\sum_{j=n}^{+\infty} \frac{c_j}{\lambda_j^*} \uparrow 0 \quad (n \to +\infty).$$

By Lemma 1 for all $x \in [\varphi_n, \varphi_{n+1}]$

$$\mu(x, F) = a_n^* \exp\{x\lambda_n^*\}.$$ Hence, for all $x \in [\varphi_n, \varphi_{n+1}]$ and $n \geq 2$

$$\ln \mu(x, F) = \ln a_n^* + x\lambda_n^* \leq L_{n-1} + \lambda_n^* \sum_{j=n+1}^{+\infty} \frac{c_j}{\lambda_j^*} + \varphi_{n+1}^* \lambda_n^* = L_{n-1}$$

(16) and for $m, 2 \leq m \leq n - 1,$

$$\ln a_m^* + x\lambda_m^* \geq L_{m-1} + \lambda_m^* \sum_{j=m+1}^{+\infty} \frac{c_j}{\lambda_j^*} + \varphi_{m+1}^* \lambda_m^* \geq L_{m-1} \geq 1.$$ From the previous inequality for all $x \in [\varphi_n, \varphi_{n+1}]$ and $n \geq 2$ we obtain

$$M(x, F) = F(x) = \sum_{k=0}^{+\infty} a_k e^{z\lambda_k} \geq \sum_{m=1}^{n} a_m^* e^{z\lambda_m^*} \geq n = E_n h(L_n) \exp\{L_n\} > E_n \mu(x, F) h(\ln \mu(x, F))$$

and (9) follows. From (15) we have $L_{n-1} \leq \Phi(1/|\varphi_n|) \quad (n \geq 2).$ Therefore, from (16) for all $x \in [\varphi_n, \varphi_{n+1}]$ and $n \geq 2$

$$\ln \mu(x, F) \leq L_{n-1} \leq \Phi(1/|\varphi_n|) \leq \Phi(1/|x|).$$

This complete the proof of Theorem 1. \hfill \square

**Proof of Theorem 2.** Assume that condition (5) is fulfilled but there exist a function $d \in L_0$ and a sequence $x_j \uparrow 0 \quad (j \to +\infty)$ such that

$$(\forall j \geq 1) : \quad M(x_j, F) \geq \mu(x_j, F) h\left(\frac{1}{|x_j|} d\left(\frac{1}{|x_j|}\right)\right).$$

(17)

For $x < 0$ we set $n_1(x) = \min\{n : \psi(\lambda_n) \geq (1 + \theta + \delta) \frac{1}{|x|}\}$, $\delta > 0$. Then

$$n_1(x) \uparrow +\infty \quad (x \uparrow 0) \quad \text{and} \quad \psi(\lambda_{n_1(x)-1}) < (1 + \theta + \delta) \frac{1}{|x|}.$$ (18)

Moreover, using (3) and definition of $n_1(x)$ we have

$$\sum_{n=n_1(x)}^{+\infty} a_n \exp\{x\lambda_n\} \leq \sum_{n=n_1(x)}^{+\infty} \exp\left\{- (\theta + \delta) \frac{\lambda_n}{\psi(\lambda_n)}\right\} = \sum_{n=n_1(x)-1}^{+\infty} \exp\left\{- (\theta + \delta) \frac{t}{\psi(t)}\right\} dn(t).$$

Condition (7) implies

$$\int_{t_0}^{+\infty} \exp\left\{- (\theta + \delta) \frac{t}{\psi(t)}\right\} dn(t) < +\infty,$$
hence by (18) we have
\[
\sum_{n=n_1(x)}^{+\infty} |a_n| \exp\{x\lambda_n\} = o(1) \quad (x \uparrow 0).
\]

The condition \(\sup\{|a_n|: n \geq 0\} = +\infty\) implies \(\mu(x, F) \uparrow +\infty \quad (x \uparrow 0)\). Thus from (19) we obtain
\[
\sum_{n=n_1(x)}^{+\infty} |a_n| \exp\{x\lambda_n\} < \frac{1}{2} \mu(x, F) \quad (x \in [\sigma_1, 0)).
\]

Since for all fixed \(m\) and \(x < 0\)
\[
\sum_{n=0}^{m} |a_n| \exp\{x\lambda_n\} \leq \sum_{n=0}^{m} |a_n|,
\]
there exists a continuous function \(n_2 = n_2(x)\) such that \(n_2(x) \nrightarrow +\infty \quad (x \uparrow 0)\) and
\[
\sum_{n=0}^{[n_2(x)]} |a_n| \exp\{x\lambda_n\} \leq \frac{1}{2} \mu(x, F) \quad (x \in [\sigma_2, 0)),
\]
where \([a]\) means the most integer such that \([a] \leq a\). Let \(\sigma_3 = \max\{\sigma_1, \sigma_2\}\). Then from the previous inequality and (20) for \(x \in [\sigma_3, 0)\) we have
\[
M(x, F) < \mu(x, F) + \sum_{n=[n_2(x)]+1}^{n_1(x)-1} |a_n| \exp\{x\lambda_n\} \leq (n_1(x) - [n_2(x)]) \mu(x, F) \leq (n_1(x) - n_2(x) + 1) \mu(x, F).
\]

Now choose a function \(l_1 \in L\) such that
\[
l_1(n_1(x) - 1) \leq n_2(x) - 2 \quad (x \in [\sigma_3, 0]).
\]
Then \(n_1(x) - n_2(x) + 1 = n_1(x) - 1 - (n_2(x) - 2) \leq n_1(x) - 1 - l_1(n_1(x) - 1)\), and for all function \(l_2 \in L\) by condition (5)
\[
M(x, F) < (n_1(x) - 1 - l_1(n_1(x) - 1)) \mu(x, F) \leq h(l_2(n_1(x) - 1) \psi(\lambda_{n_1(x)-1})) \mu(x, F) \leq h(l_2(n_1(x) - 1)(1 + \theta + \delta) / |x|) \mu(x, F), \quad (x \in [\sigma_4, 0)).
\]

Without loss of generality, we may assume that \(d(1/|x_j|) \uparrow +\infty \quad (j \rightarrow +\infty)\). Let \(n_1^*(x)\) be a continuous function such that \(n_1(x) - 1 \leq n_1^*(x) \leq n_1(x) \quad (x < 0)\). At last, we choose a function \(l_2 \in L\) such that for \(x = x_j \quad (j \geq 1)\)
\[
l_2(n_1^*(x))(1 + \theta + \delta) = d(1/|x|).
\]
Then from inequality (21) for \(x = x_j \quad (j \geq 1)\) we obtain
\[
M(x, F) < \mu(x, F) h(d(1/|x|)/|x|),
\]
and this contradicts (17).
REFERENCES

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