Corrections to the paper “Sharp estimates of the growth of the Poisson-Stieltjes integral in the polydisc” by I.E.Chyzhykov, O.A.Zolota

The proof of the sufficiency of the theorem in [1] contains a gap, because the estimate of the Poisson kernel on the set $Q_k = E_k \setminus E_{k-1}$ (p.194) does not hold. The authors do not know whether the sufficiency statement is true in the general case, but here we give the proof under the stronger assumption $0 < \beta_j < 2$, $j \in \{1, \ldots, n\}$ instead of $\frac{1}{\beta_1} + \cdots + \frac{1}{\beta_n} > \frac{1}{2}$ as it was stated in [1].

**Theorem.** Let $\mu$ be a finite Borel measure on $T^n$, $n \in \mathbb{N}$, $0 < \beta_j < 2$, $1 \leq j \leq n$. In order that
\[
\left| \int_{T^n} \mathcal{P}(z,w) \, d\mu(w) \right| \leq C \delta^{1-\frac{1}{\beta_j}}, \quad 0 < \delta < 1, \quad |z_j| = 1 - \delta^{\frac{1}{\beta_j}},
\]
hold for some positive constant $C$, it is sufficient, and for nonnegative $\mu$ it is necessary that $\mu \in H^{(\beta_1, \ldots, \beta_n)}$.

**Proof.** Sufficiency. For fixed $\delta \in (0, 1)$ we set $r_j = 1 - \delta^{\frac{1}{\beta_j}}$, and let $z_j = r_j e^{i\varphi_j}$, $1 \leq j \leq n$. For $1 \leq j \leq n$ we denote $E_k = \left\{ (e^{i\theta_1}, \ldots, e^{i\theta_n}) : |\theta_j - \varphi_j| \leq (2^k \cdot \delta)^{\frac{1}{\beta_j}}, 1 \leq j \leq n\right\}$, $0 \leq k \leq N_j$, where $N_j = \left\lceil \log_2 \frac{n^{\beta_j}}{\delta} \right\rceil + 1$, and
\[
R_{m_1 \ldots m_n} = \left\{ (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in T^n : (2^{m_j-1}\delta)^{\frac{1}{\beta_j}} \leq |\theta_j - \varphi_j| \leq (2^{m_j}\delta)^{\frac{1}{\beta_j}}, \quad \text{if } m_j > 0, \right\}
\]
\[
\left| \theta_j - \varphi_j \right| \leq \delta^{\frac{1}{\beta_j}}, \quad \text{if } m_j = 0,
\]
Then $T^n = \bigcup_{m_1=0}^{N_1} \ldots \bigcup_{m_n=0}^{N_n} R_{m_1 \ldots m_n}$. As in [2] (proof of Theorem 3.1) we have for fixed $k$ that:
\[
\left| \int_{R_{m_1 \ldots m_n}} \mathcal{P}(z,w) \, d\mu(w) \right| \leq |\mu| \left( R_{m_1 \ldots m_n} \right) \cdot \prod_{j=1}^{n} \mathcal{P}(|z|, \tilde{w}),
\]
where $z \in U^n$, $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_n)$, $\tilde{w}_j = e^{i\tilde{\theta}_j}$, $\tilde{\theta}_j = (2^{k-1} \cdot \delta)^{\frac{1}{\beta_j}}, 1 \leq j \leq n$.

The following estimates are known ([2])
\[
P_0 \left( r e^{i\varphi}, e^{i\theta} \right) \leq \frac{\pi^2}{(|\varphi - \theta|^2)} (1 - r), \quad P_0 \left( r e^{i\varphi}, e^{i\theta} \right) \leq \frac{2}{1 - r}.
\]

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Applying the first estimate, we get
\[ P_0(\|z\|, \tilde{w}_j) \leq \frac{\pi^2}{\theta_j^2} (1 - r_j) = \frac{\pi^2}{(2^{k-1} \cdot \delta)^{\frac{r_j}{2}}} (1 - r_j), k \geq 1. \]

Using the second inequality in (2), we obtain
\[ \left| \int_{R_{0\ldots0}} P(z, w) d\mu(w) \right| \leq |\mu|(R_{0\ldots0}) \cdot \frac{2^n}{(1 - r_1) \cdots (1 - r_n)}. \]

Using the definition of the class \( H^{(\beta_1, \ldots, \beta_n)} \) we have that \( |\mu|(R_{0\ldots0}) \leq C \delta \). The equalities \( 1 - r_j = \delta^{\frac{1}{\beta_j}}, 1 \leq j \leq n \) imply
\[ \left| \int_{R_{0\ldots0}} P(z, w) d\mu(w) \right| \leq C \delta^{1 - \left( \frac{1}{\beta_1} + \cdots + \frac{1}{\beta_n} \right)}. \]

Then we estimate the integral over the sets \( R_{m_1 \ldots m_n} \). Since, \( R_{m_1 \ldots m_n} \subset E_{\max\{m_1, \ldots, m_n\}} \), we get that \( |\mu|(R_{m_1 \ldots m_n}) \leq C \cdot 2^{m_1 + \cdots + m_n} \delta. \)

Therefore, for \( N = \max N_j, 1 \leq j \leq n \)
\[ \left| \sum_{m_1=1}^{N} \cdots \sum_{m_n=1}^{N} \int_{R_{m_1 \ldots m_n}} P(z, w) d\mu(w) \right| \leq \sum_{m_1=1}^{N} \cdots \sum_{m_n=1}^{N} |\mu|(R_{m_1 \ldots m_n}) \cdot \prod_{j=1}^{n} \frac{\pi^2 (1 - r_j)}{(2^{m_j - 1} \delta)^{-\frac{r_j}{2}}} \leq \]
\[ \leq C \sum_{m_1=1}^{N} \cdots \sum_{m_n=1}^{N} 2^{m_1} \cdots 2^{m_n} \cdot \delta \cdot \prod_{j=1}^{n} \left( \frac{\delta^{\frac{1}{\beta_j}}}{2^{m_j - 1} \delta^{-\frac{r_j}{2}}} \right) \leq \]
\[ \leq C \delta^{1 - \left( \frac{1}{\beta_1} + \cdots + \frac{1}{\beta_n} \right)} \sum_{m_1=1}^{N} \left( 2^{1 - \frac{2}{\beta_1}} \right)^{m_1} \cdots \sum_{m_n=1}^{N} \left( 2^{1 - \frac{2}{\beta_n}} \right)^{m_n} < C \delta^{1 - \frac{1}{\pi^2}}. \]
\[ \square \]

**REFERENCES**


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