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# ON SPACES OF FUZZY METRICS

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We consider the space of all fuzzy metrics in the sense of George and Veeramani that are compatible with the topology of a compact metrizable space. It is proved that this space of fuzzy metrics is an $\ell^2$-manifold.

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Рассматривается пространство всех нечетких метрик в смысле George и Veeramani, совместимых с топологией компактного метризуемого пространства. Доказано, что такое пространство нечетких метрик является $\ell^2$-многообразием.

**1. Introduction.** It is well-known that different function spaces are homeomorphic to some model spaces of infinite-dimensional topology. This note is devoted to the topology of the space of fuzzy metrics on a compact metrizable space (see, e.g., [1], where a space of retractions of a unit segment is considered and also a related paper [2]). The main result states that the space of fuzzy metrics on an infinite compact metrizable space is a manifold modeled on the separable Hilbert space $\ell^2$. Note that we use the notion of a fuzzy metric space in the form of George and Veeramani ([3]), which is a modification of that introduced by Kramosil and Michalek ([8]). The theory of fuzzy metric spaces is developing rapidly; in some aspects it is even more interesting and complicated that the theory of metric spaces.

**2. Result.** We start with necessary definitions from the theory of fuzzy metric spaces. See, e.g. [7] for details and examples.

A continuous operation $(a,b) \mapsto a * b\colon [0,1] \times [0,1] \to [0,1]$ is called a t-norm, if $*$ is associative, commutative, monotonic and 1 is its neutral element.

A function $M\colon X \times X \times (0,\infty) \to [0,1]$ is said to be a fuzzy metric on a set $X$, if it satisfies the following conditions: (i) $M(x,y,t) > 0$; (ii) $M(x,y,t) = 1$ if and only if $x = y$; (iii) $M(x,y,t) = M(y,x,t)$; (iv) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$; (v) the function $M(x,y,-)\colon (0,\infty) \to [0,1]$ is continuous.

The triple $(X, M, *)$ is called a fuzzy metric space ([3, 4]). If condition (iv) in the definition of a fuzzy metric $M\colon X \times X \times (0,\infty) \to [0,1]$ is replaced with the stronger condition (iv') $M(x,y,t) * M(y,z,t) \le M(x,z,t)$, then this function is called a fuzzy ultrametric.

For every $x \in X$, every $r > 0$ and $t > 0$ let

$$B(x,r,t) = \{y \in X | M(x,y,t) > 1 - r\}.$$

The sets $B(x,r,t)$ form a base of a metrizable topology on $X$.

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By $\vee$ (respectively $\wedge$) we denote the operation max (respectively min). Note that $\wedge$ is a $t$-norm. In the sequel, all considered fuzzy metrics concern the t-norm $\wedge$.

We denote by $\mathcal{FM}(X)$ the set of all fuzzy metrics on $X$ which generate its topology. The main result of this note is a description of topology of this space.

We are going to describe a special structure in the space $\mathcal{FM}(X)$ (see the definition below). Given $c_1, \dots, c_n \in [0,1]$ with $c_1 \vee \dots \vee c_n = 0$ and $M_1, \dots, M_n \in \mathcal{FM}(X)$, let $M = \bigwedge_{i=1}^n c_i \vee M_i$.

**Lemma 1.** $M \in \mathcal{FM}(X)$.

*Proof.* Let us verify the properties from the definition of a fuzzy metric.

1) For any $x, y \in X$ and $t > 0$, we have

$$M(x,y,t) = \bigwedge_{i=1}^{n} c_i \vee M_i(x,y,t) \geq \bigwedge_{i=1}^{n} M_i(x,y,t) > 0.$$

2) Without loss of generality, one may assume that $c_1 = 0$. Then, for any $x, y \in X$ and $t > 0$, we have

$$M(x,x,t) \geq c_1 \vee M_1(x,x,t) = 0 \vee 1 = 0.$$

3) Clearly, $M(x,y,t) = M(y,x,t)$.

4) Let $x, y, z \in X$ and $t, s > 0$. Then

$$M(x,y,t) \bigwedge M(y,z,s) = \left( \bigwedge_{i=1}^{n} c_i \vee M_i(x,y,t) \right) \bigwedge \left( \bigwedge_{i=1}^{n} c_i \vee M_i(y,z,s) \right) =$$

$$= \bigwedge_{i=1}^{n} (c_i \vee M_i(x,y,t)) \bigwedge (c_i \vee M_i(y,z,s)) = \bigwedge_{i=1}^{n} \left( c_i \vee \left( M_i(x,y,t) \bigwedge M_i(y,z,s) \right) \right) \leq$$

$$\leq \bigwedge_{i=1}^{n} (c_i \vee M_i(x,z,t+s)) = M(x,z,t+s).$$

5) Clearly, the function $t \mapsto M(x,y,t)$ is continuous as the minimum of finite family of continuous functions.

Therefore, $M$ is a fuzzy metric on $X$. □

Define a metric $d$ on $\mathcal{FM}(X)$ by the following formula:

$$d(M', M'') = \sup_{i \in \mathbb{N}} \min\{1/n, \sup\{|\ln M'(x,y,t) - \ln M''(x,y,t)| \\ \mid x, y \in X,\ 1/(i+1) \leq t \leq i+1\}\}.$$

A $c$-structure on a topological space $X$ is an assignment to every nonempty finite subset $A \subset X$ a contractible subspace $F(A) \subset X$ such that $F(A) \subset F(B)$ whenever $A \subset B$. A pair $(X, F)$, where $F$ is a $c$-structure on $X$ is called a $c$-space.

The following notions are introduced by C. Horvath (see [5]). A subset $E \subset X$ is called an $F$-set if $F(A) \subset E$ for any finite $A \subset E$. A metric space $(X, d)$ is said to be a metric $l.c.$-space if all open balls are $F$-sets and all open $r$-neighborhoods of $F$-sets are also $F$-sets.

We say that a metric space $(X, d)$ satisfies the Strong Discrete Approximation Property if, for any continuous map $f\colon \bigsqcup_{i=1}^{\infty} I^n \to X$ and any continuous function $\alpha\colon X \to (0, \infty)$, there exists a continuous map $g\colon \bigsqcup_{i=1}^{\infty} I^n \to X$ such that $d(f(x), g(x) < \alpha(f(x))$, for every $x \in \bigsqcup_{i=1}^{\infty} I^n$, and the family $\{f(I^n) \mid n \in \mathbb{N}\}$ is discrete.

Define an *l.c.*-structure on $\mathcal{FM}(X)$ as follows:

$$F(M_1, \dots, M_n) = \left\{ \bigwedge_{i=1}^{n} c_i \vee M_i \mid c_1, \dots, c_n \in [0,1],\ c_1 \vee \dots \vee c_n = 0 \right\},$$

for any finite subset $\{M_1, \dots, M_n\} \subset \mathcal{FM}(X)$.

**Lemma 2.** *For every $M_1, \dots, M_n \in \mathcal{FM}(X)$, the set $F(M_1, \dots, M_n)$ is contractible.*

*Proof.* Define a map $H\colon F(M_1, \dots, M_n) \times [0,1] \to F(M_1, \dots, M_n)$ by the formula

$$H\left( \bigwedge_{i=1}^{n} c_i \vee M_i, t \right) = \bigwedge_{i=1}^{n} t \vee c_i \vee M_i.$$

It is clear that $H$ is well-defined and is a homotopy contracting $F(M_1, \dots, M_n)$ to the point $M_1 \bigwedge \dots \bigvee M_n$. □

By ANR we denote the class of absolute neighborhood retracts in the class of metrizable spaces.

**Theorem 1.** *The space $\mathcal{FM}(X)$ is an ANR-space for every compact metrizable space $X$.*

*Proof.* The result follows from the existence of *l.c.*-structure on $\mathcal{FM}(X)$ and a result of C. D. Horvath ([6]). □

**Theorem 2.** *For every compact metrizable infinite space $X$, the space $\mathcal{FM}(X)$ is an $\ell^2$-manifold.*

*Proof.* We will use Toruńczyk's characterization theorem for $\ell^2$-manifolds ([9]). The theorem states that a separable nowhere locally compact ANR-space that satisfies the Strongly Discrete Approximation Property is an $\ell^2$-manifold. First we prove that the space $\mathcal{FM}(X)$ is nowhere locally compact.

Let $N$ denote the fuzzy metric on the unit segment $[0,1]$ defined by the formula $N(a,b,t) = t(t + |a-b|)^{-1}$ (see, e.g., [7]). For any $M \in \mathcal{FM}(X)$, let $M'$ denote the fuzzy metric on the set $X \times [0,1]$ defined as follows: $M'((x,a),(y,b),t) = \min\{M(x,y,t),(a,b,t)\}$. Note that then, clearly, $M'((x,0),(y,0),t) = M(x,y,t)$.

Assume, on the contrary, that $X$ is not nowhere locally compact. Then there exists $M \in \mathcal{FM}(X)$ and $i \in \mathbb{N}$ such that the closed $(1/i)$-ball of $M$ (we denote it by $\bar{B}_{1/i}(M)$) is not compact. There exists $\varepsilon > 0$ such that, for any continuous function $f\colon X \to [0,\varepsilon] \subset [0,1]$, the fuzzy metric $M_f$ on $X$ defined by the formula $M_f(x,y,t) = M'((x,f(x)),(y,f(y)),t)$ belongs to $\bar{B}_{1/i}(M)$.

Let $x_0$ be a nonisolated point of the space $X$ and $(x_i)$ a sequence in $X$ converging to $x_0$. For any $i$, pick a neighborhood $U_i$ of $x_i$ so that the sequence $(U_i)$ converges to $x_0$. For any $i$, let $f_i\colon X \to [0,\varepsilon]$ be a continuous function such that $f_i(x_i) = \varepsilon$ and $f_i|(X \setminus U_i) = 0$.

Because of assumed compactness of $\bar{B}_{1/i}(M)$, the sequence $(M_{f_i})$ contains a convergent subsequence. Without loss of generality, one may assume that the sequence $(M_{f_i})$ converges to an element $M_0$ in $\mathcal{FM}(X)$. Then

$$1 > \frac{t}{t+\varepsilon} = N(0,\varepsilon) \le \lim_{i\to\infty} M'((x_0,0),(x_i,\varepsilon),t) = \lim_{i\to\infty} M_{f_i}(x_0,x_i,t) = \lim_{i\to\infty} M_0(x_0,x_i,t) =$$
$$= M_0(x_0,x_0,t) = 1$$

and we obtain a contradiction.

The technique based on embedding in the product $X \times [0,1]$ and using the defined above fuzzy metrics $M_f$ is used in the verification of the Strong Discrete Approximation Property for the space $M \in \mathcal{FM}(X)$. Because of technical difficulties we skip this part of proof. □

An analogous result can be obtained for the spaces of fuzzy ultrametrics in the case of a zero-dimensional compact metrizable space $X$. In the above proof one should replace the unit interval $[0,1]$ with the fuzzy metric $N$ with the Cantor set endowed with the fuzzy ultrametric.

**3. Remarks.** Our methods do not work for finite fuzzy metric spaces. However, there are simple arguments for the case of a two-point fuzzy metric space. Indeed, every fuzzy metric $M$ on $\{x, y\}$ is completely determined by a nondecreasing function $t \mapsto M(x, y, t)$ and therefore the set of fuzzy metrics is homeomorphic to the set of all nondecreasing functions from $(0, \infty)$ to $(0, 1)$, which is homeomorphic to $\ell^2$. We leave as an open question the cases of finite fuzzy metric spaces as well as fuzzy metric spaces for different choices of t-norms.

A fuzzy metric is called stationary of it does not depend on $t$. An analogous question can be formulated also for the space of stationary fuzzy metrics.

Finally, we do not know how to prove the contractibility of the space of fuzzy metrics. Once we do that, the considered space is homeomorphic to the separable Hilbert space $\ell^2$.

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