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CONVERGENCE OF STOCHASTIC PROCESS WITH MARKOV SWITCHINGS

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It has been established sufficient conditions for the convergence of a multi-dimensional stochastic process in the case of dependence of the regression function on the environment, which is described by Markov switchings. It has been obtained the generator of a limiting process, which is a stochastic diffusion process in the sense of the classical definition.

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Установлены достаточные условия сходимости многомерного стохастического процесса в случае зависимости функции регрессии от внешней среды, которая описывается марковскими переключениями. Получен генератор предельного процесса, который является стохастическим диффузионным процессом в классическом определении.

1. Introduction. There is a number of papers devoted to the stochastic differential equations, in which stability and convergence to a limiting diffusion processes is considered (see e.g. [1, 2]). Most of proofs of stochastic processes conditions are based on central limit theorems in function spaces.

The papers by V. S. Korolyuk, A. F. Turbin ([3]) and A. S. Swishchuk ([4]) are dedicated to the ascertainment of stability conditions of stochastic diffusion processes with Markov switching by the small parameter method.

By developing the small parameter method we establish sufficient conditions for the convergence of stochastic differential equations solutions with Markov switching to the diffusional processes by the construction of two-dimensional Markov process generators.

2. Problem statement and designation. Let $C(u,x), u \in \mathbb{R}^d$, be a regression function. The second variable x of the regression function describes the influence of external factors that are described by a uniform ergodic Markov processes $x(t), t \geq 0$ in a measurable phase states space $(X, \mathbf{X})$. The Markov process generator is defined by the equality:

$$Q\varphi(x) = q(x) \int_X P(x,dy) \left[\varphi(y) - \varphi(x)\right], \tag{1}$$

in the Banach space $\mathbf{B}(X)$ of real bounded continuous functions $\varphi(x), x \in X$, with the norm

$$\|\varphi(x)\| = \sup_{x \in X} |\varphi(x)|,$$

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where $P(x,B), x \in X, B \in \mathbf{X}$ is the stochastic kernel ([5]), $q(x) = g^{-1}(x)$, $g(x) = E\theta_x$, θ_x is the sojourn time of the Markov process in the state x, and $q(x)$ is the intensity of sojourn time in the state $x, x \in X$.

The stationary distribution $\pi(B), B \in \mathbf{X}$, of a Markov process $x(t), t \geq 0$ is defined by the equalities

$$\pi(dx)q(x) = q\rho(dx),\ q = \int_X \pi(dx)q(x),$$

where $\rho(B), B \in \mathbf{X}$, is the stationary distribution of the embedded Markov chain $x_n = x(\tau_n), n \geq 0$, where $\tau_{n+1} = \tau_n + \theta_{n+1}, n \geq 0$, and

$$\mathbb{P}(\theta_{n+1} \leq t | x_n = x) = F_x(t) = \mathbb{P}(\theta_x \leq t).$$

For the generator Q of a Markov process $x(t), t \geq 0$, the potential is determined by the relation $R_0 = \Pi - (\Pi + Q)^{-1}$, where $\Pi\varphi(x) = \int_X \pi(dx)\varphi(x)$ is the projection on the zeros subspace of the operator $Q\colon N_Q = \{\varphi\colon Q\varphi = 0\}$ ([6]).

The continuous stochastic process by the regression functions $C(u,x)$ in an ergodic Markov environment is defined by the stochastic differential equation:

$$du^\varepsilon(t) = C(u^\varepsilon(t), x(t/\varepsilon))dt + \sigma(u^\varepsilon(t))dw(t),\ u^\varepsilon(0) = u_0, \tag{2}$$

where u is a random evolution, x is a Markov process, w is a Wiener process that depends on a time t, and ε is a small series parameter.

The average regression function is defined by the equality:

$$C(u) = \int_X \pi(dx)C(u,x).$$

2. The convergence of a stochastic process.

Theorem 1. *Let the regression function $C(u,\cdot)$ and the variation $\sigma(u)$ satisfy the following conditions*

$C1\colon C(u,\cdot) \in C^2(\mathbb{R}^d)$,

$C2\colon \sigma(u) \in C^2(\mathbb{R}^d)$.

Then the solution $u^\varepsilon(t), t \geq 0$, of equation (2) converges weakly to the limit diffusion process $\zeta(t), t \geq 0$ as $\varepsilon \to 0$, which is defined by the generator

$$\boldsymbol{L}\varphi(u) = C(u)\varphi'(u) + (1/2)\sigma^2(u)\varphi''(u), \varphi(u) \in C^4(R^d),$$

where $\sigma^2(u) := \sigma^*(u)\sigma(u)$.

Corollary 1. *The diffusion process $\zeta(t), t \geq 0$, is the solution of the stochastic differential equation*

$$d\zeta(t) = C(\zeta(t))dt + \sigma(\zeta(t))dw(t).$$

Theorem 2. *Let the regression function $C(u,\cdot)$ and the variation $\sigma(u)$ satisfy the following conditions*

$C1\colon C(u,\cdot) \in C^2(\mathbb{R}^d)$,

$C2^*\colon \sigma(u,\cdot) \in C^2(\mathbb{R}^d)$.

Then the solution $u^\varepsilon(t), t \geq 0$, of the equation

$$du^\varepsilon(t) = C(u^\varepsilon(t), x(t/\varepsilon))dt + \sigma(u^\varepsilon(t), x(t))dw(t),\ \ u^\varepsilon(0) = u_0, \tag{3}$$

converges weakly to the limit diffusion process $\zeta(t), t \geq 0$ as $\varepsilon \to 0$, which is defined by the generator

$$\boldsymbol{L}\varphi(u) = C(u)\varphi'(u) + (1/2)\sigma^2(u)\varphi''(u), \varphi(u) \in C^4(\mathbb{R}^d),$$

where

$$\sigma^2(u) := \int_X \pi(dx)\sigma^2(u, x), \sigma^2(u, x) = \sigma^*(u, x)\sigma(u, x). \tag{4}$$

First, to prove Theorem 1 we construct the stochastic process generator and exactly its asymptotic representation.

3. Properties of the procedure generator.

Lemma 1 ([7])**.** *The coupled Markov process generator*

$$u^\varepsilon(t), x_t^\varepsilon = x(t/\varepsilon), t \geq 0, \tag{5}$$

on the Banach space $\boldsymbol{B}(\mathbb{R}^d, X)$ of real-valued functions $\varphi(u, x) \in C^{2,0}(\mathbb{R}^d, X)$ is represented as follows

$$\boldsymbol{L}^\varepsilon \varphi^\varepsilon(u, x) = \varepsilon^{-1} Q\varphi(u, x) + \boldsymbol{L}(x)\varphi(u, x),$$

where

$$\boldsymbol{L}(x)\varphi(u, x) = [\boldsymbol{C}(x) + \boldsymbol{S}]\varphi(u, x), \tag{6}$$

$\boldsymbol{C}(x)\varphi(u, x) = C(u, x)\varphi'(u, x),$
$\boldsymbol{S}\varphi(u, x) = (1/2)\sigma^2(u)\varphi''(u, x).$

Proof. The Markov process generator on a test-functions $\varphi(u, x)$ is defined by the equality

$$\mathbf{L}_t^\varepsilon \varphi(u, x) = \lim_{\Delta \to 0} \frac{1}{\Delta}[E[\varphi(u^\varepsilon(t + \Delta), x_{t+\Delta}^\varepsilon)|u^\varepsilon(t) = u, x_t^\varepsilon = x] - \varphi(u, x)]. \tag{7}$$

Let's find the conditional expectation

$$E[\varphi(u^\varepsilon(t + \Delta), x_{t+\Delta}^\varepsilon)|u^\varepsilon(t) = u, x_t^\varepsilon = x] = E\varphi(u^\varepsilon(t + \Delta), x_{t+\Delta}^\varepsilon) = E\varphi(u + \Delta u, x_{t+\Delta}^\varepsilon).$$

First step is to integrate stochastic differential equation (2)

$$u^\varepsilon(t) = u^\varepsilon(0) + \int_0^t C(u^\varepsilon(s), x_s^\varepsilon)ds + \int_0^t \sigma^2(u^\varepsilon(s))dw(s).$$

Now calculate Δu as difference $u^\varepsilon(t + \Delta)$ and $u^\varepsilon(t)$

$$\Delta u = \int_t^{t+\Delta} C(u^\varepsilon(s), x_s^\varepsilon)ds + \int_t^{t+\Delta} \sigma^\varepsilon(u^\varepsilon(s))dw(s).$$

Let us denote $\mu_\Delta := \int_t^{t+\Delta} \sigma(u^\varepsilon(s))dw(s)$.

We obtain the following form of the conditional expectation

$$E[\varphi(u^\varepsilon(t+\Delta), x^\varepsilon_{t+\Delta})|u^\varepsilon(t)=u, x^\varepsilon_t = x] =$$

$$= E\varphi(u^\varepsilon(t+\Delta), x^\varepsilon_{t+\Delta}) =$$

$$= E[\varphi(u+\Delta u, x)]I(\theta_x > \varepsilon^{-1}\Delta) + E[\varphi(u+\Delta u, x^\varepsilon_{t+\Delta})]I(\theta_x \le \varepsilon^{-1}\Delta) + o(\Delta).$$

The distribution function on the sojorn time θ_x has the exponential distribution, i.e. there are representations

$$I(\theta_x > \varepsilon^{-1}\Delta) = e^{-\varepsilon^{-1}q(x)\Delta} = 1 - \varepsilon^{-1}q(x)\Delta + o(\Delta),$$

and

$$I(\theta_x \le \varepsilon^{-1}\Delta) = 1 - e^{-\varepsilon^{-1}q(x)\Delta} = \varepsilon^{-1}q(x)\Delta + o(\Delta).$$

Hence, we have

$$E\varphi(u^\varepsilon(t+\Delta), x^\varepsilon_{t+\Delta}) =$$

$$= E[\varphi(u+\Delta u, x)] + \varepsilon^{-1}q(x)\big\{E[\varphi(u+\Delta u, x^\varepsilon_{t+\Delta})] - E[\varphi(u+\Delta u, x)]\big\}\Delta + o(\Delta).$$

Now rewrite $\varepsilon^{-1}q(x)E[\varphi(u+\Delta u, x^\varepsilon_{t+\Delta})]\Delta$ using the Taylor formula

$$\varepsilon^{-1}q(x)E[\varphi(u+\Delta u, x^\varepsilon_{t+\Delta})]\Delta = \varepsilon^{-1}q(x)E[\varphi(u, x^\varepsilon_{t+\Delta}) + \varphi'(u, x^\varepsilon_{t+\Delta})\Delta u + o(\Delta)]\Delta =$$

$$= \varepsilon^{-1}q(x)E[\varphi(u, x^\varepsilon_{t+\Delta})]\Delta + \varepsilon^{-1}q(x)E[\varphi'(u, x^\varepsilon_{t+\Delta})\Delta u]\Delta + o(\Delta).$$

According to the Wiener process conditional expectation properties, which can be written as follows [8, Chapter 1, §3 p.42] $E\mu_\Delta = 0$, $E\mu_\Delta\mu_\Delta = \int_t^{t+\Delta}\sigma^2(u(s))ds = \sigma^2(u(t))\Delta$, we obtain

$$\varepsilon^{-1}q(x)E[\varphi'(u, x^\varepsilon_{t+\Delta})\Delta u]\Delta =$$

$$= \varepsilon^{-1}q(x)E[\varphi'(u, x^\varepsilon_{t+\Delta})C(u^\varepsilon(t), x^\varepsilon_t)\Delta]\Delta + \varepsilon^{-1}q(x)E[\varphi'(u, x^\varepsilon_{t+\Delta})\mu_\Delta]\Delta =$$

$$= \varepsilon^{-1}q(x)E[\varphi'(u, x^\varepsilon_{t+\Delta})C(u^\varepsilon(t), x^\varepsilon_t)]\Delta^2 + \varepsilon^{-1}q(x)E\mu_\Delta E[\varphi'(u, x^\varepsilon_{t+\Delta})]\Delta = o(\Delta).$$

Thus

$$E\varphi(u^\varepsilon(t+\Delta), x^\varepsilon_{t+\Delta}) = E[\varphi(u+\Delta u, x)] + \varepsilon^{-1}q(x)\big\{E[\varphi(u, x^\varepsilon_{t+\Delta})] - E[\varphi(u+\Delta u, x)]\big\}\Delta + o(\Delta).$$

Using the Taylor formula, we have

$$E\varphi(u^\varepsilon(t+\Delta), x^\varepsilon_{t+\Delta}) =$$

$$= E[\varphi(u+\Delta u, x)] + \varepsilon^{-1}q(x)\big\{E[\varphi(u, x^\varepsilon_{t+\Delta})] - E[\varphi(u, x) + \varphi'(u, x)\Delta]\big\}\Delta + o(\Delta).$$

Substituting $x^\varepsilon_{t+\Delta}$ for y we obtain the Markov process generator (1). So

$$E\varphi(u^\varepsilon(t+\Delta), x^\varepsilon_{t+\Delta}) = E[\varphi(u + C(u^\varepsilon(t), x^\varepsilon_t)\Delta + \mu_\Delta, x)] + \varepsilon^{-1}Q\varphi(u, x)\Delta + o(\Delta).$$

Let us add and subtract the expression $\varphi(u + C(u^\varepsilon(t), x^\varepsilon_t)\Delta, x)$ in the expectation and for convenience substitute $u + C(u^\varepsilon(t), x^\varepsilon_t)\Delta$, for z. So we get

$$E\varphi(u^\varepsilon(t+\Delta), x^\varepsilon_{t+\Delta}) = E[\varphi(z+\mu_\Delta, x) - \varphi(z, x) + \varphi(z, x)] + \varepsilon^{-1}Q\varphi(u, x)\Delta + o(\Delta) =$$

$$E[\varphi'(z, x)\mu_\Delta + (1/2)\varphi''(z, x)\mu_\Delta^2 + o(\mu_\Delta^2) + \varphi(z, x)] + \varepsilon^{-1}Q\varphi(u, x)\Delta + o(\Delta).$$

By using conditional expectation Wiener process properties, the expectation takes the form

$$E\varphi(u^\varepsilon(t+\Delta), x^\varepsilon_{t+\Delta}) = E[\varphi(z,x) + o(\mu^2_\Delta)] + (1/2)\sigma^2(u)\varphi''(z,x)\Delta + \varepsilon^{-1}Q\varphi(u,x)\Delta + o(\Delta).$$

From (7) we obtain the coupled Markov process generator

$$\mathbf{L}^\varepsilon_t\varphi(u,x) = \lim_{\Delta\to 0}\frac{1}{\Delta}\big\{E[\varphi(u + C(u^\varepsilon(t), x^\varepsilon_t)\Delta, x) - \varphi(u,x) + o(\mu^2_\Delta)] + (1/2)\sigma^2(u)\varphi''(z,x)\Delta +$$

$$+\varepsilon^{-1}Q\varphi(u,x)\Delta + o(\Delta)\big\} =$$

$$= \varepsilon^{-1}Q\varphi(u,x) + C(u,x)\varphi'(u,x) + (1/2)\sigma^2(u)\varphi''(u,x).$$

So we obtain the statement of Lemma 1. □

Lemma 2. *The limit generator* $\mathbf{L}^\varepsilon$ *on the test-functions* $\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x), \varphi(u) \in C^4(R^d)$ *is represented as*

$$\mathbf{L}^\varepsilon\varphi^\varepsilon(u,x) = \boldsymbol{L}\varphi(u) + \varepsilon\theta(x)\varphi(u), \tag{8}$$

where $\|\theta(x)\varphi(u)\| < M, M < \infty$.

Proof. The generator $\boldsymbol{L}^\varepsilon_t$ on the test-functions $\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x)$ is presentet as

$$\boldsymbol{L}^\varepsilon\varphi^\varepsilon(u,x) = \varepsilon^{-1}Q\varphi^\varepsilon(u,x) + \mathbf{L}(x)\varphi^\varepsilon(u,x) =$$

$$= \varepsilon^{-1}Q\varphi(u) + Q\varphi_1(u,x) + \mathbf{L}(x)\varphi(u) + \varepsilon\mathbf{L}(x)\varphi_1(u,x).$$

From $\varphi(u) \in N_Q$ it follows that $Q\varphi(u) = 0$.

Considering the singular perturbation problem solution we obtain the equality

$$Q\varphi_1(u,x) + \mathbf{L}(x)\varphi(u) = \mathbf{L}\varphi(u), \tag{9}$$

where limit generator $\mathbf{L}\varphi(u)$ takes the form ([5, p.143])

$$\mathbf{L}\varphi(u) = \int_X \pi(dx)\mathbf{L}(x)\varphi(u). \tag{10}$$

Using (9) we have $Q\varphi_1(u,x) = [\mathbf{L} - \mathbf{L}(x)]\varphi(u)$. Let's denote $\tilde{\mathbf{L}}(x) = \mathbf{L}(x) - \mathbf{L}$. Thus $\varphi_1(u,x) = R_0\tilde{\mathbf{L}}(x)\varphi(u)$.

Let us consider the expression $\mathbf{L}(x)\varphi_1(u,x)$. Using the structure of $\varphi_1(u,x)$ we have

$$\mathbf{L}(x)\varphi_1(u,x) = \mathbf{L}(x)R_0\tilde{\mathbf{L}}(x)\varphi(u) = \theta(x)\varphi(u),$$

where $\theta(x) = \mathbf{L}(x)R_0\tilde{\mathbf{L}}(x)$.

From the properties of function $\varphi(u)$ and conditions $C1$ and $C2$ we obtain, that

$$\|\theta(x)\varphi(u)\| < M, \quad M < \infty.$$

□

4. Proof of the theorems. Using formula (8) and the pattern theorem [5, Chapter 6, p.197] we get the statement of Theorem 1.

The proof of Theorem 2 is realized according to the proof scheme of Theorem 1 taking into account the expression for the generator $\mathbf{L}(x)$ in formula (6)

$$\mathbf{L}(x)\varphi(u,x) = [\mathbf{C}(x) + \mathbf{S}(x]\varphi(u,x),$$

where $\mathbf{S}(x) = (1/2)\sigma^2(u,x)\varphi''(u,x)$, and the representation of the limit generator $\mathbf{L}$ in (10)

$$\mathbf{L}\varphi(u) = \mathbf{C}(x)\varphi'(u) + (1/2)\sigma^2(u)\varphi''(u),$$

where $\sigma^2(u)$ is defined by equality (4).

Conclusion. These results can be applied to solve problems of large deviation and asymptotically small diffusion ([8]), using a small series parameter.

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