УДК 515.12

L. I. Karchevska, T. M. Radul

TRIVIAL I^τ-FIBRATIONS OF THE MULTIPLICATION MAPS FOR MONADS $\mathbb{O}$, $\mathbb{OH}$ AND $\mathbb{OS}$

L. I. Karchevska, T. M. Radul. *Trivial I^τ-fibrations of the multiplication maps for monads $\mathbb{O}$, $\mathbb{OH}$ and $\mathbb{OS}$*, Mat. Stud. **37** (2012), 193–202.

In this paper we investigate when the multiplication maps of monads $\mathbb{O}$, $\mathbb{OH}$ and $\mathbb{OS}$ are trivial fibrations with fibers homeomorphic to a Tychonov cube or a Hilbert cube.

Л. И. Карчевская, Т. Н. Радул. *Тривиальные I^τ-расслоения отображений умножения для монад $\mathbb{O}$, $\mathbb{OH}$ и $\mathbb{OS}$* // Мат. Студії. – 2012. – Т.37, №2. – C.193–202.

В данной статье мы рассматриваем условия, при которых отображения умножения для монад $\mathbb{O}$, $\mathbb{OH}$ и $\mathbb{OS}$ являются тривиальными расслоениями со слоями гомеоморфными тихоновскому или гильбертовому кубу.

1. Introduction. Geometric properties of various functors have been studied extensively over the past few decades ([8]). Researches concern studying the question of how functors affect properties of spaces and maps between them as well as the investigation of properties of maps involved in the structures generated by functors (i.e. monad multiplication maps, structural mappings of algebras).

This research concerns the monads $\mathbb{O}$, $\mathbb{OH}$, $\mathbb{OS}$ generated by functors of order-preserving, positively homogeneous and semiadditive functionals respectively and is to answer the question when multiplication maps for these monads are trivial fibrations with fibers homeomorphic to the Tychonov cube.

Results of M. Zarichnyi on the inclusion hyperspaces monad (see [18]), for instance, show that for a continuum X the multiplication map $\mu_{\mathsf{G}}X$ for this monad is homeomorphic to the projection map $pr_{\mathsf{G}(X)}\colon I^\tau \times \mathsf{G}(X) \to \mathsf{G}(X)$ iff X is openly generated and χ-homogeneous. In this research we obtain a similar condition for multiplication maps of monads $\mathbb{O}$ and $\mathbb{OH}$ (X is not necessarily connected in our case).

2. Definitions and facts. In this section we shall recall some necessary definitions and results from infinite-dimensional topology as well as define the objects of our investigation — monads of order-preserving and positively homogeneous functionals and name some of their properties.

Since in what follows we will deal with endofunctors in the category Comp, we assume all spaces to be compact Hausdorff (briefly compacta) and maps to be continuous, unless explicitly stated otherwise or the property should be established.

By $w(X)$ we denote the weight of a space X, and by $\chi(x, X)$ the character at a point $x \in X$. We call X *χ-homogeneous* if for every $x, y \in X$ we have $\chi(x, X) = \chi(y, X)$.

2010 *Mathematics Subject Classification:* 54B30, 57N20.
Keywords: Trivial fibration, Tychonov cube, monad.

We say that a subspace X of a space Y is a *retract* of Y if there exists a mapping $r\colon Y \to X$ with $r|_X = \mathrm{id}_X$. A space X is called an *absolute retract* (briefly an AR), if for every embedding $i\colon X \hookrightarrow Y$ the subspace $i(X)$ is a retract of Y.

Recall that a *τ-system*, where τ is any cardinal number, is a continuous inverse system consisting of compacta of weight $\le \tau$ and epimorphisms over a τ-complete indexing set. As usual, ω stands for the countable cardinal number. A compactum X is called *openly generated*, if it can be represented as the limit of some ω-system with open bonding mappings ([15]).

For a compact Hausdorff space X by $C(X)$ we denote the Banach space of all continuous real-valued functions on X with the sup-norm $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}$. By c_X, where $c \in \mathbb{R}$, we denote the constant function: $c_X(x) = c$ for all $x \in X$.

Let $\nu\colon C(X) \to \mathbb{R}$ be a functional. We say that ν is:

- *normed*, if $\nu(1_X) = 1$;
- *weakly additive*, if for any $\phi \in C(X)$ and $c \in \mathbb{R}$ we have $\nu(\phi + c_X) = \nu(\phi) + c$;
- *order-preserving*, whenever for any $\varphi, \psi \in C(X)$ such that $\varphi(x) \le \psi(x)$ for all $x \in X$ (i.e. $\varphi \le \psi$) the inequality $\nu(\varphi) \le \nu(\psi)$ holds;
- *positively homogeneous*, if for any $\varphi \in C(X)$ and any real $t \ge 0$ we have $\nu(t\varphi) = t\nu(\varphi)$;
- *semiadditive*, if $\nu(\varphi + \psi) \le \nu(\varphi) + \nu(\psi)$.

Now for any space X denote $\mathsf{V}X = \prod_{\varphi \in C(X)} [\min \varphi, \max \varphi]$. For any mapping $f\colon X \to Y$ let $\mathsf{V}(f)$ be a mapping such that $\mathsf{V}(f)(\nu)(\varphi) = \nu(\varphi \circ f)$ for any $\nu \in \mathsf{V}X,\ \varphi \in C(Y)$. Defined in that way, V forms a covariant functor in the category *Comp*.

Let $X \in \mathit{Comp}$ be an arbitrary space. We define the following sets of functionals:

1. $\mathsf{O}X$ is the set of functionals satisfying 1)–3) (order-preserving functionals),
2. $\mathsf{OH}X$ is the set of all functionals on $C(X)$ which satisfy properties 1)–4) (positively homogenous functionals),
3. $\mathsf{OS}X$ is the set of functionals on $C(X)$ which satisfy properties 1)–5) (semiadditive functionals),
4. $\mathsf{P}X$ stands for the set of all functionals on $C(X)$ which are normed ($\|\mu\| = 1$), positive ($\mu(\varphi) \ge 0$ for all $\varphi \ge 0$) and linear.

Let F stand for one of $\mathsf{O}, \mathsf{OH}, \mathsf{OS}, \mathsf{P}$. The space $\mathsf{F}(X)$ is considered as a subspace of $\mathsf{V}(X)$. For any function $f\colon X \to Y$, the map $\mathsf{F}(f)\colon \mathsf{F}X \to \mathsf{F}Y$ is the restriction of $\mathsf{V}(f)$ on the corresponding space $\mathsf{F}X$. Then F forms a covariant functor in *Comp*, which is a subfunctor of V.

Let us note that the defined functors form a chain

$$\mathsf{P} \subset \mathsf{OS} \subset \mathsf{OH} \subset \mathsf{O} \subset \mathsf{V}.$$

A *monad* in the category *Comp* is a triple $\mathsf{F} = (\mathsf{F}, \eta, \mu)$, where $\eta\colon Id_{Comp} \to \mathsf{F}$ and $\mu\colon \mathsf{F}^2 \to \mathsf{F}$ are natural transformations such that the following equalities hold: 1) $\mu X \circ \eta \mathsf{F}X = \mu X \circ \mathsf{F}(\eta X) = \mathrm{id}_{\mathsf{F}X}$; 2) $\mu X \circ \mu \mathsf{F}(X) = \mu X \circ \mathsf{F}(\mu X)$ ([5]).

The abovementioned functors generate monads. If F is one of V, O, OH, OS, P, the identity and multiplication maps are defined as follows. The natural transformation $\eta\colon Id_{Comp} \to \mathsf{F}$ is given by $\eta X(x)(\varphi) = \varphi(x)$ for any $x \in X$ and $\varphi \in C(X)$, and the natural transformation $\mu\colon \mathsf{F}^2 \to \mathsf{F}$ given by $\mu X(\nu)(\varphi) = \nu(\pi_\varphi)$, where $\pi_\varphi\colon \mathsf{F}X \to \mathbb{R}$, $\pi_\varphi(\lambda) = \lambda(\varphi)$.

Results on categorical and topological properties of functors O, OH and OS can be found in [4], [9], [11]–[13].

Recall that a map $f\colon X\to Y$ is called *soft* if for any space Z and its closed subset A, any functions $\psi\colon A\to X$, $\Psi\colon Z\to Y$ with $\Psi|A=f\circ\psi$ there's mapping $G\colon Z\to X$ such that $G|_A=\psi$ and $\Psi=f\circ G$.

In particular, we shall use the following statements about functors O and OH:

Theorem 1 ([9], [11]). *Functors O and OH preserve open maps.*

Theorem 2 ([9], [11], [13]). *For a functor $\mathsf{F}\in\{\mathsf{O},\mathsf{OH}\}$ and a compact space X the following conditions are equivalent:*

- *X is openly generated;*
- *the space $\mathsf{F}X$ is an AR;*
- *the multiplication map $\mu_{\mathsf{F}}X\colon \mathsf{F}^2X\to\mathsf{F}X$ is soft.*

Theorem 3 ([12]). *Let $f\colon X\to Y$ be open. $\mathsf{O}(f)$ has a degenerate fiber if and only if f has.*

Theorem 4 ([12]). *An openly generated compactum X is χ-homogeneous if and only if $\mathsf{O}X$ is.*

We also note that the analogous to theorems 3 and 4 statements hold in the case of functor OH and their proofs are just the same as in case of O.

Finally, let us recall the definition of an I^τ-fibration and the criteria of an I^τ-fibration.

A map $f\colon X\to Y$ is called an *I^τ-fibration* if it is homeomorphic to the projection map $p_Y\colon I^\tau\times Y\to Y$. Note that a map with all fibers homeomorphic to I^τ is not necessarily an I^τ-fibration (see [2], [17] for counterexamples).

The following theorem is the well-known Torunczyk-West criterion of a Q-fibration (by Q we denote the Hilbert cube $[0,1]^\omega$):

Theorem 5 ([17]). *A soft mapping $f\colon X\to Y$ of metric AR-compacta is homeomorphic to Q-fibration if and only if it satisfies the condition of disjoint approximation: for any $\varepsilon>0$ there are mappings $g_1,g_2\colon X\to X$ such that $g_1(X)\cap g_2(X)=\varnothing$, $d(g_i,id_X)<\varepsilon$, $f\circ g_i=f$.*

In the case of noncompact spaces X and Y we shall use the following fact.

Theorem 6 ([6]). *Let $f\colon X\to Y$ be a soft and perfect map between locally compact ANR-spaces. If for any cover $\mathcal{U}\in\operatorname{cov}(X)$ there exist $\mathcal{U}$-close to id_X mappings $f_1,f_2\colon X\to X$ such that $f_1(X)\cap f_2(X)=\varnothing$ and $f\circ f_i=f$, $i=\overline{1,2}$, then f is a trivial Q-fibration.*

In the case of an arbitrary τ, the criterion of an I^τ-fibration contains a generalization of the condition of the Torunczyk-West theorem.

Let us give necessary definitions first.

A map $p\colon X\to Y$ is said to have the *property of disjoint τ-approximation* if for any family $\mathcal{F}$ of functionally open covers of X with $|\mathcal{F}|<\tau$ there are two maps $f_1,f_2\colon X\to X$ such that

- $f_1(X)\cap f_2(X)=\varnothing$,
- $p\circ f_i=f_i$ for every $i\in\{1,2\}$,
- each map f_i, $i\in\{1,2\}$, is $\mathcal{U}$-near to the identity map id_X for each open cover $\mathcal{U}\in\mathcal{F}$.

Theorem 7 ([2])**.** *A soft mapping $f\colon X\to Y$ between AR-compacta with fibers of weight $\le\tau$ is a trivial I^τ-fibration if and only if f satisfies the condition of disjoint λ-approximation for any $\lambda<\tau$.*

The following statement provides a sufficient condition under which a map satisfies the condition of disjoint λ-approximation.

Lemma 1 ([14])**.** *Let $f\colon X\to Y$ be the limit projection p_1 of a λ-spectrum $\{X_\alpha,p_\alpha,\mathcal{A}\}$ such that the index set $\mathcal{A}$ has the least element 1, all limit projections allow two disjoint sections. Then f satisfies the condition of disjoint λ-approximation.*

3. Trivial fibrations of mappings μ_{O} and μ_{OH}. For the sake of convenience, we consider the cases $\tau=\omega$ and $\tau>\omega$ separately. Let us first consider the case of the countable τ.

Define a metric on the space $\mathsf{O}X$ and its subfunctors for any metrizable compactum X in the following way. In the case when X is metrizable, the space $C(X)$ of all continuous functions on X is separable. Choose any dense in $C(X)$ countable set $\{\varphi_i\}_{i\in\mathbb{N}}$. We can assume that the function 0_X is not in $\{\varphi_i\}_{i\in\mathbb{N}}$. Put $d_{\mathsf{O}}(\lambda,\nu)=\sum_{i=1}^{\infty}\frac{|\lambda(\varphi_i)-\nu(\varphi_i)|}{\|\varphi_i\|\cdot 2^i}$. Then d_O is an admissible metric on $\mathsf{O}X$. Indeed, take any $B_\varepsilon(\nu)=\{\lambda\in\mathsf{O}X|d_{\mathsf{O}}(\lambda,\nu)<\varepsilon\}$. Choose a number $n_0\in\mathbb{N}$ such that the inequality $\sum_{i=n_0}^{\infty}\frac{1}{2^{i-1}}<\frac{\varepsilon}{2}$ holds. Then $O\Big(\nu;\varphi_1,\dots,\varphi_{n_0};\frac{\varepsilon}{2}\cdot\Big(\frac{1}{\sum_{i=n_0}^{\infty}\frac{1}{\|\varphi_i\|\cdot 2^i}}\Big)\Big)\subset B_\varepsilon$. Hence, d_{O} generates a topology on $\mathsf{O}X$.

Before coming to the proof of the theorem let us recall how to extend a positively homogeneous functional on a single function (see [4]).

Suppose that a set $A\subset C(X)$ is such that $0_X\in A$, $t\varphi+c_X\in A$ for any $\varphi\in A$, $t>0$ and $c\in\mathbb{R}$. Consider any positively homogeneous functional ν on A and some function $\psi\in C(X)\backslash A$. If we want to extend ν to the space $A\cup\{t\psi+c_X|c\in\mathbb{R},t>0\}$, the only possible values $\nu(\psi)$ are in the segment $[\sup\{\nu(\varphi)|\varphi\in A,\varphi\le\psi\},\ \inf\{\nu(\varphi)|\varphi\in A,\varphi\ge\psi\}]$.

Theorem 8. *The mapping $\mu_{\mathsf{OH}}X$ is a Q-fibration for any metrizable space X which contains more than one point.*

Proof. Assume X is metrizable and not one-point. To prove our theorem, we shall use the Torunczyk-West criterion (Theorem 5). It means that for any $\varepsilon>0$ we have to find two mappings $g_1,g_2\colon\mathsf{OH}^2X\to\mathsf{OH}^2X$ which are both ε-close to $id_{\mathsf{OH}^2(X)}$ and preserve the fibers of $\mu_{\mathsf{OH}}X$.

Choose some dense in $C(\mathsf{OH}(X))$ countable set $D=\{\Phi_i\}_{i\in\mathbb{N}}$. Since $\mathsf{OH}X$ is connected and not one-point, we may assume that for any function $\Phi_i\in D$ the sets $\Phi_i^{-1}(\max\Phi_i)$ and $\Phi_i^{-1}(\min\Phi_i)$ are infinite and disjoint. For the rest of the proof we let the metric d_{OH^2X} to be as in the beginning of the section and use the dense subset D of $C(\mathsf{OH}X)$.

Fix any $\varepsilon>0$. There exists some $n_0\in\mathbb{N}$ such that

$$\sum_{i=n_0}^{\infty}\frac{|\Lambda(\Phi_i)-M(\Phi_i)|}{\|\Phi_i\|\cdot 2^i}<\frac{\varepsilon}{2}$$

for any $\Lambda,M\in\mathsf{OH}^2X$.

Now for any function $\Phi_i,\ i=\overline{1,n_0}$ pick two points $s_i,v_i\in\mathsf{OH}(X)$ such that $\Phi_i(s_i)=\max\{\Phi_i(x)|x\in\mathsf{OH}(X)\}$ and $\Phi_i(v_i)=\min\{\Phi_i(x)|x\in\mathsf{OH}(X)\}$. Denote $S=\{s_i|i=\overline{1,n_0}\},\ I=\{v_i|i=\overline{1,n_0}\}$. Due to the choice of the set D we may assume that the sets S

and I are disjoint, and also that $\inf \mathsf{OH}(X) \notin S$, $\sup \mathsf{OH}(X) \notin I$. Here for any set $A \subset \mathsf{OH}X$ by $\inf A$ we denote the functional given by the condition $\inf A(\varphi) = \{\inf \nu(\varphi) | \nu \in A\}$ for $\varphi \in C(X)$. The functional $\inf A$ is defined similarly.

Choose a function $\Phi_0 \colon \mathsf{OH}(X) \to \mathbb{R}$ such that $\Phi_0(I \cup \inf \mathsf{OH}(X)) \subset \{1\}$ and $\Phi_0(S \cup \sup \mathsf{OH}(X)) \subset \{0\}$. Denote $Y = \{\pi_\varphi \mid \varphi \in C(X)\} \cup \{t\tilde{\Phi}_i + c_{\mathsf{OH}(X)} | \ c \in \mathbb{R}, t > 0, i = \overline{1, n_0}\}$. Take any functional $M \in \mathsf{OH}^2(X)$. Let $M_0 = \{\Lambda \in \mathsf{OH}^2(X) \mid \Lambda|_Y = M|_Y,\ \Lambda(\Phi_0) = 0\}$, and $M_1 = \{\Lambda \in \mathsf{OH}^2(X) \mid \Lambda|_Y = M|_Y,\ \Lambda(\Phi_0) = 1\}$.

Due to the choice of function Φ_0, we have that $M_0 \neq \varnothing$, $M_1 \neq \varnothing$. Indeed, for the first thing note that $\Phi_0 \notin Y$. Secondly, if we take any function $\Phi \in Y$ such that $\Phi \geq t\Phi_0$ ($\Phi \leq t\Phi_0$), where $t > 0$, then the inequality $M(\Phi) \geq t$ ($M(\Phi) \leq 0$ respectively) holds. Hence, if we define a functional $\Lambda \colon Y \cup \{t\Phi_0 + c_{OHX} | t > 0,\ c \in \mathbb{R}\} \to \mathbb{R}$ by the conditions $\Lambda|_Y = M|_Y$, $\Lambda(t\Phi_0 + c_{\mathsf{OH}X}) = ta + c$, where $a \in [0, 1]$, then we obtain a positively homogeneous functional which can be extended to the whole space $C(\mathsf{OH}X)$.

Let us show that the mappings $G_0, G_1 \colon \mathsf{OH}^2X \to \exp \mathsf{OH}^2X$ defined by $G_0(M) = M_0$, $G_1(M) = M_1$ are continuous. Indeed, take any sequence $\{M_n\}_{n\in\mathbb{N}} \subset \mathsf{OH}^2X$ that converges to some $M \in \mathsf{OH}^2(X)$. We may assume that there exists $A = \lim_{n\to\infty}(M_n)_0$. We must show that the equality $M_0 = A$ holds. The inclusion $A \subset M_0$ is obvious. Let us show that the inclusion $M_0 \subset A$ takes place. Assuming the opposite, we get that there are $\Lambda \in M_0$ and some function $\Phi \in C(\mathsf{OH}(X))$ such that $\Lambda(\Phi) = a > \sup A(\Phi)$ or $\Lambda(\Phi) = a < \inf A(\Phi)$ (this follows from the fact that all $(M_n)_0$ are OH-convex, i.e. for any $V \in \mathsf{OH}^2X$ with $\inf(M_n)_0 \leq V \leq \sup(M_n)_0$ we have $V \in (M_n)_0$, hence so is their limit. Suppose the first case holds. Note that, since $\mu_{\mathsf{OH}}X$ is open, the sequence $\{\mu_{\mathsf{OH}}X^{-1}(\mu_{\mathsf{OH}}X(M_n))\}$ converges to $\mu_{\mathsf{OH}}X^{-1}(\mu_{\mathsf{OH}}X(M))$. Hence, $\sup\{M_n(\pi_\varphi) \mid \pi_\varphi \leq \Phi,\ \varphi \in C(X)\}$ and $\inf\{M_n(\pi_\varphi) \mid \pi_\varphi \geq \Phi,\ \varphi \in C(X)\}$ must converge to $\sup\{M(\pi_\varphi) \mid \pi_\varphi \leq \Phi,\ \varphi \in C(X)\}$ and $\inf\{M(\pi_\varphi) \mid \pi_\varphi \geq \Phi,\ \varphi \in C(X)\}$ respectively. Indeed, consider any convergent subsequence $\{\sup\{M_{n_k}(\pi_\varphi) \mid \pi_\varphi \leq \Phi,\ \varphi \in C(X)\}\}_{k\in\mathbb{N}}$ of $\{\sup\{M_n(\pi_\varphi) \mid \pi_\varphi \leq \Phi,\ \varphi \in C(X)\}\}_{n\in\mathbb{N}}$ (at least one such a subsequence must exist!). Suppose that its limit s_1 is not equal to $s = \sup\{M(\pi_\varphi) \mid \pi_\varphi \leq \Phi,\ \varphi \in C(X)\}$, say $s_1 > s$. Now note that the set $\mu_{\mathsf{OH}}X^{-1}(\nu)$ for any $\nu \in \mathsf{OH}(X)$ consists of all possible extensions of the functional $\overline{\Theta} \colon D \to \mathbb{R}$, where $D = \{\pi_\varphi \mid \varphi \in C(X)\}$ and $\overline{\Theta}(\pi_\varphi) = \nu(\varphi)$. Since any such an extension must be order-preserving, its possible values on Φ are in the closed interval $[\sup\{\overline{\Theta}(\pi_\varphi) \mid \pi_\varphi \leq \Phi,\ \varphi \in C(X)\},\ \inf\{\overline{\Theta}(\pi_\varphi) \mid \pi_\varphi \geq \Phi,\ \varphi \in C(X)\}]$. So, in our case we get that the possible value of any functional from $\lim_{k\to\infty} \mu_{\mathsf{OH}}X^{-1}(\mu_{\mathsf{OH}}X(M_{n_k}))$ (again we may assume the sequence converges) cannot be less than s_1 on Φ, whereas functionals from $\mu_{OH}X^{-1}(\mu_{\mathsf{OH}}X(M))$ are allowed to take any value up to s on Φ, hence $\{\mu_{\mathsf{OH}}X^{-1}(\mu_{\mathsf{OH}}X(M_{n_k}))\}_{k\in\mathbb{N}}$ doesn't converge to $\mu_{\mathsf{OH}}X^{-1}(\mu_{\mathsf{OH}}X(M))$, a contradiction with the openness of $\mu_{\mathsf{OH}}X$. The same reasonings could be applied in the case of the sequence $\{\inf\{M_n(\pi_\varphi) \mid \pi_\varphi \geq \Phi,\ \varphi \in C(X)\}\}_{n\in\mathbb{N}}$.

Take now any $\delta > 0$. There exists $k_0 \in \mathbb{N}$ such that $|\sup\{M_n(\pi_\varphi)|\ \pi_\varphi \leq \Phi, \varphi \in C(X)\} - \sup\{M(\pi_\varphi)|\ \pi_\varphi \leq \Phi, \varphi \in C(X)\}| < \delta$, $|\inf\{M_n(\pi_\varphi)|\pi_\varphi \geq \Phi, \varphi \in C(X)\} - \inf\{M(\pi_\varphi)|\pi_\varphi \geq \Phi, \varphi \in C(X)\}| < \delta$ and $|M(\Phi_i) - M_n(\Phi_i)| < \delta$, $i = \overline{1, n_0}$ for all $n \geq k_0$. Hence, we get the inequalities $|\sup\{M_n(\Psi) \mid \Psi \in Y \cup \{\Phi_0\},\ \Psi \leq \Phi\} - \sup\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\},\ \Psi \leq \Phi\}| < \delta$ and $|\inf\{M_n(\Psi) \mid \Psi \in Y \cup \{\Phi_0\},\ \Psi \geq \Phi\} - \inf\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\},\ \Psi \geq \Phi\}| < \delta$ for sufficiently large numbers n. This means that whatever is $a = M(\Phi) \in [\sup\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\},\ \Psi \leq \Phi\},\ \inf\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\},\ \Psi \geq \Phi\}]$, we can choose $k_0 \in \mathbb{N}$ such that $[\sup\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\},\ \Psi \leq \Phi\},\ \inf\{M(\Psi) \mid \Psi \in Y \cup \{\Phi_0\},\ \Psi \geq \Phi\}] \cap (a - \frac{a - \sup A(\Phi)}{2}, a + \frac{a - \sup A(\Phi)}{2}) \neq \varnothing$ for all $n \geq k_0$, which means that we can obtain functionals from $(M_n)_-$ with values at Φ strictly larger than $\sup A(\Phi)$, a contradiction. Hence, the mappings G_0, G_1 are continuous.

Take now any $M \in \mathsf{OH}^2(X)$ and $\Lambda \in M_0$. We have that

$$d_{\mathsf{OH}}(M,\Lambda) = \sum_{i=1}^{\infty} \frac{|M(\Phi_i) - \Lambda(\Phi_i)|}{\|\Phi_i\| \cdot 2^i} < \sum_{i=1}^{n_0} \frac{|M(\Phi_i) - \Lambda(\Phi_i)|}{\|\Phi_i\| \cdot 2^i} + \frac{\varepsilon}{2} \leq$$
$$\leq \sum_{i=1}^{n_0} \frac{|M(\tilde{\Phi}_i) - \Lambda(\tilde{\Phi}_i)| + 4\varepsilon_1}{\|\Phi_i\| \cdot 2^i} + \frac{\varepsilon}{2} = \sum_{i=1}^{n_0} \frac{4\varepsilon_1}{\|\Phi_i\| \cdot 2^i} + \frac{\varepsilon}{2} = \varepsilon.$$

Define now $g_i \colon \mathsf{OH}^2(X) \to \mathsf{OH}^2(X)$, $i = \overline{0,1}$ by the formula $g_i(M) = \sup G_i(M)$. Functions g_0, g_1 defined that way are continuous, ε-close to the map $id_{\mathsf{OH}^2(X)}$, with disjoint images and preserve the fibers of $\mu_{\mathsf{OH}}X$. □

Now we shall consider the case of $\tau > \omega$.

Theorem 9. *Let $w(X) = \tau > \omega$. The map $\mu_{\mathsf{OH}}X \colon \mathsf{OH}^2X \to \mathsf{OH}X$ is an I^τ-fibration if and only if X is openly generated and χ-homogeneous.*

Proof. Sufficiency. Suppose that the space X is openly generated and χ-homogeneous, $w(X) = \tau > \omega$. We shall use theorem 6 in combination with lemma 1 to prove this part of the statement. Suppose that $\omega \leq \lambda < \tau$. Represent X as the limit of a λ-system $\mathcal{S} = \{X_\alpha, p_\alpha, \mathcal{A}\}$, where $\mathcal{A}$ has the minimal element 1 and X_1 is a singleton. Also we can suppose that all p_α are open. Consider $Y_\alpha = \mathsf{OH}^2(X_\alpha) \times_{\mathsf{OH}(X_\alpha)} \mathsf{OH}(X)$, and by q_α denote the diagonal product $q_\alpha = (\mathsf{OH}^2(p_\alpha), \mu_{\mathsf{OH}}X)$. We obtained a λ-system $\{Y_\alpha, q_\alpha, \mathcal{A}\}$ with the first limit projection q_1 homeomorphic to $\mu_{\mathsf{OH}}X$. Note also that every q_α can be assumed to be soft since so is $\mu_{\mathsf{OH}}X$. We shall prove that each q_α allows two disjoint sections.

First let us show that the fibers of each q_α are infinite. Indeed, consider any $(\Lambda, \nu) \in Y_\alpha$. Then $\mu_{\mathsf{OH}}X_\alpha(\Lambda) = \mathsf{OH}(p_\alpha)(\nu)$. Denote $D = \{\pi_\psi \mid \psi \in C(X)\} \cup \{\Phi \circ \mathsf{OH}(p_\alpha) \mid \Phi \in C(\mathsf{OH}(X_\alpha))\}$. All mappings q_α, being soft ([9]), are surjective. Hence, there is at least one functional $\Theta \colon C(\mathsf{OH}(X)) \to \mathbb{R}$ such that $q_\alpha(\Theta) = (\Lambda, \nu)$. Then $\Theta(\pi_\psi) = \nu(\psi)$, $\Theta(\Phi \circ \mathsf{OH}(p_\alpha)) = \Lambda(\Phi)$.

Our present aim is to find a function $\Phi_0 \in C(\mathsf{OH}(X))$ such that there would exist at least two distinct extensions of $\Theta|_D$ to the space $D \cup \{t\Phi_0 + c_{\mathsf{OH}(X)} \mid t > 0,\ c \in \mathbb{R}\}$.

Since X is χ-homogeneous, openly generated (this yields $\chi(X) = w(X)$ by Lemma 4 of [12]) and $w(X_\alpha) < w(X)$, the mapping p_α does not have one-point fibers, and so does not $\mathsf{OH}(p_\alpha)$ (Theorem 3). Let us make the following denotations: $S = \{\sup \mathsf{OH}(p_\alpha)^{-1}(\lambda) \mid \lambda \in \mathsf{OH}(X_\alpha)\}$, $I = \{\inf \mathsf{OH}(p_\alpha)^{-1}(\lambda) \mid \lambda \in \mathsf{OH}(X_\alpha)\}$. Both these sets are closed due to openness of $\mathsf{OH}(p_\alpha)$ and operations $\sup, \inf \colon \exp \mathsf{OH}X \to \mathsf{OH}X$ being continuous. Now define $\Phi_0 \in C(\mathsf{OH}X)$ to be a function with $\Phi_0(S) = 0$ and $\Phi_0(I) = 1$. Suppose that $\Phi \circ \mathsf{OH}(p_\alpha) \leq \Phi_0$. Since $\Phi \circ \mathsf{OH}(p_\alpha)$ is constant on the fibers of $\mathsf{OH}(p_\alpha)$, this implies $\Phi \circ \mathsf{OH}(p_\alpha) \leq 0$, hence $\Phi \leq 0$ and $\Theta(\Phi \circ \mathsf{OH}(p_\alpha)) = \Lambda(\Phi) \leq 0$. Similarly, $\Theta(\Phi \circ \mathsf{OH}(p_\alpha)) \geq 1$ for any $\Phi \circ \mathsf{OH}(p_\alpha) \geq \Phi_0$. Now pick any $\psi \in C(X)$ with $\pi_\psi \leq \Phi_0$. We have that $\nu \in \mathsf{OH}(p_\alpha)^{-1}(\lambda)$ for some $\lambda \in \mathsf{OH}(X_\alpha)$. Then $\Theta(\pi_\psi) = \pi_\psi(\nu) \leq \pi_\psi(\sup(\mathsf{OH}(p_\alpha)^{-1}(\lambda)) \leq \Phi_0(\sup(\mathsf{OH}(p_\alpha)^{-1}(\lambda))) = 0$. Similarly, $\Theta(\pi_\psi) \geq 1$ for all $\pi_\psi \geq \Phi_0$. Also, it is obvious that Φ_0 does not belong to D, hence, if we define $\Theta(t\Phi_0 + c_{\mathsf{OH}(X)}) = ta + c$, where $a \in [0, 1]$, we shall obtain a positively homogeneous functional on $D \cup \{t\Phi_0 + c_{\mathsf{OH}(X)} \mid t > 0,\ c \in \mathbb{R}\}$, which we can extend to the whole space $C(\mathsf{OH}X)$ according to [4]. Therefore, we have shown that Θ has at least two extensions from D, hence the fibers of q_α are not singletons. So, for any $(\Lambda,\nu)\in Y_\alpha$ define $g_1=\inf q_\alpha^{-1}(\Lambda,\nu)$, $g_2=\sup q_\alpha^{-1}(\Lambda,\nu)$. The mappings g_1, g_2 are continuous disjoint sections for q_α.

Hence, the mapping $\mu_{\mathsf{OH}}X$ satisfies the condition of Lemma 1, and by Theorem 6 it is an I^τ-fibration.

Necessity. Since $\mu_{\mathsf{OH}}X$ is an I^τ-fibration, we have that it is soft. The softness of $\mu_{\mathsf{OH}}X$ implies that X is openly generated (Theorem 2). The space X must be χ-homogeneous, since, if we suppose the opposite, we get that OH^2X is not χ-homogeneous (theorem 4), hence, there exist some $\Lambda\in\mathsf{OH}^2X$ with $\chi(\Lambda,\mathsf{OH}^2X)=\tau^{'}<\tau$, and therefore $\mu_{\mathsf{OH}}X^{-1}(\mu_{\mathsf{OH}}X(\Lambda))$ is not homeomorphic to I^τ. □

Note. Proofs of Theorems 8 and 9 are the same in the case of monad $\mathbb{O}$. Note that the proof of Theorem 8 (a part of it which concerns the choice of the function Φ_0) could be a bit easier for monad $\mathbb{O}$. Indeed, the function $\Phi_0\colon \mathsf{O}X\to\mathbb{R}$ such that $\Phi_0(\inf\mathsf{O}X)=2\alpha$ and $\Phi_0(\sup\mathsf{O}X)=-2\alpha$, where $\alpha=\max\{\sup\Phi_i-\inf\Phi_i|\ i=\overline{1,n_0}\}$ would do. Also, for any $M\in\mathsf{O}^2X$ we take $M_0=\{\Lambda\in\mathsf{O}^2(X)\mid \Lambda|_Y=M|_Y,\ \Lambda(\Phi_0)=-\alpha\}$, and $M_1=\{\Lambda\in\mathsf{O}^2X\mid\Lambda|_Y=M|_Y,\ \Lambda(\Phi_0)=\alpha\}$, where $Y=\{\pi_\varphi\mid\varphi\in C(X)\}\cup\{\Phi_i+c_{\mathsf{O}X}|\ c\in\mathbb{R}, i=\overline{1,n_0}\}$. But in case of OH argumentation in the proof of Theorem 8 with Φ_0, M_0 and M_1 as just described fails.

4. Q-fibrations of the mapping $\mu_{\mathsf{OS}}X$. In this section we shall discuss when the multiplication map $\mu_{\mathsf{OS}}X$ restricted to a certain subset of OS^2X is a trivial Q-fibration.

Let us first remark that the functor OS is isomorphic to the composition of the functors cc and P. The isomorphism is generated by the map $hX\colon\mathsf{ccP}X\to\mathsf{OS}X$ given by

$$hX(A)(\varphi)=\sup\{\nu(\varphi)|\nu\in A\},\quad \varphi\in C(X),\ A\in\mathsf{ccP}X.$$

It was shown in [3] that hX is a homeomorphism for any X, and, moreover, a natural transformation between OS and ccP.

Some properties of the functor cc were studied in [1]. For any convex compact X, $\mathsf{cc}X$ is defined to be the set of all nonempty closed convex subsets of X, $\mathsf{cc}X$ is considered as the subspace of $\exp X$. For any affine mapping $f\colon X\to Y$ the function $\mathsf{cc}(f)$ is given by $\mathsf{cc}(f)(A)=f(A)$ where $A\in\mathsf{cc}X$. If X is a metric space with a metric d, an admissible metric on $\mathsf{exp}X$, and hence on $\mathsf{cc}X$, can be given by the Hausdorff metric

$$d_H(A,C)=\inf\{\varepsilon>0|A\subset B_\varepsilon(C)\text{ and }C\subset B_\varepsilon(A)\}.$$

Here $B_\varepsilon(A)=\{x\in X|d(x,A)<\varepsilon\}$. Also, by $B(x,\delta)$ we denote the δ-ball around x.

Consider the following technical lemma.

Lemma 2. *For any metric convex compactum X the union map $\cup\colon\mathsf{cc}^2X\to\mathsf{cc}X$ is open.*

Proof. We shall prove the continuity of the inverse mapping. Take any $A\in\mathsf{cc}X$ and any sequence A_n in $\mathsf{cc}X$ which converges to A. For any $\delta>0$ there exists $n_0\in\mathbb{N}$ such that for any $n\geq n_0$ we have $d_H(A,A_n)<\delta$.

Take any $\mathcal{A}\in\mathsf{cc}^2(X)$ with the property $\cup\mathcal{A}=A$. For any $n\geq n_0$ define $\mathcal{A}_n$ to be the convex hull of the set consisting of elements of the form $C'=A_n\cap\mathrm{cl}(B_\delta(C))$, where $C\in\mathcal{A}$. Then the distance between $\mathcal{A}$ and $\mathcal{A}_n$ is less than 2δ.

Hence, $\cup^{-1}(A_n)$ converges to $\cup^{-1}(A)$. □

Proposition 1. *The map $\mu_{\mathsf{OS}}X$ is homeomorphic to the composition $\cup\mathsf{cc}b$.*

Hence, openness of the map $\mu_{\mathsf{OS}}X$ depends on the barycentrical openness of the compactum $\mathsf{OS}X \cong \mathsf{ccP}X$.

Denote

$$\mathcal{K} = b^{-1}(E(\mathsf{OS}X)) \subset \mathsf{ccP}(\mathsf{OS}X),$$

where $E(\mathsf{OS}X)$ stands for the set of extremal points of $\mathsf{OS}X$ and $b\colon \mathsf{P}(\mathsf{OS}X) \to \mathsf{OS}X$ is the barycenter mapping (see [7] for more details). Put $M = \mathsf{ccP}(\mathsf{OS}X) \setminus \mathcal{K}$.

Theorem 10. *For any metric compactum X such that the barycenter map b is open, the map $\mu_{\mathsf{OS}}X|_M$ is a trivial Q-fibration.*

Proof. First note that $\mu_{\mathsf{OS}}X^{-1}(\mu_{\mathsf{OS}}X(M)) = M$. Since b and $\cup$ are open and the functor cc preserves open maps, the map μ_{OS} is soft, hence the restriction $\mu_{\mathsf{OS}}X|_M$ is also soft. Openness of b gives us the fact that $b|_{\mathsf{P}(\mathsf{OS}X)\setminus\mathcal{K}}$ is a trivial Q-fibration. To see that $\mu_{\mathsf{OS}}X|_M$ is a trivial Q-fibration, we have to verify the conditions of Theorem 6.

Evidently, M and $\mu_{\mathsf{OS}}X(M)$ are both locally compact ANRs as open subsets of compact absolute retracts $\mathsf{ccP}(\mathsf{OS}X)$ and $\mathsf{OS}X$ respectively. The map $\mu_{\mathsf{OS}}|_M$ is perfect as a restriction of a perfect map to a full preimage.

At last, we have to check the disjoint approximation property. Take any cover $\mathcal{U} \in \mathrm{cov}(M)$. Without loss of generality, we may assume $\mathcal{U}$ to be locally finite. Then there exists a continuous mapping $\epsilon\colon X \to (0,1]$ such that the cover $\{B(A,\epsilon(A))\}_{A\in M}$ is inscribed into $\mathcal{U}$. Further, represent M as a countable union

$$M = \bigcup_{n=1}^{\infty} F_n, \text{ where } F_n = \Big\{A \in \mathsf{ccP}(\mathsf{OS}X) | d(A,\mathcal{K}) \geq \frac{1}{n}\Big\}.$$

For any element $\nu \in M$ there exists $n_\nu \in \mathbb{N}$ such that whenever $A \in \mathsf{ccP}(\mathsf{OS}X)$ contains ν then it belongs to F_{n_ν}. Put $\epsilon_\nu = \min_{A\in F_{n_\nu}} \epsilon(A)$. Since $b|_{\mathsf{P}(\mathsf{OS}X)\setminus\mathcal{K}}$ is a trivial Q-fibration ([7]), we can choose functions

$$f_1, f_2\colon \mathsf{P}(\mathsf{OS}X) \setminus \mathcal{K} \to \mathsf{OS}X \setminus E(\mathsf{OS}X)$$

which preserve the fibers of $b|_{\mathsf{P}(\mathsf{OS}X)\setminus\mathcal{K}}$, have disjoint images and are $\mathcal{V}$-close to the identity map, where $\mathcal{V} = \{B(\nu,\epsilon_\nu)\}_{\nu\in\mathsf{P}(\mathsf{OS}X)\setminus\mathcal{K}}$. We can extend f_1 and f_2 to the whole $\mathsf{P}(\mathsf{OS}X)$ by setting $f_1(\nu) = f_2(\nu) = \nu$ for any $\nu \in \mathcal{K}$.

We define functions $F_1, F_2\colon M \to M$ the following way. For any $A \in M$ we put $F_i(A) = \mathrm{conv} f_i(A)$.

We have that $d_H(A, F_i(A)) \leq \max_{\nu\in A} \epsilon_\nu = \max_{\mu\in A} \min_{B\in F_{n_\nu}} \epsilon(B) \leq \epsilon(A)$. Hence, both mappings are $\mathcal{U}$-close to the identity mapping. Also, they preserve the fibers of $\mu_{\mathsf{OS}}X|_M$ due to the choice of f_1 and f_2.

Finally, $F_1(A) \neq F_2(B)$ for any convex sets A and B. This can be seen as follows. Due to the choice of f_1 and f_2, the sets $f_1(A)$ and $f_2(B)$ can intersect only by elements from $\mathcal{K}$. Since f_1 and f_2 preserve the fibers of b, there exists an extremal point $a \in \mathrm{conv} f_1(A) \setminus (f_2(B) \cup \mathcal{K})$. On the other hand, the assumption $\mathrm{conv} f_1(A) = \mathrm{conv} f_2(B)$ yields the inclusion $a \in f_2(B)$, a contradiction. Hence, $F_1(M) \cap F_2(M) = \varnothing$. □

Denote by D the two-point discrete space. Since OSD is affinely homeomorphic to 2-dimensional simplex, the barycenter map $b\colon P(OSD) \to OSD$ is open (we say shortly that the convex compactum OSD is barycentrically open). Hence we obtain:

Corollary. *The map $\mu_{OS}D|_M$ is a trivial Q-fibration.*

The general situation depends on the barycentrical openess of the compactum $OSX \cong ccPX$. It is well known that compacta of the form PX are barycentrically open. Does this yield barycentric openness of $ccP(X)$? In fact, there is a more general

Question. *Let K be a convex compactum such that its barycenter mapping $b_K\colon P(K) \to K$ is open. Is the barycenter map $b_{ccK}\colon P(ccK) \to ccK$ open?*

From results of [10] it follows that this question is equivalent to the following one: given openness of the map $\varphi_K\colon K \times K \to K$ defined by

$$\varphi_K(x, y) = \frac{1}{2}x + \frac{1}{2}y,$$

is the respective map φ_{ccK} open?

5. Acknowledgements. The authors would like to express their grateful acknowledgement to the referee for diligent proofreading, useful comments that much helped to improve the text.

REFERENCES

1. L. Bazylevych, D. Repovš, M. Zarichnyi, *Hyperspace of convex compacta of nonmetrizable compact convex subspaces of locally convex spaces*, Topology and its Applications, **155** (2008), 764–772.
2. A. Chigogidze, *Trivial fibrations with fibers homeomorphic to the Tychonov cube*, Mat. Zametki, **39** (1986), 747–756.
3. D. Davletov, G. Djabbarov, *Functor of semiadditive functionals*, Methods Funct. Anal. Topol., **14** (2008), 314–322.
4. G. Djabbarov, *Categorical properties of the functor of weakly additive positively-homogeneous functionals*, Uzb. math. jour., **2** (2006), 20–28.
5. S. Eilenberg, J. Moore, *Adjoint functors and triples*, J. Math., **9** (1965), 381–389.
6. V. Fedorchuk, A. Chigogidze, Absolute retracts and infinitedimensional manifolds, Nauka, Moskow, 1992, 231p.
7. V. Fedorchuk, *Probability measures in topology*, Uspekhi Mat. Nauk, **46** (1991), 41–80. (in Russian)
8. V. Fedorchuk, M. Zarichnyii, *Covariant functors in the category of thopological spaces*, Results of Science and Technics. Algebra. Topology. Geometry., Moscow. VINITI, **28** 47–95. (in Russian)
9. L. Karchevska, *On geometric properties of functors of positively homogeneous and semiadditive functionals*, Ukr. Mat. Zurnal, **62** (2010), 1351–1359. (in Ukrainian)
10. R. O'Brien, *On the openness of the barycentre map*, Math. Ann., **223** (1976), 207–212.
11. T. Radul, *On the functor of order-preserving functionals*, Comment. Math. Univ. Carolin, **39** (1998), 609–615.
12. T. Radul, *Topology of the space of order-preserving functionals*, Bull. Pol. Acad. Sci. Math., **47** (1999), 53–60.
13. T. Radul, *Convexities generated by L-monads*, Applied Categorical Structures, **19** (2011), 729–739.
14. T. Radul, M. Zarichnyi, *Monads in the category of compacta*, Uspekhi Mat. Nauk, **50** (1995), 83–106.
15. E. Schepin, *Functors and uncountable powers of compacta*, Uspekhi Mat. Nauk, **36** (1981), 3–62 (in Russian).
16. A. Teleiko, M. Zarichnyi, Categorical Topology of Compact Hausdorff Spaces, VNTL Publishers, Lviv, 1999.
17. H. Torunczyk, J. West, Fibrations and Bundles with Hilbert Cube Manifold Fibers, Preprint, 1982.

18. M. Zarichnyi, *Absolute extensors and geometry of monad multiplication maps in the category of compacta*, Mat. Sbornik, **9** (1991), 1261–1280. (in Russian)

Department of Mechanics and Mathematics
Lviv National University
crazymaths@ukr.net

Institute of Mathematics
Casimirus the Great University, Bydgoszcz, Poland
and Department of Mechanics and Mathematics
Lviv National University, Ukraine
tarasradul@yahoo.co.uk

Received 09.08.2011
Revised 20.02.2012