
Using multiplicative polynomials on algebras it is proved an analogue of Hilbert Nullstellensatz for the case of an infinite-dimensional real Banach space.


С использованием мультипликативных полиномов на алгебрах, в статье доказан аналог теоремы Гильберта о нулях для бесконечномерного вещественного банахового пространства.

1. Notations and preliminaries. Let X be a Banach space over the field \( \mathbb{K} \) of real \( \mathbb{R} \) or complex \( \mathbb{C} \) numbers. A map \( P: X \to \mathbb{K} \) is an \( n \)-homogeneous polynomial if there is an \( n \)-linear form \( B_P: X^n \to \mathbb{K} \) such that \( P(x) = B_P(x, \ldots, x) \) for every \( x \in X \). A finite sum of homogeneous polynomials is just a polynomial. We denote by \( \mathcal{P}(X) \) the algebra of all continuous polynomials on \( X \) and let \( \mathcal{P}_0(X) \) be a subalgebra of \( \mathcal{P}(X) \). We use the notation \( (P_1, \ldots, P_n) \) for the ideal in \( \mathcal{P}_0(X) \) generated by polynomials \( P_1, \ldots, P_n \) in \( \mathcal{P}_0(X) \).

Let \( A \) be a Banach algebra over \( \mathbb{K} \). A polynomial \( D: A \to \mathbb{K} \) is said to be multiplicative if \( D(ab) = D(a)D(b) \) for all \( a, b \in A \). It is known ([1]) that every multiplicative polynomials must be homogeneous.

For an ideal \( J \in \mathcal{P}_0(X) \), \( V(J) \) denotes the zero of \( J \), that is, the common set of zeros of all polynomials in \( J \). Let \( G \) be a subset of \( X \). Then \( I(G) \) denotes the hull of \( G \), that is, a set of all polynomials in \( \mathcal{P}_0(X) \) vanishing on \( G \). The set \( \text{Rad} \ J \) is called the radical of \( J \) if \( P^k \in J \) for some positive integer \( k \) implies \( P \in \text{Rad} \ J \). \( P \) is called a radical polynomial if it can be represented as a product of mutually different irreducible polynomials.

A subalgebra \( A_0 \) of an algebra \( A \) is called factorial if for every \( f \in A_0 \) the equality \( f = f_1f_2 \) implies that \( f_1 \in A_0 \) and \( f_2 \in A_0 \).

This paper is devoted to applications of multiplicative polynomials to polynomial algebras of infinite many variables. In particular, we prove a version of Hilbert Nullstellensatz for real polynomials on a real Banach space.

2. Admissible matrix algebras. Let us denote by \( \mathfrak{F}(\mathbb{K}^n) \) the free algebra with \( n \) generators \( \{a_1, a_2, \ldots, a_n\} \). The map

\[
S_n: \sum c_{k_1\ldots k_m}a_{k_1}a_{k_2}\cdots a_{k_m} \mapsto \sum c_{k_1\ldots k_m}t_{k_1}t_{k_2}\cdots t_{k_m},
\]

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where \((t_1, t_2, \ldots, t_n) \in \mathbb{K}^n\), \(c_{k_1 k_2 \ldots k_n} \in \mathbb{K}\), is a homomorphism from \(\mathfrak{F}(\mathbb{K}^n) = K[t_1, t_2, \ldots, t_n]\) onto the algebra of polynomial \(\mathcal{P}(\mathbb{K}^n)\) of \(n\) variables over \(\mathbb{K}\).

For a given polynomial \(P \in \mathcal{P}(\mathbb{K}^n)\) we consider the following set of elements in \(\mathfrak{F}(\mathbb{K}^n)\),

\[ S_n^{-1}(P) := \{ F \in \mathfrak{F}(\mathbb{K}^n) : S_n(F) = P \}. \]

Let us denote by \(M_k(\mathbb{K})\) the algebra of \(k \times k\) square matrices over \(\mathbb{K}\). We say that a finite sequence of matrices \((A_1, A_2, \ldots, A_n)\) is admissible for \(\mathcal{P}(\mathbb{K}^n)\) if for every \(P \in \mathcal{P}(\mathbb{K}^n)\)

\[ \det(F_1(A_1, A_2, \ldots, A_n)) = \det(F_2(A_1, A_2, \ldots, A_n)) \]

for any two elements \(F_1, F_2 \in S_n^{-1}(P)\). We say that a subalgebra \(A \subset M_k(\mathbb{K})\) is admissible for \(\mathcal{P}(\mathbb{K}^n)\) if every sequence \((A_1, A_2, \ldots, A_n) \subset M_k(\mathbb{K})\) is admissible for \(\mathcal{P}(\mathbb{K}^n)\). It is clear that if \((A_1, A_2, \ldots, A_n)\) is admissible for \(\mathcal{P}(\mathbb{K}^n)\), then the algebra generated by \((A_1, A_2, \ldots, A_n)\) is admissible for \(\mathcal{P}(\mathbb{K}^n)\).

Let \(X\) be a linear space over \(\mathbb{K}\) with a Hamel basis \((e_{\alpha})\), \(\alpha \in \mathfrak{A}\) for some index set \(\mathfrak{A}\). Then \(A \subset M_k(\mathbb{K})\) is admissible for \(\mathcal{P}(X)\) if for every finite sequence \(\{e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_n}\} \subset \mathfrak{A}\), \(A\) is admissible for \(\mathcal{P}(V_n)\), where \(V_n\) is the linear span of \(\{e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_n}\}\).

Let \(w\) be an element in the algebraic tensor product \(A \otimes X\). Then \(w\) can be represented as a finite sum \(\sum A_j \otimes x_j\), where \(A_j \in A\) and \(x_j \in X\). Since every vector in \(X\) have a finite sum representation by bases vectors \((e_{\alpha})\), \(w\) can be uniquely represented as a finite sum

\[ w = \sum A_{\alpha_j} \otimes e_{\alpha_j}. \]

Thus, if \(A\) is admissible for \(\mathcal{P}(X)\), then for every \(w \in A \otimes X\) there is a well defined map \(\hat{w}: \mathcal{P}(X) \to \mathbb{K}\) acting by

\[ \hat{w}(P) = \det\left( P\left( \sum A_{\alpha_j} \otimes e_{\alpha_j} \right) \right). \]

Let \(X\) be a Banach space over \(\mathbb{K}\) and \(\mathcal{P}_c(X) \subset \mathcal{P}(X)\) be the space of all continuous polynomials. The next proposition follows from properties of determinants.

**Theorem 1.** If \(A \subset M_k(\mathbb{K})\) is admissible for \(\mathcal{P}(X)\), then for every \(w \in A \otimes X\), \(\hat{w}\) is a multiplicative \(k\)-homogeneous polynomial on \(\mathcal{P}(X)\).

Note that in general case the map \(\hat{w}(P)\) may be discontinuous.

For a given matrix \(A \in M_n(\mathbb{K})\) we denote by \(\mathcal{A}_A\) the commutative subalgebra in \(M_n(\mathbb{K})\) generated by \(A\).

**Theorem 2.** Let \(A \in M_n(\mathbb{K})\). Then for every \(w \in \mathcal{A}_A \otimes X\) the map \(\hat{w}: \mathcal{P}_c(X) \to \mathbb{C}\) is continuous.

**Proof.** According to [3] for a given \(w \in \mathcal{A}_A \otimes X\) there exists a functional calculus of \(w\) in the algebra \(\mathcal{P}_c(X)\) such that for every \(P \in \mathcal{P}_c(X)\) we can define \(P(w) \in M_n(\mathbb{K})\) and the map \(P \mapsto P(w)\) is a continuous homomorphism from \(\mathcal{P}_c(X)\) to \(M_n(\mathbb{K})\). It is easy to see that \(\hat{w}(P) = \det(P(w))\) and \(\hat{w}\) is continuous.

Let \(\mathcal{A}_0\) be a subalgebra of \(M_k(\mathbb{K})\) such that for any \(A \in \mathcal{A}_0\), \(CAC^{-1}\) is an uptriangular matrix for some fixed \(C\). Then \(\mathcal{A}_0\) is \(\mathcal{P}(X)\) admissible for every linear space \(X\) and

\[ \hat{w}(P) = \det\left( P\left( \sum_{j=1}^{n} A_{\alpha_j} \otimes e_{\alpha_j} \right) \right) = \]

\[ = P\left( \sum_{j=1}^{n} \lambda_{\alpha_j} e_{\alpha_j} \right) P\left( \sum_{j=1}^{n} \lambda_{\alpha_j}^2 e_{\alpha_j} \right) \cdots P\left( \sum_{j=1}^{n} \lambda_{\alpha_j}^k e_{\alpha_j} \right), \tag{1} \]
Theorem 3. Let $P_0(X)$ be a factorial subalgebra in $P(X)$ which contains all liner functionals and $J = (P_1, \ldots, P_n)$. Then $\text{Rad} J \subset P_0(X)$ and
\[ I[V(J)] = \text{Rad} J \text{ in } P_0(X). \]

We can prove an analogue to the previous theorem for the case of an infinite-dimensional real Banach space.

Here we need a correspondence between a complex number $a + bi$ and the matrix $A = \begin{pmatrix} b & a \\ -a & b \end{pmatrix} \in M_2(\mathbb{R})$, namely, $|a + bi| = \det A$ and $a + bi, a - bi$ are the eigenvalues of matrix $A$. Let us denote the set of matrices of such a form by $A_0$. Obviously, the algebra $A_0$ is $P(X)$ admissible because all elements of $A_0$ can be simultaneously reduced to the triangle form. We will write this form by $CA_{\alpha_j} C^{-1}$.

So let $X$ be a real Banach space and subalgebra $A_0 \subset M_2(\mathbb{R})$ be admissible for $P_c(X)$. Thus, there exists $w$ in $A_0 \otimes X$ such that
\[ \hat{w}(P) = P \left( \sum_{j=1}^{\infty} \lambda_{\alpha_j}^1 e_{\alpha_j} \right) P \left( \sum_{j=1}^{\infty} \lambda_{\alpha_j}^2 e_{\alpha_j} \right), \tag{2} \]
where $\lambda_{\alpha_j}^1, \lambda_{\alpha_j}^2$ are the diagonal elements of $CA_{\alpha_j} C^{-1}$, $A_{\alpha_j} \in A_0$, $j = 1, 2, \ldots$.

Theorem 4. Let $P_1, P_2, \ldots, P_m$ be polynomials in $P_c(X)$ for a real Banach space $X$ and $J = (P_1, P_2, \ldots, P_m)$ be the ideal generated by these polynomials. Let
\[ \bigcap_{k=1}^{m} \{ w \in A_0 \otimes X: \hat{w}(P_k) = 0 \} = \emptyset. \]

Then $J$ contains $1$, that is, there are polynomials $Q_1, Q_2, \ldots, Q_m \in P_c(X)$ such that
\[ \sum_{k=1}^{m} P_k Q_k = 1. \]

Proof. By (2) we have
\[ \hat{w}(P) = P \left( \sum_{j=1}^{\infty} \lambda_{\alpha_j} e_{\alpha_j} \right) P \left( \sum_{j=1}^{\infty} \overline{\lambda_{\alpha_j}} e_{\alpha_j} \right), \]
where $\lambda_{\alpha_j}, \overline{\lambda_{\alpha_j}}$ are the diagonal elements of $CA_{\alpha_j} C^{-1}$, $A_{\alpha_j} \in A_0$. Then we can write
\[ \bigcap_{k=1}^{m} \left\{ (\lambda_{\alpha_j})_{j=1}^{\infty}: P_k \left( \sum_{j=1}^{\infty} \lambda_{\alpha_j} e_{\alpha_j} \right) P_k \left( \sum_{j=1}^{\infty} \overline{\lambda_{\alpha_j}} e_{\alpha_j} \right) = 0 \right\} = \emptyset, \]
that is,
\[ \bigcap_{k=1}^{m} \left\{ (\lambda_{\alpha_j})_{j=1}^{\infty}: P_k \left( \sum_{j=1}^{\infty} \lambda_{\alpha_j} e_{\alpha_j} \right) P_k \left( \sum_{j=1}^{\infty} \lambda_{\alpha_j} e_{\alpha_j} \right) = 0 \right\} = \emptyset. \]
The last condition is equivalent to the following one
\[ \bigcap_{k=1}^{m} \left\{ (\lambda_{\alpha_j})_{j=1}^\infty : P_k \left( \sum_{j=1}^\infty \lambda_{\alpha_j} e_{\alpha_j} \right) = 0 \right\} = \emptyset. \]

So it means that the polynomials \( P_1, P_2, \ldots, P_m \) have no common complex zeros. Then due to Theorem 3 \( J \) contains 1, that is, there exist polynomials \( Q'_1, Q'_2, \ldots, Q'_m \in \mathcal{P}_c(X_\mathbb{C}) \) such that
\[ \sum_{k=1}^{m} P_k Q'_k = 1, \text{ where } X_\mathbb{C} \text{ is the complexification of } X. \]

Let \( Q'_k = U_k + iV_k \), be the decomposition of \( Q'_k \), onto the real and imaginary parts, \( k = 1, \ldots, m \). Then
\[ \sum_{k=1}^{m} P_k V_k = \text{Im}(1) = 0. \]

That is \( \sum_{k=1}^{m} U_k P_k = 1 \). Hence, we just have to set \( Q_k = U_k, k = 1, \ldots, m \).

**Corollary 1.** If \( P_1 \) and \( P_2 \) are irreducible polynomials of \( \mathcal{P}_c(X) \) and
\[ \{ w_1 \in \mathcal{A}_0 \otimes X : \hat{w}(P_1) = 0 \} = \{ w_2 \in \mathcal{A}_0 \otimes X : \hat{w}(P_2) = 0 \}, \]
then \( P_1 = cP_2 \), where \( c = \text{const} \).

**Theorem 5.** For any ideal \( J \in \mathcal{P}(\mathbb{R}^n), J = (P_1, P_2, \ldots, P_m) \), we define
\[ \mathcal{V}(J) = \{ w \in \mathcal{A}_0 \otimes X : \hat{w}(P_k) = 0, k = 1, \ldots, m \}, \]
\[ \mathcal{I}(\mathcal{V}(J)) = \{ P_k \in \mathcal{P}(X) : \hat{w}(P_k) = 0, w \in \mathcal{V}(J) \}. \]

Then \( \text{Rad } J \subset \mathcal{P}(X) \) and \( \mathcal{I}(\mathcal{V}(J)) = \text{Rad } J \).

**Proof.** We know already that
\[ \hat{w}(P) = P \left( \sum_{j=1}^\infty \lambda_{\alpha_j} e_{\alpha_j} \right) \overline{P \left( \sum_{j=1}^\infty \lambda_{\alpha_j} e_{\alpha_j} \right)}, \]
where \( \lambda_{\alpha_j}, \overline{\lambda_{\alpha_j}} \) are the diagonal elements of \( CA_{\alpha_j}C^{-1}, A_{\alpha_j} \in \mathcal{A}_0 \). Then we can write
\[ \mathcal{I}(\mathcal{V}(J)) = \left\{ P_k \in \mathcal{P}_c(X) : P_k \left( \sum_{j=1}^\infty \lambda_{\alpha_j} e_{\alpha_j} \right) \right\} = \left\{ P_k \in \mathcal{P}_c(X) : \hat{w}(P_k) = 0, w \in \mathcal{V}(J) \right\}. \]

Hence \( \mathcal{I}(\mathcal{V}(J)) = \text{Rad } J \). \qed

**REFERENCES**


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