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MATRIX ALGEBRAIC SETS OF INFINITE DIMENSION

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Using multiplicative polynomials on algebras it is proved an analogue of Hilbert Nullstellensatz for the case of an infinite-dimensional real Banach space.

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С использованием мультипликативных полиномов на алгебрах, в статье доказан аналог теоремы Гильберта о нулях для бесконечномерного вещественного банахового пространства.

1. Notations and preliminaries. Let X be a Banach space over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers. A map $P\colon X \to \mathbb{K}$ is an *n-homogeneous polynomial* if there is an n-linear form $B_P\colon X^n \to \mathbb{K}$ such that $P(x) = B_P(x, \dots x)$ for every $x \in X$. A finite sum of homogeneous polynomials is just a *polynomial*. We denote by $\mathcal{P}(X)$ the algebra of all continuous polynomials on X and let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. We use the notation $(P_1, \dots, P_n)$ for the ideal in $\mathcal{P}_0(X)$ generated by polynomials $P_1, \dots, P_n$ in $\mathcal{P}_0(X)$.

Let A be a Banach algebra over $\mathbb{K}$. A polynomial $D\colon A \to \mathbb{K}$ is said to be *multiplicative* if $D(ab) = D(a)D(b)$ for all $a, b \in A$. It is known ([1]) that every multiplicative polynomials must be homogeneous.

For an ideal $J \in \mathcal{P}_0(X)$, $V(J)$ denotes the *zero* of J, that is, the common set of zeros of all polynomials in J. Let G be a subset of X. Then $I(G)$ denotes the *hull* of G, that is, a set of all polynomials in $\mathcal{P}_0(X)$ vanishing on G. The set Rad J is called the *radical* of J if $P^k \in J$ for some positive integer k implies $P \in \text{Rad } J$. P is called a *radical* polynomial if it can be represented as a product of mutually different irreducible polynomials.

A subalgebra A_0 of an algebra A is called *factorial* if for every $f \in A_0$ the equality $f = f_1 f_2$ implies that $f_1 \in A_0$ and $f_2 \in A_0$.

This paper is devoted to applications of multiplicative polynomials to polynomial algebras of infinite many variables. In particular, we prove a version of Hilbert Nullstellensatz for real polynomials on a real Banach space.

2. Admissible matrix algebras. Let us denote by $\mathfrak{F}(\mathbb{K}^n)$ the free algebra with n generators $\{\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n\}$. The map

$$S_n\colon \sum c_{k_1 k_2 \cdots k_m} \mathfrak{a}_{k_1} \mathfrak{a}_{k_2} \cdots \mathfrak{a}_{k_m} \mapsto \sum c_{k_1 k_2 \cdots k_m} t_{k_1} t_{k_2} \cdots t_{k_m},$$

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where $(t_1, t_2, \dots, t_n) \in \mathbb{K}^n,\ c_{k_1 k_2 \cdots k_m} \in \mathbb{K}$, is a homomorphism from $\mathfrak{F}(\mathbb{K}^n) = K[t_1, t_2, \dots, t_n]$ onto the algebra of polynomial $\mathcal{P}(\mathbb{K}^n)$ of n variables over $\mathbb{K}$.

For a given polynomial $P \in \mathcal{P}(\mathbb{K}^n)$ we consider the following set of elements in $\mathfrak{F}(\mathbb{K}^n)$,

$$S_n^{-1}(P):=\{F \in \mathfrak{F}(\mathbb{K}^n)\colon S_n(F) = P\}.$$

Let us denote by $M_k(\mathbb{K})$ the algebra of $k \times k$ square matrices over $\mathbb{K}$. We say that a finite sequence of matrices $(A_1, A_2, \dots, A_n)$ is *admissible* for $\mathcal{P}(\mathbb{K}^n)$ if for every $P \in \mathcal{P}(\mathbb{K}^n)$

$$\det(F_1(A_1, A_2, \dots, A_n)) = \det(F_2(A_1, A_2, \dots, A_n))$$

for any two elements $F_1, F_2 \in S_n^{-1}(P)$. We say that a subalgebra $\mathcal{A} \subset M_k(\mathbb{K})$ is admissible for $\mathcal{P}(\mathbb{K}^n)$ if every sequence $(A_1, A_2, \dots, A_n) \subset M_k(\mathbb{K})$ is admissible for $\mathcal{P}(\mathbb{K}^n)$. It is clear that if $(A_1, A_2, \dots, A_n)$ is admissible for $\mathcal{P}(\mathbb{K}^n)$, then the algebra generated by $(A_1, A_2, \dots, A_n)$ is admissible for $\mathcal{P}(\mathbb{K}^n)$.

Let X be a linear space over $\mathbb{K}$ with a Hamel basis (e_α), $\alpha \in \mathfrak{A}$ for some index set $\mathfrak{A}$. Then $\mathcal{A} \subset M_k(\mathbb{K})$ is *admissible* for $\mathcal{P}(X)$ if for every finite sequence $\{e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_n}\}$, $\mathcal{A}$ is admissible for $\mathcal{P}(V_n)$, where V_n is the linear span of $\{e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_n}\}$.

Let w be an element in the algebraic tensor product $\mathcal{A} \otimes X$. Then w can be represented as a finite sum $\sum A_j \otimes x_j$, where $A_j \in \mathcal{A}$ and $x_j \in X$. Since every vector in X have a finite sum representation by bases vectors (e_α), w can be uniquely represented as a finite sum

$$w = \sum A_{\alpha_j} \otimes e_{\alpha_j}.$$

Thus, if $\mathcal{A}$ is admissible for $\mathcal{P}(X)$, then for every $w \in \mathcal{A} \otimes X$ there is a well defined map $\widehat{w}\colon \mathcal{P}(X) \to \mathbb{K}$ acting by

$$\widehat{w}(P) = \det\Big(P\Big(\sum A_{\alpha_j} \otimes e_{\alpha_j}\Big)\Big).$$

Let X be a Banach space over $\mathbb{K}$ and $\mathcal{P}_c(X) \subset \mathcal{P}(X)$ be the space of all continuous polynomials. The next proposition follows from properties of determinants.

Theorem 1. *If $\mathcal{A} \subset M_k(\mathbb{K})$ is admissible for $\mathcal{P}(X)$, then for every $w \in \mathcal{A} \otimes X$, $\widehat{w}$ is a multiplicative k-homogeneous polynomial on $\mathcal{P}(X)$.*

Note that in general case the map $\widehat{w}(P)$ may be discontinuous.

For a given matrix $A \in M_n(\mathbb{K})$ we denote by $\mathcal{A}_A$ the commutative subalgebra in $M_n(\mathbb{K})$ generated by A.

Theorem 2. *Let $A \in M_n(\mathbb{K})$. Then for every $w \in \mathcal{A}_A \otimes X$ the map $\widehat{w} : \mathcal{P}_c(X) \to \mathbb{C}$ is continuous.*

Proof. According to [3] for a given $w \in \mathcal{A}_A \otimes X$ there exists a functional calculus of w in the algebra $\mathcal{P}_c(X)$ such that for every $P \in \mathcal{P}_c(X)$ we can define $P(w) \in M_n(\mathbb{K})$ and the map $P \mapsto P(w)$ is a continuous homomorphism from $\mathcal{P}_c(X)$ to $M_n(\mathbb{K})$. It is easy to see that $\widehat{w}(P) = \det(P(w))$ and $\widehat{w}$ is continuous. □

Let $\mathcal{A}_0$ be a subalgebra of $M_k(\mathbb{K})$ such that for any $A \in \mathcal{A}_0$, CAC^{-1} is an uptriangular matrix for some fixed C. Then $\mathcal{A}_0$ is $\mathcal{P}(X)$ admissible for every linear space X and

$$\widehat{w}(P) = \det\left(P\left(\sum_{j=1}^{n} A_{\alpha_j} \otimes e_{\alpha_j}\right)\right) =$$

$$= P\left(\sum_{j=1}^{n} \lambda_{\alpha_j}^{1} e_{\alpha_j}\right) P\left(\sum_{j=1}^{n} \lambda_{\alpha_j}^{2} e_{\alpha_j}\right) \cdots P\left(\sum_{j=1}^{n} \lambda_{\alpha_j}^{k} e_{\alpha_j}\right), \tag{1}$$

where $\lambda^1_{\alpha_j}, \lambda^2_{\alpha_j}, \dots, \lambda^k_{\alpha_j}$ are the diagonal elements of $CA_{\alpha_j}C^{-1}$.

3. Matrix zeros of real polynomials. Due to the Hilbert Nullstellensatz Theorem the radical of each ideal in $\mathcal{P}(\mathbb{C}^n)$ is completely defined by its zero. Moreover this result is still true for *finitely* generated ideals in an infinite-dimensional complex Banach space. For example, in [2, 4] the following theorem is proved.

Theorem 3. *Let $\mathcal{P}_0(X)$ be a factorial subalgebra in $\mathcal{P}(X)$ which contains all liner functionals and $J = (P_1, \dots, P_n)$. Then* Rad $J \subset \mathcal{P}_0(X)$ *and*

$$I[V(J)] = \text{Rad } J \quad \textit{in } \mathcal{P}_0(X).$$

We can prove an analogue to the previous theorem for the case of an infinite-dimensional real Banach space.

Here we need a correspondence between a complex number $a + bi$ and the matrix $A = \begin{pmatrix} b & a \\ -a & b \end{pmatrix} \in M_2(\mathbb{R})$, namely, $|a+bi| = \det A$ and $a+bi$, $a-bi$ are the eigenvalues of matrix A. Let us denote the set of matrices of such a form by $\mathcal{A}_0$. Obviously, the algebra $\mathcal{A}_0$ is $\mathcal{P}(X)$ admissible because all elements of $\mathcal{A}_0$ can be simultaneously reduced to the triangle form. We will write this form by $CA_{\alpha_j}C^{-1}$.

So let X be a real Banach space and subalgebra $\mathcal{A}_0 \subset M_2(\mathbb{R})$ be admissible for $\mathcal{P}_c(X)$. Thus, there exists w in $\mathcal{A}_0 \otimes X$ such that

$$\widehat{w}(P) = P\left(\sum_{j=1}^{\infty} \lambda^1_{\alpha_j} e_{\alpha_j}\right) P\left(\sum_{j=1}^{\infty} \lambda^2_{\alpha_j} e_{\alpha_j}\right), \tag{2}$$

where $\lambda^1_{\alpha_j}, \lambda^2_{\alpha_j}$ are the diagonal elements of $CA_{\alpha_j}C^{-1}$, $A_{\alpha_j} \in \mathcal{A}_0$, $j = 1, 2 \dots$.

Theorem 4. *Let $P_1, P_2, \dots, P_m$ be polynomials in $\mathcal{P}_c(X)$ for a real Banach space X and $J = (P_1, P_2, \dots, P_m)$ be the ideal generated by these polynomials. Let*

$$\bigcap_{k=1}^{m} \{w \in \mathcal{A}_0 \otimes X \colon \widehat{w}(P_k) = 0\} = \varnothing.$$

Then J contains 1, *that is, there are polynomials $Q_1, Q_2, \dots, Q_m \in \mathcal{P}_c(X)$ such that*

$$\sum_{k=1}^{m} P_k Q_k = 1.$$

Proof. By (2) we have

$$\widehat{w}(P) = P\left(\sum_{j=1}^{\infty} \lambda_{\alpha_j} e_{\alpha_j}\right) P\left(\sum_{j=1}^{\infty} \overline{\lambda_{\alpha_j}} e_{\alpha_j}\right),$$

where $\lambda_{\alpha_j}, \overline{\lambda_{\alpha_j}}$ are the diagonal elements of $CA_{\alpha_j}C^{-1}$, $A_{\alpha_j} \in \mathcal{A}_0$. Then we can write

$$\bigcap_{k=1}^{m} \left\{ (\lambda_{\alpha_j})_{j=1}^{\infty} \colon P_k\left(\sum_{j=1}^{\infty} \lambda_{\alpha_j} e_{\alpha_j}\right) P_k\left(\sum_{j=1}^{\infty} \overline{\lambda_{\alpha_j}} e_{\alpha_j}\right) = 0 \right\} = \varnothing,$$

that is,

$$\bigcap_{k=1}^{m} \left\{ (\lambda_{\alpha_j})_{j=1}^{\infty} \colon P_k\left(\sum_{j=1}^{\infty} \lambda_{\alpha_j} e_{\alpha_j}\right) \overline{P_k\left(\sum_{j=1}^{\infty} \lambda_{\alpha_j} e_{\alpha_j}\right)} = 0 \right\} = \varnothing.$$

The last condition is equivalent to the following one

$$\bigcap_{k=1}^{m}\left\{(\lambda_{\alpha_j})_{j=1}^{\infty}\colon P_k\left(\sum_{j=1}^{\infty}\lambda_{\alpha_j}e_{\alpha_j}\right)=0\right\}=\varnothing.$$

So it means that the polynomials $P_1, P_2 \dots, P_m$ have no common complex zeros. Then due to Theorem 3 J contains 1, that is, there exist polynomials $Q'_1, Q'_2 \dots, Q'_m \in \mathcal{P}_c(X_{\mathbb{C}})$ such that

$$\sum_{k=1}^{m} P_k Q'_k = 1, \text{ where } X_{\mathbb{C}} \text{ is the complexification of } X.$$

Let $Q'_k = U_k + iV_k$, be the decomposition of Q'_k, onto the real and imaginary parts, $k = 1, \dots, m$. Then

$$\sum_{k=1}^{m} P_k V_k = \operatorname{Im}(1) = 0.$$

That is $\sum_{k=1}^{m} U_k P_k = 1$. Hence, we just have to set $Q_k = U_k$, $k = 1, \dots, m$. □

Corollary 1. *If P_1 and P_2 are irreducible polynomials of $\mathcal{P}_c(X)$ and*

$$\{w_1 \in \mathcal{A}_0 \otimes X\colon \widehat{w_1}(P_1) = 0\} = \{w_2 \in \mathcal{A}_0 \otimes X\colon \widehat{w_2}(P_2) = 0\},$$

then $P_1 = cP_2$, where $c =$ const.

Theorem 5. *For any ideal $J \in \mathcal{P}(\mathbb{R}^n)$, $J = (P_1, P_2, \dots, P_m)$, we define*

$$\mathcal{V}(J) = \{w \in \mathcal{A}_0 \otimes X\colon \widehat{w}(P_k) = 0,\ k = 1, \dots, m\},$$
$$\mathcal{I}(\mathcal{V}(J)) = \{P_k \in \mathcal{P}(X)\colon \widehat{w}(P_k) = 0,\ w \in \mathcal{V}(J)\}.$$

Then Rad $J \subset \mathcal{P}(X)$ *and* $\mathcal{I}(\mathcal{V}(J)) = \operatorname{Rad} J$.

Proof. We know already that

$$\widehat{w}(P) = P\left(\sum_{j=1}^{\infty}\lambda_{\alpha_j}e_{\alpha_j}\right)\overline{P\left(\sum_{j=1}^{\infty}\lambda_{\alpha_j}e_{\alpha_j}\right)},$$

where $\lambda_{\alpha_j}, \overline{\lambda_{\alpha_j}}$ are the diagonal elements of $CA_{\alpha_j}C^{-1}$, $A_{\alpha_j} \in \mathcal{A}_0$. Then we can write

$$I(V(J)) = \left\{P_k \in \mathcal{P}_c(X)\colon P_k\left(\sum_{j=1}^{\infty}\lambda_{\alpha_j}e_{\alpha_j}\right)\right\} = \left\{P_k \in \mathcal{P}_c(X)\colon \widehat{w}(P_k) = 0, w \in \mathcal{V}(J)\right\}.$$

Hence $\mathcal{I}(\mathcal{V}(J)) = \operatorname{Rad} J$. □

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