I. E. Chyzhykov, O. A. Zolota

ON SHARPNESS OF GROWTH ESTIMATES OF CAUCHY-STIELTJES INTEGRALS IN THE UNIT DISC AND THE POLYDISC


The aim of the note is to show sharpness of results from [1] and [2] on the growth of Cauchy-Stieltjes integral in the unit disc and the polydisc. We start with notation. For \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \ n \in \mathbb{N}, \) let \( |z| = \max\{|z_j| : 1 \leq j \leq n\} \) be the polydisc norm. Denote by \( U^n = \{z \in \mathbb{C}^n : |z| < 1\} \) the unit polydisc and \( T^n = \{z \in \mathbb{C}^n : |z_j| = 1, 1 \leq j \leq n\} \) the skeleton. For \( z \in U^n, \) \( z_j = r_je^{i\psi_j}, w = (w_1, \ldots, w_n) \in T^n, \) \( w_j = e^{i\theta_j}, 1 \leq j \leq n \) we write \( C_{\alpha}(z,w) = \prod_{j=1}^n C_{\alpha_j}(z_j,w_j), \) where \( \alpha = (\alpha_1, \ldots, \alpha_n), \ \alpha_j > 0, \ 1 \leq j \leq n, \) \( C_{\alpha_j}(z_j,w_j) = \frac{1}{(1-z_j\bar{w}_j)^{\alpha_j}} \) is the generalized Cauchy kernel for the unit disc, \( C_{\alpha_j}(0,w_j) = 1. \)

For \( \psi = (\psi_1, \ldots, \psi_n) \in [-\pi; \pi]^n, \gamma = (\gamma_1, \ldots, \gamma_n) \in [0; \pi]^n \) we define the Stolz angle \( S(\psi,\gamma) = S(\psi_1,\gamma_1) \times \cdots \times S(\psi_n,\gamma_n) \) in the polydisc, where \( S(\psi_j,\gamma_j) \) is the Stolz angle for the unit disc with the vertex \( e^{i\psi_j}, \)

\[ S(\psi_j,\gamma_j) = \{z_j \in \mathbb{C} : |z_j - e^{i\psi_j}| \leq A(\gamma_j)(1-r_j), 1 \leq j \leq n, \ \ A(\gamma_j) = \sqrt{1 + 4\tan^2\gamma_j/2}. \]

Let \( \omega : \mathbb{R}_+^n \to \mathbb{R}_+ \) be a semi-additive continuous increasing function in each variable vanishing if at least one of the arguments equals zero. We call \( \omega \) a modulus of continuity. We denote \( \tau = [-\pi; \pi). \) A Borel set \( E \subset T^n \) is called a set of positive \( \omega \)-capacity if there exists a nonnegative measure \( \nu \) on \( T^n \) such that

\[ \int_E d\nu = \int_{T^n} d\nu = 1 \]

and

\[ \sup_{x \in \mathbb{R}^n} \int_{\tau^n} d\nu(e^{it_1}, \ldots, e^{it_n})/\omega(|t_1-x_1|, \ldots, |t_n-x_n|) < +\infty. \]

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Otherwise, $E$ is called a set of zero $\omega$-capacity.

Basic properties of the sets of zero $\omega$-capacity can be found in [2].

The following theorem, which generalizes a result of M. M. Sheremeta ([1]) for several variables, is proved in [2].

**Theorem A.** Let $\alpha_j > 0, \beta_j > 0, \gamma_j \in [0, \pi), 1 \leq j \leq n, n \in \mathbb{N}, \omega$ be a modulus of continuity satisfying

$$
\int_0^1 \ldots \int_0^1 \frac{\omega(t_1, \ldots, t_n)}{t_1^{\alpha_1+1} \ldots t_n^{\alpha_n+1}} dt_1 \ldots dt_n = +\infty.
$$

Let $\mu$ be a complex-valued Borel measure on $T^n$ with $|\mu|(T^n) < +\infty$. Then

$$
\left| \int_{T^n} C_\alpha(z, w) d\mu(w) \right| = o \left( \log^{n-1} \delta \cdot \frac{1}{|z_1-e^{i\psi_1}|} \ldots \frac{1}{|z_n-e^{i\psi_n}|} \right), \quad \delta \to 0^+,
$$

where $|z_j| = 1 - \delta^{\frac{1}{p_j}}, z \in S(\psi, \gamma)$, for $(e^{i\psi_1}, \ldots, e^{i\psi_n}) \in T^n$ except, possibly, a set of zero $\omega$-capacity.

In particular, for $\omega(t_1, \ldots, t_n) = t_1^{p_1} \ldots t_n^{p_n}, 0 < p_j < \alpha_j$ we have that

$$
\left| \int_{T^n} C_\alpha(z, w) d\mu(w) \right| = o \left( \log^{n-1} \delta \prod_{j=1}^n |z_j - e^{i\psi_j}|^{p_j-\alpha_j} \right) = o \left( \log^{n-1} \delta \cdot \delta^{\sum_{j=1}^n \frac{p_j-\alpha_j}{p_j}} \right), \quad \delta \to 0^+,
$$

$|z_j| = 1 - \delta^{\frac{1}{p_j}}, z \in S(\psi, \gamma)$, holds outside a set of zero $\omega$-capacity.

In the case $n = 1$, as it is shown in [1], Theorem A holds without the multiplier $\log^{\frac{1}{\delta}}$, and we have the following corollary.

**Corollary.** Let $\alpha > 0, p \in (0, [\alpha)], h \in \mathbb{R}, \mu$ be a complex-valued Borel measure on $T$ with $|\mu|(T) < +\infty$. Then

$$
\lim_{|z| \to 1, z \in S(\psi, \gamma)} \log^+ \left| \int_T C_\alpha(z, w) d\mu(w) \right| - \log(1 - |z|) \leq \alpha - p
$$

(1)

holds outside, possibly, a set of $e^{i\psi}$ of zero $\omega$-capacity, where $\omega(t) = t^p |\log t|^h$.

We construct examples which show that Theorem 2 from [1] and Theorem A are sharp in some sense.

First, we consider the case $n = 1$ in detail.

We construct a Cantor type set $E$ on the segment $[0; \frac{\pi}{2}]$ ([3]). Given a positive number $\xi < \frac{1}{2}$, on the segment $[0; \frac{\pi}{2}]$ we mark two non-overlapping “white” intervals of length $\frac{\pi}{2} \xi$, such that the first interval has its left endpoint at 0, and the second interval has its right endpoint at $\frac{\pi}{2}$. Two “white” intervals are separated by one “black” interval of length $\frac{\pi}{2} (1 - 2\xi)$. Such a dissection is called a $\xi$-dissection of the given segment. Let us remove the “black” interval. We have got the set $E_1$. On the second step we make $\xi^2$-dissection of each “white” interval left and remove “black” intervals to obtain the set $E_2 \subset E_1$. If we proceed infinitely in the same way we get a perfect set $E = \cap_{m=1}^\infty E_m$ (see [3] for details).
Let \( \omega(t) = t^s \cdot |\log t|^h \), where \( s = \log_2 2 \). We use Theorem 3 (Ch. IV [4]): let \( E \) be an \( n \)-dimensional Cantor set such that \( E_m \) is obtained at the \( m \)-th step and consists of \( 2^{mn} \) cubes with sides of lengths \( l_m \). The set \( E \) has positive \( \omega \)-capacity if and only if \( \sum_{m=1}^{\infty} \frac{2^{-mn}}{\omega(l_m)} < +\infty \).

Applying this criterion with \( n = 1 \), \( l_m = \frac{\pi}{2^m} \), we deduce that a set \( E \) has positive \( \omega \)-capacity if and only if

\[
\sum_{m=1}^{\infty} \frac{1}{|\log \frac{\pi}{2} + m\log \xi|^h} < +\infty,
\]

i.e., if \( h > 1 \) then the set \( E \) is of positive \( t^s \cdot |\log t|^h \)-capacity. The condition \( \int_0^1 t^{-\alpha-1} \omega(t) \, dt = \infty \) holds for \( \alpha \geq s \). Let \( F(t) \) be the function associated with a Cantor type set \( E \) (see [3]). We define it such that \( F(0) = 0 \), \( F(\frac{\pi}{2}) = 1 \) and extend it to \((-\infty; +\infty)\) by the formulas \( F(t) = 1 \) as \( \frac{\pi}{2} < t < +\infty \) and \( F(t) = 0 \) as \(-\infty < t < 0 \). The function \( F \) is continuous, nondecreasing, and constant on every interval contiguous to the set \( E \).

We show that on the set \( E \) of values \( \psi \), \( z = re^{i\psi} \)

\[
\left| \int_{-\pi}^{\pi} C_\alpha(z, e^{-it}) \, dF(t) \right| \geq \begin{cases} 
K_1(1-r)^{s-\alpha}, & s < \alpha, \\
K_1, & s = \alpha \end{cases} \quad r \uparrow 1
\]

holds, where \( K_1 \) is a positive constant.

For the function \( F(t) \) we define the modulus of continuity at the point \( t_0 \) by

\[
\varkappa(\lambda; t_0, F) = \sup_{|t_2 - t_0| < \lambda} |F(t_2) - F(t_0)|.
\]

In [3, p. 24] it is shown that

\[
\varkappa\left(\frac{\pi}{2}, t_0, F\right) = \left(\frac{1}{2}\right)^{m-2}.
\]

Hence, using the monotony of \( \varkappa \) for we obtain that \( \varkappa(\lambda; t_0, F) \asymp \lambda^s, \lambda \to 0^+ \).

We consider the function

\[
f(z) = \int_0^{\frac{\pi}{2}} \frac{dF(t)}{(1 - ze^{-it})^{\alpha}}, \quad |z| < 1,
\]

where \( s < \alpha \leq 1 \). It is clear that \( \omega \)-capacity of the set \( E \cap \left[0; \frac{\pi}{4}\right] \) is positive.

We estimate \( |f(z)| \) from below. For \( z = re^{i\theta}, t_0 \in E \cap \left[0; \frac{\pi}{4}\right] \) we deduce

\[
\text{Re} \left( re^{i\theta} \right) = \frac{\pi}{2} \cos \left( \alpha \arg \frac{1 - re^{-i(t-t_0)}}{1 - re^{-i(t-t_0)}^h} \right) dF(t) = \int_0^{\frac{\pi}{2}} \frac{\cos \left( \alpha \arctg \frac{r |t-t_0|}{1 - r \cos (t-t_0)} \right)}{|1 - re^{-i(t-t_0)}^h|^\alpha} dF(t).
\]

Denote \( \arctg \frac{r |t-t_0|}{1 - r \cos (t-t_0)} = \gamma \). It is clear that \( \gamma \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). Therefore, the integrated function is nonnegative. Then for \( 0 \leq |t - t_0| \leq 1 - r \) we get consequently

\[
tg \gamma = \frac{r \sin |t - t_0|}{1 - r \cos (t-t_0)} \leq \frac{r \sin (1 - r)}{1 - r} \leq 1, \quad \cos \alpha \gamma \geq \cos \left( \frac{\alpha \pi}{4} \right) > 0.
\]
Then we estimate \( |1 - re^{-i(t-t_0)}|^\alpha \) for \( |t - t_0| \leq 1 - r \):

\[
|1 - re^{-i(t-t_0)}|^\alpha = \left(1 + r^2 - 2r \cos (t - t_0)\right)^{\alpha/2} = \left((1 - r)^2 + 4r \cdot \sin^2 \left(\frac{t - t_0}{2}\right)\right)^{\alpha/2} \leq \left((1 - r)^2 + (1 - r)^2\right)^{\alpha/2} \leq K_2 \cdot (1 - r)^\alpha.
\]

Hence, we obtain that

\[
|f (re^{it_0})| \geq \int_{t_0}^{t_0+1-r} \frac{\cos(\alpha \frac{t}{2})}{|1 - re^{-i(t-t_0)}|^\alpha} dF (t) \geq K_3 \int_{t_0}^{t_0+1-r} \frac{dF (t)}{(1 - r)^\alpha} = \frac{K_3 \kappa (1 - r; t_0, F)}{(1 - r)^\alpha}.
\]

Using the asymptotic for \( \kappa \) we deduce

\[
|f (re^{it_0})| \geq \begin{cases} 
K_1 (1 - r)^{s-\alpha}, & s \leq \alpha, \\
K_1, & s = \alpha, \\
0, & t_0 \in E.
\end{cases}
\]

(2)

On the other hand, Theorem 2 from [1] yields that the inequality

\[
\int_{-\pi}^{\pi} C_\alpha (z, e^{-it}) dg (t) = o \left( \log \frac{1}{|1 - ze^{i\psi}|} \cdot |1 - ze^{i\psi}|^{s-\alpha} \right), \quad |z| \to 1
\]

holds for all \( \psi \in [-\pi; \pi] \) except, possibly, a set of zero \( t^s \log t^h \)-capacity, where \( \alpha > s > 0 \), \( g \) is a function of bounded variation on \( [-\pi; \pi] \).

It follows from (2) that

\[
\lim_{|z| \to 1, z \in \partial (E)} \log^+ \left| \frac{f_T C_\alpha (z, e^{it}) dF (t)}{-\log (1 - |z|)} \right| \geq \alpha - s
\]

holds for \( t \in E \), which shows sharpness of the corollary.

To show sharpness of Theorem A in the multidimensional case we consider the case \( n = 2 \). Let \( E = E_1 \times E_2 \), where \( E_j \) is the Cantor type set on the segment \([0; \frac{\alpha}{2}]\) constructed by using \( \xi_j \)-dissections, \( \xi_j < \frac{1}{4} \), \( j \in 1, 2 \). Let \( F_j (t_j) \) be the corresponding Cantor type function associated with the set \( E_j \). We define this function so that \( F_j (0) = 0 \), \( F_j (\frac{\alpha}{2}) = 1 \), \( F_j (t_j) = 1 \) as \( \frac{\alpha}{2} < t_j < +\infty \) and \( F_j (t_j) = 0 \) as \( -\infty < t_j < 0 \). Similarly to the one-dimensional case the modulus of continuity of \( F_j (t_j) \) satisfies \( \kappa (\lambda; t_0, F_j) \approx \lambda^{s_j} \cdot \log \lambda \). Let \( \mu_F \) be the Stieltjes measure associated with \( F_j (t_j) \) and \( \mu_F = \mu_{F_1} \otimes \mu_{F_2} \) be the product of the measures \( \mu_1 \) and \( \mu_2 \) (Ch. V, §6.2, [5]), in particular, the measure of the rectangle \([a_1, b_1] \times [a_2, b_2] \) is calculated by the formula \( \mu_F ([a_1, b_1] \times [a_2, b_2]) = (F_1 (b_1) - F_1 (a_1)) \cdot (F_2 (b_2) - F_2 (a_2)). \)

Let \( \omega (t_1, t_2) = t_1^{s_1} \log t_1^{h_1} \cdot t_2^{s_2} \log t_2^{h_2} \). Similarly to the case \( n = 1 \) the set \( E_j \) has positive \( \omega \)-capacity if and only if \( h_j > 1 \). It follows from the definition of \( \omega \)-capacity that the set \( E \) has positive \( \omega \)-capacity with \( \omega (t_1, t_2) = t_1^{s_1} \log t_1^{h_1} \cdot t_2^{s_2} \log t_2^{h_2} \) if \( h_j > 1 \).

We define the modulus of continuity of the positive measure \( \mu_F \) at the point \((t_0_1, t_0_2)\)

\[
\kappa (\lambda_1, \lambda_2; (t_0_1, t_0_2), \mu_F) = \mu_F (\{[t_0_1 - \lambda_1, t_0_1 + \lambda_1] \times [t_0_2 - \lambda_2, t_0_2 + \lambda_2]\}).
\]

Hence, \( \kappa (\lambda_1, \lambda_2; (t_0_1, t_0_2), \mu_F) \approx \lambda_1^{s_1} \cdot \lambda_2^{s_2} \), where \((t_0_1, t_0_2) \in E \).
Theorem 1. Let \( \alpha_1, \alpha_2 > 0, \) \( \alpha_1 + \alpha_2 < 1, \) \( s_j < \alpha_j, \) \( j = 1, 2. \) Then there exists a set \( E = \{ (e^{i\psi_1}, e^{i\psi_2}) \} \subset T^2 \) of positive \( \omega \)-capacity with \( \omega (t_1, t_2) = t_1^{-s_1} |\log t_1|^{\alpha_1} \cdot t_2^{-s_2} |\log t_2|^{\alpha_2}, \) a positive measure \( \mu \) on \( T^2 \) and a constant \( K_1 > 0 \) such that

\[
\left| \int_{T^2} C_\alpha (z, w) d\mu (w) \right| \geq \begin{cases} 
K_4 (1 - r_1)^{s_1 - \alpha_1} \cdot (1 - r_2)^{s_2 - \alpha_2}, & s_j < \alpha_j, \\
K_4 (1 - r_1)^{s_1 - \alpha_1}, & s_1 < \alpha_1, \\
K_4 (1 - r_2)^{s_2 - \alpha_2}, & s_2 < \alpha_2, \\
K_4, & s_1 = \alpha_1, \\
K_4, & s_j = \alpha_j,
\end{cases}
\]

where \( z_j = r_j e^{i\psi_j}, r_j \uparrow 1, s_j = \log \frac{1}{r_j}. \)

Proof. Let

\[
f (z_1, z_2) = \int_{[0, \frac{\pi}{2}]^2} \frac{d\mu_F (t_1, t_2)}{\prod_{j=1}^2 (1 - z_j e^{-i t_j})^{\alpha_j}}.
\]

For \( z_j = r_j e^{i\theta_j}, (t_01, t_02) \in E \cap \left[ 0, \frac{\pi}{4} \right]^2 \) we estimate \( |f (z_1, z_2)| \) from below similarly to the one-dimensional case:

\[
|f (z_1, z_2)| \geq \text{Re} f (z_1, z_2) = \text{Re} \int_{[0, \frac{\pi}{2}]^2} \frac{d\mu_F (t_1, t_2)}{\prod_{j=1}^2 |1 - z_j e^{-i t_j}|^{\alpha_j}} = \text{Re} \int_{[0, \frac{\pi}{2}]^2} \frac{\prod_{j=1}^2 \exp \left( i \alpha_j \arg \frac{1}{1 - z_j e^{-i t_j}} \right) d\mu_F (t_1, t_2)}{\prod_{j=1}^2 |1 - z_j e^{-i t_j}|^{\alpha_j}} = \int_{[0, \frac{\pi}{2}]^2} \frac{\cos \left( \alpha_1 \arg \frac{1}{1 - z_1 e^{-i t_1}} + \alpha_2 \arg \frac{1}{1 - z_2 e^{-i t_2}} \right) d\mu_F (t_1, t_2)}{\prod_{j=1}^2 |1 - r_j e^{-i(t_j - t_0)}|^{\alpha_j}} \geq K_5 \int_{[t_01, t_01 + 1 - r_1] \times [t_02, t_02 + 1 - r_2]} \frac{d\mu_F (t_1, t_2)}{\prod_{j=1}^2 (1 - r_1)^{\alpha_1} (1 - r_2)^{\alpha_2}} \geq K_5 \prod_{j=1}^2 (1 - r_j)^{\alpha_j} \cdot \forall (1 - r_1, 1 - r_2; (t_01, t_02), \mu_F) \times \prod_{j=1}^2 (1 - r_j)^{s_j - \alpha_j}.
\]

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Faculty of Mechanics and Mathematics,
Ivan Franko National University of Lviv
chyzykov@yahoo.com

Institute of Physics, Mathematics and Computer Science,
Drohobych Ivan Franko State Pedagogical University
o.zolota@gmail.com

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