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ON SHARPNESS OF GROWTH ESTIMATES OF CAUCHY-STIELTIES INTEGRALS IN THE UNIT DISC AND THE POLYDISC

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The examples showing sharpness of results of M. M. Sheremeta and the second author are constructed.

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Построены примеры на точность результатов М. М. Шереметы и второго автора.

The aim of the note is to show sharpness of results from [1] and [2] on the growth of Cauchy-Stieltjes integral in the unit disc and the polydisc. We start with notation. For $z=(z_1,\dots,z_n)\in\mathbb{C}^n,\ n\in\mathbb{N}$, let $|z|=\max\{|z_j|:1\le j\le n\}$ be the polydisc norm. Denote by $U^n=\{z\in\mathbb{C}^n:|z|<1\}$ the unit polydisc and $T^n=\{z\in\mathbb{C}^n:|z_j|=1,1\le j\le n\}$ the skeleton. For $z\in U^n, z_j=r_je^{i\varphi_j}, w=(w_1,\dots,w_n)\in T^n, w_j=e^{i\theta_j}, 1\le j\le n$ we write $C_\alpha(z,w)=\prod_{j=1}^n C_{\alpha_j}(z_j,w_j)$, where $\alpha=(\alpha_1,\dots,\alpha_n),\ \alpha_j>0,\ 1\le j\le n$, $C_{\alpha_j}(z_j,w_j)=\frac{1}{(1-z_j\bar w_j)^{\alpha_j}}$ is the generalized Cauchy kernel for the unit disc, $C_{\alpha_j}(0,w_j)=1$.

For $\psi=(\psi_1,\dots,\psi_n)\in[-\pi;\pi]^n,\ \gamma=(\gamma_1,\dots,\gamma_n)\in[0;\pi)^n$ we define the Stolz angle $S(\psi,\gamma)=S(\psi_1,\gamma_1)\times\ldots\times S(\psi_n,\gamma_n)$ in the polydisc, where $S(\psi_j,\gamma_j)$ is the Stolz angle for the unit disc with the vertex $e^{i\psi_j}$,

$$S(\psi_j,\gamma_j)=\{z_j\in\mathbb{C}\colon |z_j-e^{i\psi_j}|\le A(\gamma_j)(1-r_j)\},\ 1\le j\le n,\quad A(\gamma_j)=\sqrt{1+4\mathrm{tg}^2\frac{\gamma_j}{2}}.$$

Let $\omega\colon\mathbb{R}_+^n\to\mathbb{R}_+$ be a semi-additive continuous increasing function in each variable vanishing if at least one of the arguments equals zero. We call ω a modulus of continuity. We denote $\tau=[-\pi;\pi)$. A Borel set $E\subset T^n$ is called a *set of positive ω-capacity* if there exists a nonnegative measure ν on T^n such that

$$\int_E d\nu=\int_{T^n}d\nu=1$$

and

$$\sup_{x\in\mathbb{R}^n}\int_{\tau^n}\frac{d\nu(e^{it_1},\dots,e^{it_n})}{\omega(|t_1-x_1|,\dots,|t_n-x_n|)}<+\infty.$$

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Otherwise, E is called a *set of zero ω-capacity.*

Basic properties of the sets of zero ω-capacity can be found in [2].

The following theorem, which generalizes a result of M. M. Sheremeta ([1]) for several variables, is proved in [2].

Theorem A. *Let $\alpha_j > 0, \beta_j > 0, \gamma_j \in [0, \pi)$, $1 \le j \le n$, $n \in \mathbb{N}$, ω be a modulus of continuity satisfying*

$$\int_0^1 \dots \int_0^1 \frac{\omega(t_1, \dots, t_n)}{t_1^{\alpha_1+1} \cdot \dots \cdot t_n^{\alpha_n+1}} dt_1 \dots dt_n = +\infty.$$

Let μ be a complex-valued Borel measure on T^n with $|\mu|(T^n) < +\infty$. Then

$$\left| \int_{T^n} C_\alpha(z, w)\, d\mu(w) \right| = o\left(\log^n \frac{1}{\delta} \cdot \int_{|z_1 - e^{i\psi_1}|}^1 \dots \int_{|z_n - e^{i\psi_n}|}^1 \frac{\omega(t_1, \dots, t_n)\, dt_1 \dots dt_n}{t_1^{\alpha_1+1} \cdot \dots \cdot t_n^{\alpha_n+1}} \right), \quad \delta \to 0^+,$$

where $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $z \in S(\psi, \gamma)$, for $\left(e^{i\psi_1}, \dots, e^{i\psi_n}\right) \in T^n$ except, possibly, a set of zero ω-capacity.

In particular, for $\omega(t_1, \dots, t_n) = t_1^{p_1} \cdot \dots \cdot t_n^{p_n}$, $0 < p_j < \alpha_j$ we have that

$$\left| \int_{T^n} C_\alpha(z, w)\, d\mu(w) \right| = o\left(\log^n \frac{1}{\delta} \prod_{j=1}^n \left| z_j - e^{i\psi_j} \right|^{p_j - \alpha_j} \right) = o\left(\log^n \frac{1}{\delta} \cdot \delta^{\sum_{j=1}^n \frac{p_j - \alpha_j}{\beta_j}} \right), \quad \delta \to 0^+,$$

$|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $z \in S(\psi, \gamma)$, holds outside a set of zero ω-capacity.

In the case $n = 1$, as it is shown in [1], Theorem A holds without the multiplier $\log \frac{1}{\delta}$, and we have the following corollary.

Corollary. *Let $\alpha > 0$, $p \in (0, \alpha]$, $h \in \mathbb{R}$, μ be a complex-valued Borel measure on T with $|\mu|(T) < +\infty$. Then*

$$\varlimsup_{\substack{|z| \to 1, \\ z \in S(\psi, \gamma)}} \frac{\log^+ \left| \int_T C_\alpha(z, w) d\mu(w) \right|}{-\log(1 - |z|)} \le \alpha - p \tag{1}$$

holds outside, possibly, a set of $e^{i\psi}$ of zero ω-capacity, where $\omega(t) = t^p |\log t|^h$.

We construct examples which show that Theorem 2 from [1] and Theorem A are sharp in some sense.

First, we consider the case $n = 1$ in detail.

We construct a Cantor type set E on the segment $\left[0; \frac{\pi}{2}\right]$ ([3]). Given a positive number $\xi < \frac{1}{2}$, on the segment $\left[0; \frac{\pi}{2}\right]$ we mark two non-overlapping "white" intervals of length $\frac{\pi}{2}\xi$, such that the first interval has its left endpoint at 0, and the second interval has its right endpoint at $\frac{\pi}{2}$. Two "white" intervals are separated by one "black" interval of length $\frac{\pi}{2}(1 - 2\xi)$. Such a dissection is called a *ξ-dissection* of the given segment. Let us remove the "black" interval. We have got the set E_1. On the second step we make *ξ^2-dissection* of each "white" interval left and remove "black" intervals to obtain the set $E_2 \subset E_1$. If we proceed infinitely in the same way we get a perfect set $E = \bigcap_{m=1}^\infty E_m$ (see [3] for details).

Let $\omega(t) = t^s \cdot |\log t|^h$, where $s = \log_{\frac{1}{\xi}} 2$. We use Theorem 3 (Ch. IV [4]): *let E be an n-dimensional Cantor set such that E_m is obtained at the m-th step and consists of 2^{mn} cubes with sides of lengths l_m. The set E has positive ω-capacity if and only if* $\sum\limits_m \frac{2^{-mn}}{\omega(l_m)} < +\infty$.

Applying this criterion with $n = 1$, $l_m = \frac{\pi}{2}\xi^m$, we deduce that a set E has positive ω-capacity if and only if

$$\sum_{m=1}^{\infty} \frac{1}{\left|\log \frac{\pi}{2} + m\log \xi\right|^h} < +\infty,$$

i.e., if $h > 1$ then the set E is of positive $t^s \cdot |\log t|^h$-capacity. The condition $\int_0^1 t^{-\alpha-1}\omega(t)\,dt = \infty$ holds for $\alpha \geq s$. Let $F(t)$ be the function associated with a Cantor type set E (see [3]). We define it such that $F(0) = 0$, $F(\frac{\pi}{2}) = 1$ and extend it to $(-\infty;\ +\infty)$ by the formulas $F(t) = 1$ as $\frac{\pi}{2} < t < +\infty$ and $F(t) = 0$ as $-\infty < t < 0$. The function F is continuous, nondecreasing and constant on every interval contiguous to the set E.

We show that on the set E of values ψ, $z = re^{i\psi}$

$$\left| \int_{-\pi}^{\pi} C_\alpha\left(z, e^{-it}\right) dF(t) \right| \geq \begin{cases} K_1(1-r)^{s-\alpha}, & s < \alpha, \\ K_1, & s = \alpha \end{cases}, \quad r \uparrow 1$$

holds, where K_1 is a positive constant.

For the function $F(t)$ we define the modulus of continuity at the point t_0 by

$$\varkappa(\lambda; t_0, F) = \sup_{|t_j - t_0| < \lambda} |F(t_1) - F(t_2)|.$$

In [3, p. 24] it is shown that

$$\varkappa\left(\frac{\pi}{2}\xi^m;\ t_0,\ F\right) = \left(\frac{1}{2}\right)^{m-2}.$$

Hence, using the monotony of $\varkappa$ for we obtain that $\varkappa(\lambda;\ t_0,\ F) \asymp \lambda^s$, $\lambda \to 0^+$.

We consider the function

$$f(z) = \int_0^{\frac{\pi}{2}} \frac{dF(t)}{(1 - ze^{-it})^\alpha}, \quad |z| < 1,$$

where $s < \alpha \leq 1$. It is clear that ω-capacity of the set $E \cap \left[0; \frac{\pi}{4}\right]$ is positive.

We estimate $|f(z)|$ from below. For $z = re^{it_0}, t_0 \in E \cap \left[0; \frac{\pi}{4}\right]$ we deduce

$$\mathrm{Re} f\left(re^{it_0}\right) = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\alpha \arg \frac{1}{1 - re^{-i(t-t_0)}}\right)}{\left|1 - re^{-i(t-t_0)}\right|^\alpha} dF(t) = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\alpha \,\mathrm{arctg}\, \frac{r \sin|t-t_0|}{1 - r\cos(t-t_0)}\right)}{\left|1 - re^{-i(t-t_0)}\right|^\alpha} dF(t).$$

Denote $\mathrm{arctg}\, \frac{r\sin|t-t_0|}{1-r\cos(t-t_0)} = \gamma$. It is clear that $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, the integrated function is nonnegative. Then for $0 \leq |t - t_0| \leq 1 - r$ we get consequently

$$\mathrm{tg}\, \gamma = \frac{r \sin|t - t_0|}{1 - r\cos(t - t_0)} \leq \frac{r\sin(1 - r)}{1 - r} \leq 1, \quad \cos\alpha\gamma \geq \cos\left(\alpha \frac{\pi}{4}\right) > 0.$$

Then we estimate $\left|1 - re^{-i(t-t_0)}\right|^{\alpha}$ for $|t - t_0| \le 1 - r$:

$$\left|1 - re^{-i(t-t_0)}\right|^{\alpha} = \left(1 + r^2 - 2r\cos(t - t_0)\right)^{\frac{\alpha}{2}} = \left((1-r)^2 + 4r \cdot \sin^2\frac{(t-t_0)}{2}\right)^{\frac{\alpha}{2}} \le$$
$$\le \left((1-r)^2 + (1-r)^2\right)^{\frac{\alpha}{2}} \le K_2 \cdot (1-r)^{\alpha}.$$

Hence, we obtain that

$$\left|f\left(re^{it_0}\right)\right| \ge \int\limits_{t_0}^{t_0+1-r} \frac{\cos(\alpha\frac{\pi}{4})}{\left|1 - re^{-i(t-t_0)}\right|^{\alpha}} dF(t) \ge K_3 \int\limits_{t_0}^{t_0+1-r} \frac{dF(t)}{(1-r)^{\alpha}} = \frac{K_3 \varkappa(1-r; t_0, F)}{(1-r)^{\alpha}}.$$

Using the asymptotic for $\varkappa$ we deduce

$$\left|f\left(re^{it_0}\right)\right| \ge \begin{cases} K_1(1-r)^{s-\alpha}, & s < \alpha, \\ K_1, & s = \alpha \end{cases}, \quad t_0 \in E. \tag{2}$$

On the other hand, Theorem 2 from [1] yields that the inequality

$$\left|\int\limits_{-\pi}^{\pi} C_{\alpha}\left(z, e^{-it}\right) dg(t)\right| = o\left(\log^h \frac{1}{|1 - ze^{i\psi}|} \cdot \left|1 - ze^{i\psi}\right|^{s-\alpha}\right), \quad |z| \to 1$$

holds for all ψ in $[-\pi; \pi]$ except, possibly, a set of zero $t^s|\log t|^h$-capacity, where $\alpha > s > 0$, g is a function of bounded variation on $[-\pi; \pi]$.

It follows from (2) that

$$\overline{\lim_{\substack{|z|\to 1,\\ z\in S(t,\gamma)}}} \frac{\log^+\left|\int_T C_{\alpha}(z, e^{it}) dF(t)\right|}{-\log(1 - |z|)} \ge \alpha - s$$

holds for $t \in E$, which shows sharpness of the corollary.

To show sharpness of Theorem A in the multidimensional case we consider the case $n = 2$. Let $E = E_1 \times E_2$, where E_j is the Cantor type set on the segment $[0; \frac{\pi}{2}]$ constructed by using ξ_j-*dissections*, $\xi_j < \frac{1}{2}$, $j \in 1, 2$. Let $F_j(t_j)$ be the corresponding Cantor type function associated with the set E_j. We define this function so that $F_j(0) = 0$, $F_j\left(\frac{\pi}{2}\right) = 1$, $F_j(t_j) = 1$ as $\frac{\pi}{2} < t_j < +\infty$ and $F_j(t_j) = 0$ as $-\infty < t_j < 0$. Similarly to the one-dimensional case the modulus of continuity of $F_j(t_j)$ satisfies $\varkappa(\lambda;\ t_0,\ F_j) \asymp \lambda^{s_j} \cdot |\log\lambda|^{h_j}$. Let μ_{F_j} be the Stieltjes measure associated with $F_j(t_j)$ and $\mu_F = \mu_{F_1} \otimes \mu_{F_2}$ be the product of the measures μ_1 and μ_2 (Ch. V, §6.2, [5]), in particular, the measure of the rectangle $[a_1, b_1] \times [a_2, b_2]$ is calculated by the formula $\mu_F([a_1, b_1] \times [a_2, b_2]) = (F_1(b_1) - F_1(a_1)) \cdot (F_2(b_2) - F_2(a_2))$.

Let $\omega(t_1, t_2) = {t_1}^{s_1}|\log t_1|^{h_1} \cdot {t_2}^{s_2}|\log t_2|^{h_2}$. Similarly to the case $n = 1$ the set E_j has positive $\omega_j(t_j) = {t_j}^{s_j}|\log t_j|^{h_j}$-capacity if and only if $h_j > 1$. It follows from the definition of ω-capacity that the set E has positive ω-capacity with $\omega(t_1, t_2) = {t_1}^{s_1}|\log t_1|^{h_1} \cdot {t_2}^{s_2}|\log t_2|^{h_2}$ if $h_j > 1$.

We define the modulus of continuity of the positive measure μ_F at the point (t_{01}, t_{02})

$$\varkappa(\lambda_1, \lambda_2;\ (t_{01}, t_{02}), \mu_F) = \mu_F\left(\{[t_{01} - \lambda_1, t_{01} + \lambda_1] \times [t_{02} - \lambda_2, t_{02} + \lambda_2]\}\right).$$

Hence, $\varkappa(\lambda_1, \lambda_2;\ (t_{01}, t_{02}), \mu_F) \asymp {\lambda_1}^{s_1} \cdot {\lambda_2}^{s_2}$, where $(t_{01}, t_{02}) \in E$.

Theorem 1. *Let* $\alpha_1,\ \alpha_2 > 0,\ \ \alpha_1+\alpha_2 < 1,\ s_j < \alpha_j,\ j=1,2$. *Then there exists a set* $E = \left\{\left(e^{i\psi_1}, e^{i\psi_2}\right)\right\} \subset T^2$ *of positive* ω*-capacity with* $\omega(t_1,t_2) = {t_1}^{s_1}|\log t_1|^{h_1} \cdot {t_2}^{s_2}|\log t_2|^{h_2}$, *a positive measure* μ *on* T^2 *and a constant* $K_4 > 0$ *such that*

$$\left|\int\limits_{T^2} C_\alpha(z,w)\,d\mu(w)\right| \geq \begin{cases} K_4(1-r_1)^{s_1-\alpha_1}\cdot(1-r_2)^{s_2-\alpha_2}, & s_j<\alpha_j,\\ K_4(1-r_1)^{s_1-\alpha_1}, \quad s_1<\alpha_1, & s_2=\alpha_2,\\ K_4(1-r_2)^{s_2-\alpha_2}, \quad s_2<\alpha_2, & s_1=\alpha_1,\\ K_4, & s_j=\alpha_j,\end{cases}$$

where $z_j = r_j e^{i\psi_j}$, $r_j\uparrow 1$, $s_j = \log_{\frac{1}{\xi_j}} 2$.

Proof. Let

$$f(z_1,z_2) = \int\limits_{[0,\frac{\pi}{2}]^2} \frac{d\mu_F(t_1,t_2)}{\prod\limits_{j=1}^{2}(1-z_je^{-it_j})^{\alpha_j}}.$$

For $z_j = r_je^{it_{0j}}, (t_{01},t_{02}) \in E\cap\left[0;\frac{\pi}{4}\right]^2$ we estimate $|f(z_1,z_2)|$ from below similarly to the one-dimensional case:

$$|f(z_1,z_2)| \geq \mathrm{Re} f(z_1,z_2) = \mathrm{Re}\int\limits_{[0,\frac{\pi}{2}]^2} \frac{d\mu_F(t_1,t_2)}{\prod\limits_{j=1}^{2}(1-z_je^{-it_j})^{\alpha_j}} =$$

$$= \mathrm{Re}\int\limits_{[0,\frac{\pi}{2}]^2} \frac{\prod\limits_{j=1}^{2}\exp\left(i\alpha_j\arg\frac{1}{1-z_je^{-it_j}}\right)d\mu_F(t_1,t_2)}{\prod\limits_{j=1}^{2}|1-z_je^{-it_j}|^{\alpha_j}} =$$

$$= \int\limits_{[0,\frac{\pi}{2}]^2} \frac{\cos\left(\alpha_1\arg\frac{1}{1-z_1e^{-it_1}}+\alpha_2\arg\frac{1}{1-z_2e^{-it_2}}\right)d\mu_F(t_1,t_2)}{\prod\limits_{j=1}^{2}\left|1-r_je^{-i(t_j-t_{0j})}\right|^{\alpha_j}} =$$

$$\geq \int\limits_{[0,\frac{\pi}{2}]^2} \frac{\cos((\alpha_1+\alpha_2)\frac{\pi}{2})d\mu_F(t_1,t_2)}{\prod\limits_{j=1}^{2}\left|1-r_je^{-i(t_j-t_{0j})}\right|^{\alpha_j}} \geq K_5 \int\limits_{[t_{01},t_{01}+1-r_1]\times[t_{02},t_{02}+1-r_2]} \frac{d\mu_F(t_1,t_2)}{\prod\limits_{j=1}^{2}(1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2}} =$$

$$= \frac{K_5}{\prod\limits_{j=1}^{2}(1-r_j)^{\alpha_j}}\cdot\varkappa(1-r_1,1-r_2;(t_{01},t_{02}),\mu_F) \asymp \prod_{j=1}^{2}(1-r_j)^{s_j-\alpha_j}. \qquad \square$$

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