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A FUNCTIONAL REPRESENTATION OF SUPEREXTENSION MONAD AND LINKED SYSTEM MONAD

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A functional representation of inclusion hyperspace monad was constructed in [1]. The inclusion hyperspace monad contains as submonads such important monads as the superextension monad and linked system monad. We give characterizations of a functional representation for these monads.

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В работе [1] построено функциональное представление монады гиперпространств включения. Эта монада содержит в качестве подмонад монады суперрасширения и сцепленных систем. Получена характеризация функциональных представлений данных монад.

Introduction. The algebraic aspect of the theory of functors in categories of topological spaces and continuous maps was investigated rather recently. It is based, mainly, on the existence of monad (or triple) structure in the sense of S. Eilenberg and J. Moore ([2]).

Many classical constructions lead to monads: hyperspaces, spaces of probability measures, superextensions etc. There were many investigations of monads in topological categories (see, for example, [3],[4]). But it seems that the main difficulty to obtain general results in the theory of monads is due to different nature of functors.

Many monads have a functional representation, i.e., their functorial part FX can be naturally imbedded in $\mathbb{R}^{CX}$ ([1]). A functional representation of the hyperspace functor exp is given in [5]. This representation essentially uses the linear structure on function spaces. The hyperspace functor could be included in the hyperspace monad ([4]). From the algebraic point of view, hyperspaces are free Lawson semilattices. Some functional representations of the hyperspace monad which involves the semilattice structure on function spaces are given in [6], [7]. A characterization of functional representation for the inclusion hyperspace monad $\mathbb{G}$ was given in [1]. The main goal of this paper is to obtain characterizations of functional representation for the superextension monad and linked system monad.

The paper is organized as follows: in Section 1 we give some necessary definitions, in Section 2 we obtain the main results.

1. By *Comp* we denote the category whose objects are compacta (compact Hausdorff spaces) and morphisms are continuous mappings.

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We need some definitions concerning monads (see [4] for more details). A *monad* $\mathbb{T} = (T, \eta, \mu)$ in the category *Comp* consists of an endofunctor $T\colon Comp \to Comp$ and natural transformations $\eta\colon \mathrm{Id}_{Comp} \to T$ (unity), $\mu\colon T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$. (By Id_{Comp} we denote the identity functor on the category *Comp* and T^2 is the superposition $T \circ T$ of T.)

A natural transformation $\psi\colon T \to T'$ is called a *morphism* from a monad $\mathbb{T} = (T, \eta, \mu)$ to a monad $\mathbb{T}' = (T', \eta', \mu')$ if $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ \eta T' \circ T\psi$. If all the components of ψ are monomorphisms then the monad $\mathbb{T}$ is called a *submonad* of $\mathbb{T}'$ and ψ is called a *monad embedding*.

We need the definition of the monad of order-preserving functionals $\mathbb{O}$ ([8]) which contains all considered monads as submonads.

Let X be a compactum. By $C(X)$ we denote the Banach space of all continuous functions $\phi\colon X \to \mathbb{R}$ with the sup-norm. We consider $C(X)$ with the natural order, linear structure and lattice operations $\wedge$ and $\vee$ (pointwise minimum and maximum). For each $c \in \mathbb{R}$ we denote by c_X the constant function from $C(X)$ defined by the formula $c_X(x) = c$ for each $x \in X$.

A functional $\nu\colon C(X) \to \mathbb{R}$ is called *weakly additive* if for each $c \in \mathbb{R}$ and $\phi \in CX$ we have $\nu(\phi + c_X) = \nu(\phi) + c$; *normalized* if $\nu(1_X) = 1$; *order-preserving* if for each ϕ, $\psi \in C(X)$ with $\phi \leq \psi$ we have $\nu(\phi) \leq \nu(\psi)$.

For a compactum X by OX we will denote the set of all order-preserving weakly additive normalized functionals. We consider $O(X)$ as a subspace of the space $C_p(C(X))$ of all continuous functions on $C(X)$ equipped with pointwise convergence topology.

Let X, Y be compacta and let $f\colon X \to Y$ be a continuous map. Define a map $O(f)\colon O(X) \to O(Y)$ by the formula $(O(f)(\mu))(\phi) = \mu(\phi \circ f)$, where $\mu \in O(X)$ and $\phi \in C(Y)$.

We define a mapping $\nu X\colon O^2(X) \to O(X)$ by the formula $\nu X(\alpha)(g) = \alpha(\tilde{g})$, where $\alpha \in O^2(X)$, $g \in C(X)$ and the mapping $\tilde{g}\colon O(X) \to \mathbb{R}$ is defined by the formula $\tilde{g}(\gamma) = \gamma(g)$, $\gamma \in O(X)$. It is easy to check that νX is well-defined and continuous. Let us define a map $\xi X\colon X \to OX$ by the formula $\xi X(x)(\phi) = \phi(x)$, $\phi \in C(X)$, $x \in X$.

The maps ξX and νX are the components of natural transformations $\xi\colon \mathrm{Id}_{Comp} \to O$ and $\nu\colon O^2 \to O$ and the triple $\mathbb{O} = (O, \xi, \nu)$ forms a monad on the category *Comp* ([8]).

Let us describe the inclusion hyperspace monad $\mathbb{G}$ and its submonads $\mathbb{L}$ and $\mathbb{N}$. For a compactum X by hyperspace $\exp X$ we denote the set of non-void compact subsets of X endowed with the Vietoris topology.

A base of this topology consist of the sets of the form $\langle U_1, ..., U_n\rangle = \{A \in \exp X | A \subset U_1 \cup \dots \cup U_n$ and $A \cap U_i \neq \varnothing$ for every $i\}$, where $U_1, ..., U_n$ are open subsets of X.

An element $\mathcal{A} \in \exp^2 X$ is called an *inclusion hyperspace* if for each $A \in \mathcal{A}$ and $B \in \exp X$ with $A \subset B$ we have $B \in \mathcal{A}$. Let us denote $GX = \{\mathcal{A} \in \exp^2 X \mid \mathcal{A}$ is inclusion hyperspace$\}$. We consider GX as a subset of $\exp^2 X$. For a map $f\colon X \to Y$ define a map $Gf\colon GX \to GY$ by the formula $Gf(\mathcal{A}) = \{A \in \exp Y \mid f(B) \subset A$ for some $B \in \mathcal{A}\}$, $\mathcal{A} \in GX$. Define natural transformations $\eta\colon I_{Comp} \to G$ and $\mu\colon G^2 \to G$ as follows: $\eta X(x) = \{A \in \exp X \mid x \in A\}$, $x \in X$ and $\mu X(\tilde{\mathcal{A}}) = \bigcup\{\bigcap \alpha \mid \alpha \in \tilde{\mathcal{A}}\}$, where $\tilde{\mathcal{A}} \in G^2X$. The triple $\mathbb{G} = (G, \eta, \mu)$ is a monad on the category *Comp* ([9]).

By NX we denote the subspace of GX consisting of all linked systems of closed subsets of X (a system is called *linked* if the intersection of every its two elements is nonempty). A linked system is a *maximal linked system* if it is maximal with respect to the inclusion. A subspace of all maximal linked systems in NX is called the superextension of X (written λX). The subspaces λX and NX define the submonads $\mathbb{L}$ and $\mathbb{N}$ of the monad $\mathbb{G}$ (see [10] and [11]).

The map $lX\colon GX\to OX$ is defined by the formula $lX(\mathcal{A})(\phi)=\sup\{\inf\phi A|A\in\mathcal{A}\}$, $\mathcal{A}\in GX$ and $\phi\in CX$. The maps lX are components of a monad embedding $l\colon\mathbb{G}\to\mathbb{O}$ ([1]).

We say that a functional $\nu\in OX$ *weakly preserves* $\wedge$ (*weakly preserves* $\vee$) if for each $\phi\in CX$ and $c\in\mathbb{R}$ we have $\nu(\phi\wedge c_X)=\min\{\nu(\phi),c\}$ ($\nu(\phi\vee c_X)=\max\{\nu(\phi),c\}$). For each $X\in Comp$, the subset $lX(GX)$ of OX consists of all $\nu\in OX$ which weakly preserve $\wedge$ and $\vee$ ([1]).

2. Since $\mathbb{L}$ and $\mathbb{N}$ are submonads of the monad $\mathbb{G}$, the restrictions of natural transformation l define monad imbeddings of the monads $\mathbb{L}$ and $\mathbb{N}$. The problem of characterizing of images $lX(\lambda X)$ and $lX(NX)$ arises naturally.

Definition 1. We say that a functional $\nu\in OX$ *partially preserves* $\wedge$ (*partially preserves* $\vee$) if for each $\varphi\in CX$ and $\psi\in CX$ such that $\psi|_{\varphi^{-1}[\nu(\varphi);+\infty)}\equiv\mathrm{const}$ ($\psi|_{\varphi^{-1}(-\infty;\ \nu(\varphi)]}\equiv\mathrm{const}$) we have $\nu(\varphi\wedge\psi)=\min\{\nu(\varphi),\nu(\psi)\}$ ($\nu(\varphi\vee\psi)=\max\{\nu(\varphi),\nu(\psi)\}$).

Obviously, if $\nu\in OX$ partially preserves $\wedge$ (partially preserves $\vee$), then $\nu\in OX$ weakly preserves $\wedge$ (weakly preserves $\vee$).

Theorem 1. *$lX(NX)=\{\nu\in OX\mid\nu$ partially preserves $\wedge$ and weakly preserves $\vee\}$.*

Proof. We denote $nX=\{\nu\in OX\mid\nu$ partially preserves $\wedge$ and weakly preserves $\vee\}$. Let us prove the inclusion $lX(NX)\subset nX$. Consider any $\mathcal{A}\in NX$ and put $\nu_{\mathcal{A}}=lX(\mathcal{A})$. Since $NX\subset GX$, $\nu_{\mathcal{A}}$ weakly preserves $\vee$. We should prove that $\nu_{\mathcal{A}}$ partially preserves $\wedge$.

Consider any functions $\varphi,\psi\in C(X)$ such that $\psi|_{\varphi^{-1}[\nu(\varphi),+\infty)}\equiv\mathrm{const}$. We have $\nu_{\mathcal{A}}(\varphi)=\sup\{\inf\varphi(A)|A\in\mathcal{A}\}$.

Since $\mathcal{A}$ is a compact family of compact sets, there exists $A_0\in\mathcal{A}$ such that $\inf\varphi(A_0)=\nu_{\mathcal{A}}(\varphi)$. Hence $\psi|_{A_0}\equiv c$, for some $c\in\mathbb{R}$ and we obtain $\nu_{\mathcal{A}}(\psi)\geq c$. On the other hand, since $A\cap A_0\neq\varnothing$ for each $A\in\mathcal{A}$, we have $\nu_{\mathcal{A}}(\psi)\leq c$. Thus, $\nu_{\mathcal{A}}(\psi)=c$.

Let us consider two cases:

1. $\nu_{\mathcal{A}}(\varphi)\geq c$. Then $(\varphi\wedge\psi)|_{A_0}\equiv c$, so, using previous arguments, we obtain $\nu_{\mathcal{A}}(\varphi\wedge\psi)=c=\min\{\nu_{\mathcal{A}}(\varphi),\nu_{\mathcal{A}}(\psi)\}$.

2. $\nu_{\mathcal{A}}(\varphi)\leq c$. In this case $(\varphi\wedge\psi)(A_0)\subset[\nu_{\mathcal{A}},c]$ and $\inf(\varphi\wedge\psi)(A_0)=\nu_{\mathcal{A}}$. Moreover, for any $A\in\mathcal{A}$ there exists $x\in A$ such that $\varphi(x)\leq\nu_{\mathcal{A}}(\varphi)$, hence $\inf(\varphi\wedge\psi)(A)\leq\nu_{\mathcal{A}}(\varphi)$ and $\sup\{\inf(\varphi\wedge\psi)(A)|A\in\mathcal{A}\}=\nu_{\mathcal{A}}(\varphi)$. Thus we obtain $\nu_{\mathcal{A}}(\varphi\wedge\psi)=\nu_{\mathcal{A}}(\varphi)=\min\{\nu_{\mathcal{A}}(\varphi),\nu_{\mathcal{A}}(\psi)\}$.

Now we shall prove the reverse inclusion $lX(NX)\supset nX$. Consider any $\nu\in nX$, then there exists $\mathcal{A}\in GX$ such that $lX(\mathcal{A})=\nu$. We should prove that $\mathcal{A}\in NX$. Assume the contrary. Then there exist A_1, $A_2\in\mathcal{A}$, such that $A_1\cap A_2=\varnothing$. Choose open sets V_1,V_2 such that $A_1\subset V_1$, $A_2\subset V_2$ and $V_1\cap V_2=\varnothing$.

Now we can choose a continuous function $\varphi_1\colon X\to[0,1]$ such that $\varphi_1(A_1)\subset\{1\}$ and $\varphi_1(X\backslash V_1)\subset\{0\}$. Then we have $\nu(\varphi_1)=1$. Choose another continuous function $\varphi_2\colon X\to[0,1]$, with the properties $\varphi_2(A_2)\subset\{1\}$ and $\varphi_2(X\backslash V_2)\subset\{0\}$. Then $\nu(\varphi_2)=1$.

On the other hand, we have $\varphi_1\wedge\varphi_2\equiv 0$, and then $\nu(\varphi_1\wedge\varphi_2)=0\neq\nu(\varphi_1)\wedge\nu_{\mathcal{A}}(\varphi_2)$, which is a contradiction. □

Lemma 1. *Let $\mathcal{A}\in NX$ and $A_0\in\exp X$ such that $A_0\notin\mathcal{A}$. Then there exists an open set $V\subset X$ such that $A_0\subset V$ and for any $A\in\mathcal{A}$ we have $A\setminus V\neq\varnothing$.*

Proof. Since $\mathcal{A}$ is a closed subset of $\exp X$ we can choose a basic open neighborhood $\langle V_1,\dots,V_k\rangle$ of A_0 in $\exp X$ such that $\mathcal{A}\cap\langle V_1,...,V_k\rangle=\varnothing$. Put $V=\bigcup_{i=1}^k V_i$. Then we have $A_0 \subset V$. Consider any $A \in \mathcal{A}$. Since $\mathcal{A}$ is an inclusion hyperspace, $A \cup A_0 \in \mathcal{A}$, hence $(A \cup A_0) \notin \langle V_1, ..., V_k \rangle$. On the other hand, $(A \cup A_0) \cap V_i \supset A_0 \cap V_i \neq \varnothing$ for each i. Hence, $A \setminus V = (A \cup A_0) \setminus V \neq \varnothing$. □

Theorem 2. *$lX(\lambda X) = \{\nu \in OX \mid \nu$ partially preserves $\wedge$ and $\vee\}$.*

Proof. Denote $sX = \{\nu \in OX \mid \nu$ partially preserves $\wedge$ and $\vee\}$. Let us prove the inclusion $sX \subset lX(\lambda X)$. Consider any $\nu \in sX$. Since $sX \subset nX$, by Theorem 1, there exists $\mathcal{A} \in NX$ such that $\nu = lX(\mathcal{A})$. We should prove that the linked system $\mathcal{A}$ is maximal with respect to the inclusion.

Assume the contrary. Then there exists $A_0 \in \exp X$ such that for any $A \in \mathcal{A}$ we have $A_0 \cap A \neq \varnothing$ and $A_0 \notin \mathcal{A}$. By Lemma 1, there exists an open set $V \subset X$ such that $A_0 \subset V$ and for any $A \in \mathcal{A}$ we have $A \setminus V \neq \varnothing$.

Let $\varphi\colon X \to [0;1]$ be a continuous function such that $\varphi(A_0) \subseteq 0$ and $\varphi(X \setminus V) \subseteq \{1\}$. We can choose a continuous function $\psi\colon X \to [0;1]$ such that and $\psi(\varphi^{-1}(\{0\})) \subseteq 1$ and $\psi(X \setminus V) \subseteq \{0\}$. Then we have $(\varphi \vee \psi)(x) > 0$ for each $x \in X$. Since X is compact, there exists $a > 0$ such that $(\varphi \vee \psi)(x) \geq a$ for each $x \in X$, hence $\nu(\varphi \vee \psi) \geq a$.

On the other hand, since any element $A \in \mathcal{A}$ has non-empty intersection with A_0 and $\varphi|_{A_0} \equiv 0$, we have $\nu(\varphi) = \sup\{\inf \varphi(A) | A \in \mathcal{A}\} = 0$. Similarly, since any element $A \in \mathcal{A}$ has non-empty intersection with $X \setminus V$ and $\psi|_{X \setminus V} \equiv 0$, we have $\nu(\psi) = 0$. Since ν partially preserves $\vee$, we have $\nu(\varphi \vee \psi) = 0$ and we obtain a contradiction.

Now, let us prove the inclusion $lX(\lambda X) \subset sX$. Consider any $\mathcal{A} \in \lambda X$ and put $\nu_{\mathcal{A}} = lX(\mathcal{A})$. Since $\lambda X \subset NX$, the functional $\nu_{\mathcal{A}}$ partially preserves $\wedge$. We have to prove the equality $\nu_{\mathcal{A}}(\varphi \vee \psi) = \max\{\nu_{\mathcal{A}}(\varphi), \nu_{\mathcal{A}}(\psi)\}$ for any functions $\varphi, \psi \in C(X)$ such that

$$\psi|_{\varphi^{-1}(-\infty,\ \nu_{\mathcal{A}}(\varphi)]} \equiv c$$

for some $c \in \mathbb{R}$. Since $\nu_{\mathcal{A}}$ preserves order, we have $\nu_{\mathcal{A}}(\varphi \vee \psi) \geq \max\{\nu_{\mathcal{A}}(\varphi), \nu_{\mathcal{A}}(\psi)\}$. Let us prove the inequality $\nu_{\mathcal{A}}(\varphi \vee \psi) \leq \max\{\nu_{\mathcal{A}}(\varphi), \nu_{\mathcal{A}}(\psi)\}$.

Put $B = \varphi^{-1}(-\infty, \nu_{\mathcal{A}}(\varphi)])$. Since $\nu_{\mathcal{A}}(\varphi) = \sup\{\inf \varphi(A) | A \in \mathcal{A}\}$, we have $B \cap A \neq \varnothing$ for each $A \in \mathcal{A}$. Since the system $\mathcal{A}$ is maximal, $B \in \mathcal{A}$. We have $\psi|_B \equiv c$, hence $\nu_{\mathcal{A}}(\psi) = c$.

Consider the following two cases:

1. $\nu_{\mathcal{A}}(\varphi) \leq c$. Then we have $(\varphi \vee \psi)(b) \leq c$ for each $b \in B$, hence $\nu_{\mathcal{A}}(\varphi \vee \psi) \leq c = \max\{\nu_{\mathcal{A}}(\varphi), \nu_{\mathcal{A}}(\psi)\}$.

2. $\nu_{\mathcal{A}}(\varphi) \geq c$. Then we have $(\varphi \vee \psi)(b) \leq \nu_{\mathcal{A}}$ for each $b \in B$, hence $\nu_{\mathcal{A}}(\varphi \vee \psi) \leq \nu_{\mathcal{A}} = \max\{\nu_{\mathcal{A}}(\varphi), \nu_{\mathcal{A}}(\psi)\}$. □

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