УДК 517.98, 519.1

N. A. Rozhenko

ON CHARACTERISTIC FUNCTIONS OF EQUILATERAL REGULAR STAR-TREES

N. A. Rozhenko. *On characteristic functions of equilateral regular star-trees*, Mat. Stud. **37** (2012), 65–75.

A spectral problem generated by the Sturm-Liouville equation on the edges of a regular equilateral finite star-tree with the Dirichlet boundary conditions at the pendant vertices and continuity and Kirchhoff's conditions at the interior vertices is considered. The potential in the Sturm-Liouville equations on the edges is the same on each edge and symmetric with respect to the edge midpoint. The structure of the function whose set of zeros coincides with the spectrum of such a problem (characteristic function) is described.

Н. А. Роженко. *О характеристических функциях регулярных симметрических деревьев* // Мат. Студії. – 2012. – Т.37, №1. – С.65–75.

Рассматривается спектральная задача, порожденная уравнением Штурма-Лиувилля на ребрах регулярного симметрического конечного дерева с граничными условиями Дирихле на висячих вершинах и с условиями непрерывности и Киркгоффа на внутренних вершинах. Потенциал в уравнениях Штурма-Лиувилля одинаковый на всех ребрах и симметричный относительно середины ребра. Описана структура функций, множество нулей которых совпадает со спектром задачи (характеристических функций).

1. Introduction. Quantum graph usually means quasi-one-dimensional manifold with self-adjoint differential operator on it (see [11]). Spectral and scattering problems of Sturm-Liouville or Dirac equation on such structures attract wide interest during last years because they provide relevant models of nanostructures, see [12], [13]. Usually Dirichlet or Neumann conditions are imposed at the pendant vertices of a metric graph and continuity conditions together with Kirchhoff-type conditions are stated at the interior vertices.

Among the variety of graphs important role is played by the so-called regular trees ([24]) (see Definition 1 below). The correspondence between the spectra for the discrete (combinatorial) and continuous Laplacians for a large class of so-called equilateral graphs (graphs the lengths of edges of which are equal) was given in [8], [4], [6]. It was proved there that the spectrum of the continuous Laplacian on the graph with $q_j(x) \equiv 0$, except of the branch $\frac{\pi^2 n^2}{l^2}$ (l is length of an edge, $n = 1, 2, ...$), consists of values of z such that $\cos\sqrt{z}$ belongs to the spectrum of the corresponding discrete Laplacian. The spectra of quantum graphs with external potentials were studied also in many papers (see [7], [10], [3], [12], [24], [25] and references therein). In [6] for $q_j(x) \equiv 0$ and in [18] under the assumption of the potential on the edges to be the same it was shown that for any such L_2 potential and δ-type

2010 *Mathematics Subject Classification:* 34B24, 34B45, 34L20.
Keywords: eigenvalue, characteristic function, multiplicity, recurrent relations, Dirichlet boundary conditions, Kirchhoff conditions.

boundary conditions at the vertices, up to the Dirichlet spectra on the edges, the spectrum of a quantum graph has the form $\eta^{-1}(\sigma(2\Delta))$, where $\sigma(2\Delta)$ is the spectrum of 2Δ, Δ is the corresponding discrete Laplacian, and η is the discriminant (Lyapunov function) of the Hill equation on an edge. In [9] it is shown that if the graph is not cyclically connected, then the maximal multiplicity of an eigenvalue of the corresponding operator is $\mu + g^T - p_{in}^T$, where μ is the cyclomatic number of the graph, and g^T and p_{in}^T are the number of edges and the number of interior vertices, respectively, for the tree obtained by contracting all the cycles of the graph into vertices. If the graph is cyclically connected, then the maximal multiplicity of an eigenvalue is $\mu + 1$.

We consider an equilateral oriented regular tree with the edges directed towards the root with indegree (number of incoming into a vertex edges) equal n. We assume that the potential on the edges is symmetric with respect to the midpoint of the edge and impose the Dirichlet conditions at the pendant vertices of our tree. Our aim is to describe the structure of the function whose set of zeros coincides with the spectrum of such a problem (characteristic function) and to obtain the spectrum of such problem. In this paper we generalize results for $n = 2$ from [21] to the case of an arbitrary positive integer n.

In Subsection 1.2 we describe the spectral problem on a regular equilateral star-tree S_m^n with the combinatorial distance m from the root to a pendant vertex and in Subsection 1.3 we give an operator interpretation of it which immediately implies that the spectrum of our problem is real.

In Section 2 we introduce the notion of a characteristic function, i.e. the function whose set of zeros coincides with the spectrum of our spectral problem.

In Section 3, under our assumptions, we show that the characteristic function can be given in the form $\phi(\lambda) = s^{z_m-1}(\lambda, a)c^{x_m}(\lambda, a)P_N^{(m)}(c^2(\lambda, a))$ where $s(\lambda, x)$ and $c(\lambda, x)$ are sine- and cosine-like solutions of the Sturm-Liouville equation on an edge, a is the length of an edge, m is the combinatorial distance from the root to a pendant vertex, z_m and x_m are integers, $P_N^{(m)}(z)$ is a certain polynomial. Solving the corresponding recurrent relations we obtain an explicit formulae for z_m, x_m and $P_N^{(m)}(z)$.

In Section 4 we apply the results of Section 3 to describe the spectrum of our problem and show that the spectrum is the union of the subsequences which are the zeros of the functions $s(\lambda, a)$, $c(\lambda, a)$, $(c(\lambda, a) - \sqrt{z_p})$ and $(c(\lambda, a) + \sqrt{z_p})$ where z_p are the zeros of the polynomial $P_N^{(m)}(z)$.

1.1. Main notions and auxiliary results. Let $\mathbf{v}$ be the root of a tree. For any vertex v its *generation* $\text{gen}(v)$ is defined as the combinatorial distance from the root (the number of edges in the path connecting $\mathbf{v}$ with v), in particular $\text{gen}(\mathbf{v}) = 0$. For any edge e emanating from v we define the generation as $\text{gen}(e) = \text{gen}(v)$.

Definition 1. A rooted metric tree is called *regular* if all the vertices of the same generation have equal indegrees and all the edges of the same generation are of the same length.

In this paper we deal with the following family of regular equilateral oriented trees which we call *symmetric star-trees* S_m^n.

By S_1^n we denote a tree consisting of n edges joint at the root (star graph). We direct the edges towards the root. By S_m^n we mean the tree obtained from S_{m-1}^n by attaching n edges to each of the pendant vertices of S_{m-1}^n. Each edge is directed towards the root. Thus, the degree of the root is n and the degree of each other interior vertex is $n+1$. The number of edges in our tree is $g_m = \frac{n(n^m-1)}{n-1}$. We enumerate the edges such that the distance to the root is non-increasing sequence of the edge indices. Our enumeration is arbitrary in other respects.

The local coordinates on S_m^n identifies each directed edge e_j of S_m^n with the interval $[0, a]$ and the coordinate x increases in the direction of the edge.

1.2. Spectral problem. Let the Sturm-Liouville equation

$$-y_j'' + q(x)y_j = \lambda^2 y_j, \quad x \in [0, a], \quad j = 1, 2, ..., g_m, \tag{1}$$

be defined on each edge of our tree. We impose the Dirichlet boundary conditions

$$y_j(0) = 0 \tag{2}$$

at each pendant vertex. At each interior vertex v_i which is not a root we impose the continuity conditions

$$y_{j_1}(a) = y_{j_2}(a) = ... = y_{j_n}(a) = y_j(0) \tag{3}$$

and Kirchhoff's condition

$$y'_{j_1}(a) + y'_{j_2}(a) + ... + y'_{j_n}(a) - y'_j(0) = 0. \tag{4}$$

Here j is the number of the edges outgoing away from the interior vertex v_i and $j_1, j_2, ..., j_n$ are the numbers of the edges incoming into v_i.

At the root we have

$$y_{g_m}(a) = y_{g_m-1}(a) = ... = y_{g_m-n+1}(a), \tag{5}$$

$$y'_{g_m}(a) + y'_{g_m-1}(a) + ... + y'_{g_m-n+1}(a) = 0. \tag{6}$$

Except of quantum mechanics, this problem occurs also in the theory of small transversal vibrations of nets of strings.

1.3. Differential operator. Let us equip each edge e_j $(j = 1, 2, ..., g_m)$ with a real-valued function q which belongs to $L_2(0, a)$. Now we introduce the operator $\mathcal{L}$ which we associate with the tree S_m^n equipped with the function q. First we introduce the Sturm-Liouville operation p_j on each edge e_j. Let the domain $D(p_j)$ of the differential operation p_j be the set of functions f continuous at e_j (on $[0, a]$) which possess absolutely continuous derivatives f' and therefore f'' exists a.e. on $[0, a]$. For $f \in D(p_j)$ we define the operation p_j by the equation

$$(p_j f)(x) = -\frac{d^2 f(x)}{dx^2} + q(x)f(x) \ \text{ a.e. on } \ [0, a], \ j = 1, 2, ..., g_m.$$

Let us consider vector-functions $F(x) = (f_1(x), f_2(x), ..., f_{g_m}(x))$ whose components are defined on $[0, a]$. The set of vector-functions F such that $f_j \in L^2(0, a)$, $j = 1, 2, ..., g_m$ we denote by H. Defining multiplication by constant and addition in the usual way we equip H with the inner product

$$(F, B)_H = \sum_{j=1}^{g_m} \int_0^a f_j(x)\overline{b_j(x)}dx,$$

where $B = (b_1(x), b_2(x), ..., b_{g_m}(x)) \in H$. Thus, H is a Hilbert space. It is easy to see that this space is separable.

Let $D(P)$ be the set of vector-functions $Y(x) = (y_1(x), y_2(x), ..., y_{g_m}(x))$, where $y_j \in D(p_j)$ $(j = 1, 2, ..., g_m)$. For $Y \in D(P)$ we define the operation P by the equation

$$P(Y) = ((p_1 y_1)(x), (p_2 y_2)(x), ..., (p_{g_m} y_{g_m})(x)).$$

Let J be the set of numbers of the edges incident with pendant vertices, K be the set of numbers of interior vertices, $W_i^+ = \{j\}$ is the number of the edges outgoing away from the vertex v_i and $W_i^- = \{j_1, j_2, ..., j_n\}$ the set of numbers of edges incoming into the vertex v_i $(i = 1, 2, ..., p)$, $p = g_m + 1$.

Now we are ready to construct the operator $\mathcal{L}$. Its domain D is the set of vector-functions $F = F(x) = (f_1(x), f_2(x), ..., f_{g_m}(x))$ such that:

1) $F \in (H \cap D(P))$;
2) $P(F) \in H$;
3) if v_i is a pendant vertex and $W_i^+ = \{j\}$ $(W_i^- = \varnothing)$ then $f_j(0) = 0$;
4) (continuity condition) for each $i \in K$, $j_1, j_2, ..., j_n \in W_i^-$ and $j \in W_i^+$:

$$f_{j_1}(a) = f_{j_2}(a) = ... = f_{j_n}(a) = f_j(0);$$

5) (Kirchhoff condition) for each $i \in K$, $j_1, j_2, ..., j_n \in W_i^-$ and $j \in W_i^+$:

$$\left.\frac{df_{j_1}(x)}{dx}\right|_{x=a} + \left.\frac{df_{j_2}(x)}{dx}\right|_{x=a} + ... + \left.\frac{df_{j_n}(x)}{dx}\right|_{x=a} = \left.\frac{df_j(x)}{dx}\right|_{x=0}. \tag{7}$$

By $\mathcal{L}$ we denote the operator acting in H according to $\mathcal{L}F = P(F)$ with the domain D.

It is easy to see that $\mathcal{L}$ is a selfadjoint operator in H (see, i.e. [13], [2], [5]). Since all the edges are of finite length and the real function $q \in L_2(0, a)$, the spectrum of $\mathcal{L}$ is discrete, i.e. consists of normal (isolated Fredholm) eigenvalues which accumulate only at infinity.

It should be mentioned that more general than (7) selfadjoint matching conditions were considered in [22].

1.4. Auxiliary results. In the sequel we assume that the potential is symmetric with respect to the midpoint of an edge:

Assumption. $q(x) \overset{a.e.}{=} q(a - x)$.

Consider the solution $s(\lambda, x)$ of the equation

$$-y'' + q(x)y = \lambda^2 y \tag{8}$$

which satisfies the conditions $s(\lambda, 0) = s'(\lambda, 0) - 1 = 0$ and the solution $c(\lambda, x)$ of (8) which satisfies $c(\lambda, 0) - 1 = c'(\lambda, 0) = 0$.

Due to our Assumption the following proposition is true.

Proposition 1. $s'(\lambda, a) \equiv c(\lambda, a)$.

Proof. It is known (see [17]) that $c(\lambda, a)$ is an even entire exponential type function of λ which can be presented as

$$c(\lambda, a) = \cos \lambda a + \frac{K}{\lambda} \sin \lambda a + \frac{\psi_1(\lambda)}{\lambda}, \tag{9}$$

where $K = \frac{1}{2}\int_0^a q(x)dx$, $\psi_1 \in L^a$ and L^a is the class of entire functions of exponential type $\leq a$ which belong to $L_2(-\infty, \infty)$ for real values of its argument.

Similar representation is true (see [17]) for $s'(\lambda, a)$:

$$s'(\lambda, a) = \cos \lambda a + \frac{K}{\lambda} \sin \lambda a + \frac{\psi_2(\lambda)}{\lambda}, \tag{10}$$

where $\psi_2 \in L^a$.

The set of zeros of $c(\lambda, a)$ is nothing but the spectrum of the problem

$$-y'' + q(x)y = \lambda^2 y, \quad y'(0) = y(a) = 0. \tag{11}$$

The set of zeros of $s'(\lambda, a)$ is the spectrum of the problem

$$-y'' + q(x)y = \lambda^2 y, \quad y(0) = y'(a) = 0. \tag{12}$$

Due to the Assumption, the spectra of problems (11) and (12) coincide. This means that the sets of zeros of $c(\lambda, a)$ and $s'(\lambda, a)$ coincide and due to (9) and (10) we conclude that $c(\lambda, a) \equiv s'(\lambda, a)$. □

Corollary 1. *If λ_k is a zero of $s(\lambda, a)$, then*

$$c(\lambda_k, a) = s'(\lambda_k, a) = (-1)^k. \tag{13}$$

Proof. From the Lagrange identity

$$c(\lambda, a)s'(\lambda, a) - s(\lambda, a)c'(\lambda, a) = 1$$

in view of Proposition 1 we obtain

$$c'(\lambda, a)s(\lambda, a) = c^2(\lambda, a) - 1. \tag{14}$$

This implies $c^2(\lambda_k, a) = 1$. Taking into account interlacing of the zeros of $c(\lambda, a)$ with the zeros of $s(\lambda, a)$ we arrive at (13). □

2. Characteristic functions for graphs. The *characteristic function* of the graph is an entire function whose set of zeros coincides with the spectrum of the corresponding operator $\mathcal{L}$. It is known that it can be expressed via functions $s(\lambda, a)$, $s'(\lambda, a)$, $c(\lambda, a)$ and $c'(\lambda, a)$. To show it for our star-tree S_m^n we introduce the following set of $2g_m$-dimensional vector-functions $\psi_j(\lambda, x) = \mathrm{col}\{0, 0, ...0, s(\lambda, x), 0, ..., 0\}$, where $s(\lambda, x)$ stands on the j-th position ($j = 1, 2, ..., g_m$) and $\psi_{j+g_m}(\lambda, x) = \mathrm{col}\{0, 0, ...0, c(\lambda, x), 0, ..., 0\}$, where $c(\lambda, x)$ stands on the $(j + g_m)$-th position. As in [22], we denote by L_j ($j = 1, 2, ..., 2g_m$) the linear functionals generated by the boundary and matching conditions (2)–(6). Then we obtain the matrix $\Phi(\lambda) \overset{def}{=} \|L_j(\psi_k(\lambda, x))\|_{j,k}^{2g_m}$. We define the Neumann characteristic function by setting

$$\phi_N(\lambda) \overset{def}{=} \det(\Phi(\lambda)).$$

We choose an interior vertex v_i and replace Kirchhoff's condition (4) at this vertex with

$$y_i(0) = 0.$$

Then we obtain a new matrix $\tilde{\Phi}(\lambda, v_i) \overset{def}{=} \|\tilde{L}_j(\psi_k(\lambda, x))\|_{j,k}^{2g_m}$. We introduce the Dirichlet characteristic function with respect to v_i:

$$\phi_D(\lambda, v_i) \overset{def}{=} \det(\tilde{\Phi}(\lambda, v_i)).$$

We will use the following result from [14] (it should be mentioned that a similar result was obtained earlier in [26] and for star graphs in [19], [20]): any chosen interior vertex v_i can be considered as a separating vertex which divides the tree T into two subtrees T_1 and T_2. Denote by $\phi_N^I(\lambda)$ the Neumann characteristic function of T_1 and by $\phi_D^I(\lambda, v_i)$ the Dirichlet characteristic function of T_1 with respect to v_i (i.e. with the Dirichlet boundary conditions at v_i), by $\phi_N^{II}(\lambda)$ and $\phi_D^{II}(\lambda, v_i)$ the Neumann characteristic function and Dirichlet characteristic functions with respect to v_i for T_2. Then the Neumann characteristic function $\phi_N(\lambda)$ of T and the Dirichlet characteristic function $\phi_D(\lambda, v_i)$ of T with respect to v_i can be obtained from

$$\phi_N(\lambda) = \phi_N^I(\lambda)\phi_D^{II}(\lambda, v_i) + \phi_N^{II}(\lambda)\phi_D^I(\lambda, v_i), \tag{15}$$

$$\phi_D(\lambda, v_i) = \phi_D^I(\lambda, v_i)\phi_D^{II}(\lambda, v_i). \tag{16}$$

3. Characteristic functions for symmetric star-trees. For the star-tree S_m^n we denote by $\phi_N^{(m)}(\lambda)$ the corresponding Neumann characteristic function and by $\phi_D^{(m)}(\lambda)$ the corresponding Dirichlet characteristic function with respect to the root. Being interested in the Neumann characteristic functions we consider for the sake of convenience in parallel the Dirichlet characteristic functions (with Dirichlet boundary conditions at the root $\mathbf{v}$, i.e. at the central vertex). Direct calculations show that

$$\phi_N^{(1)}(\lambda) = s^{n-1}(\lambda, a)c(\lambda, a),\ \ \phi_D^{(1)}(\lambda) = s^n(\lambda, a). \tag{17}$$

Using (15), (16) and (14) we derive from (17):

$$\phi_N^{(2)}(\lambda) = 2^{n-1}s^{n^2-1}(\lambda, a)c^{n-1}(\lambda, a)(2c^2(\lambda, a) - 1),\quad \phi_D^{(2)}(\lambda) = 2^n s^{n^2}(\lambda, a)c^n(\lambda, a). \tag{18}$$

It is clear that

$$\begin{gathered}\phi_N^{(3)}(\lambda) = 2^{n(n-1)}s^{n^3-1}(\lambda, a)c^{n(n-1)+1}(\lambda, a)(4c^2(\lambda, a) - 3)(4c^2(\lambda, a) - 1)^{n-1},\\ \phi_D^{(3)}(\lambda) \overset{def}{=} \phi_D^{(1)}(\lambda, \mathbf{v}) = 2^{n(n-1)}s^{n^3}(\lambda, a)c^{n(n-1)}(\lambda, a)(4c^2(\lambda, a) - 1)^n;\end{gathered} \tag{19}$$

$$\begin{gathered}\phi_N^{(4)}(\lambda) = 2^{n^2(n-1)}s^{n^4-1}(\lambda, a)c^{(n-1)(n^2+1)}(\lambda, a)(4c^2(\lambda, a) - 1)^{n(n-1)}\times\\ \times(8c^2(\lambda, a) - 4)^{n-1}(c^2(\lambda, a)(4c^2(\lambda, a) - 3) + (c^2(\lambda, a) - 1)(4c^2(\lambda, a) - 1)),\\ \phi_D^{(4)}(\lambda) \overset{def}{=} \phi_D^{(4)}(\lambda, \mathbf{v}) = 2^{n^2(n-1)}s^{n^4}(\lambda, a)c^{n^2(n-1)+n}(\lambda, a)\times\\ \times(4c^2(\lambda, a) - 1)^{n(n-1)}(8c^2(\lambda, a) - 4)^n.\end{gathered} \tag{20}$$

Theorem 1. *1) For each positive integer m one has*

$$\phi_N^{(m)}(\lambda) = s^{z_m-1}(\lambda, a)c^{x_m}(\lambda, a)P_N^{(m)}(c^2(\lambda, a)), \tag{21}$$

where

$$z_m = n^m, \tag{22}$$

$$x_m = \begin{cases} \frac{n^m+1}{n+1}, & \text{if } m \text{ is odd},\\ \frac{n^m-1}{n+1}, & \text{if } m \text{ is even}\end{cases}$$

and $P_N^{(m)}(z)$ is a polynomial of degree $\frac{n^{m+1}-1}{n-1} - z_m - x_m$.

2) For each positive integer m one has

$$\phi_D^{(m)}(\lambda) = s^{z_m}(\lambda, a)c^{y_m}(\lambda, a)P_D^{(m)}(c^2(\lambda, a)), \tag{23}$$

where

$$y_m = \begin{cases} \frac{n^m-n}{n+1}, & \text{if } m \text{ is odd},\\ \frac{n^m+n}{n+1}, & \text{if } m \text{ is even}\end{cases}$$

and $P_D^{(n)}(z)$ is a polynomial of degree $\frac{n^{m+1}-n}{n-1} - z_m - y_m$.

Proof. A root is a separating vertex which divides S_m^n into n identical subgraphs. Therefore, formulae (15) and (16) give us

$$\phi_N^{(m)}(\lambda) = n\psi_N^{(m)}(\lambda)(\psi_D^{(m)}(\lambda))^{n-1}, \tag{24}$$

$$\phi_D^{(m)}(\lambda) = (\psi_D^{(m)}(\lambda))^n. \tag{25}$$

Here $\psi_N^{(m)}(\lambda)$ is the characteristic function of any identical subgraph of S_m^n with the Neumann condition at the root whereas $\psi_D^{(m)}(\lambda)$ is the characteristic function of the same subgraph with the Dirichlet condition at the root. Again using (15) and (16) we obtain

$$\psi_N^{(m)}(\lambda) = c(\lambda, a)\phi_N^{(m-1)}(\lambda) + c'(\lambda, a)\phi_D^{(m-1)}(\lambda), \tag{26}$$

$$\psi_D^{(m)}(\lambda) = s(\lambda, a)\phi_N^{(m-1)}(\lambda) + s'(\lambda, a)\phi_D^{(m-1)}(\lambda). \tag{27}$$

Representations (21) and (23) with some natural degrees x_n, y_n and z_n follow by induction if we substitute (26) and (27) into (24) and (25) and make use of (14).

Let us prove (22). Using (26) and (27) we obtain from (24) $z_m = z_{m-1}n$ and from (17): $z_1 = n$, i.e. $z_m = n^m$. It is easy to check that the degree of $s(\lambda, a)$ in (21) is $z_m - 1 = n^m - 1$.

Let us find x_m and y_m. Substituting (26) and (27) into (24) and using (21) and (23) we obtain

$$\begin{aligned} x_m = \min\{&nx_{m-1} + 1, y_{m-1} + (n-1)x_{m-1}, 2y_{m-1} + (n-2)x_{m-1} + 1, ...\\ &..., (n-1)y_{m-1} + x_{m-1} + (n-2), ny_{m-1} + (n-1)\}. \end{aligned} \tag{28}$$

Substituting (26) and (27) into (25) and using (21) and (23) we obtain

$$\begin{aligned} y_m = \min\{&nx_{m-1}, (n-1)x_{m-1} + y_{m-1} + 1, (n-2)x_{m-1} + 2y_{m-1} + 2, ...\\ &..., x_{m-1} + (n-1)y_{m-1} + (n-1), ny_{m-1} + n\}. \end{aligned} \tag{29}$$

To solve recurrent relations (28), (29) with the initial conditions $x_1 = 1$, $y_1 = 0$ which follow from (17) we notice that $x_m \neq y_m$ for each m. To prove it let us suppose that $x_m = y_m$ some m. Then we consider the following two cases.

1. $x_{m-1} < y_{m-1}$, then (28) and (29) imply $x_m = nx_{m-1} + 1$ and $y_m = nx_{m-1}$. Therefore, $x_m > y_m$, a contradiction.
2. $x_{m-1} > y_{m-1}$, then from (28) and (29) we have $x_m = ny_{m-1} + (n-1) < ny_{m-1} + n = y_m$, a contradiction.

Thus, we conclude that $x_m \neq y_m$ for each m. Let $x_m > y_m$, i.e $x_m \geq y_m + 1$, for some natural m. Then (28) implies $x_{m+1} = ny_m + (n-1)$ and (29) implies $y_{m+1} = ny_m + n$ and, therefore, $x_{m+1} = y_{m+1} - 1$. On the other hand, if $y_m > x_m$, i.e. $y_m \geq x_m + 1$ then (28) implies $x_{m+1} = nx_m + 1$ and (29) implies $y_{m+1} = nx_m$ and, therefore $y_{m+1} = x_{m+1} - 1$. Taking into account that $x_1 = 1$ and $y_1 = 0$ we conclude that $x_{2m} < y_{2m}$ and $x_{2m-1} > y_{2m-1}$.

Let us consider the two cases.

1. m is even. Then $x_{m+1} = nx_m + 1$ and $y_{m+1} = x_{m+1} - 1$ and, therefore, (29) implies $y_{m+1} = nx_m = n^2y_{m-1} + n(n-1)$ what gives us $y_{m+1} - n^2y_{m-1} - n(n-1) = 0$. The solution of this recurrent relation satisfying the initial conditions $y_2 = n$ (see (18)) and $y_4 = n^3 - n^2 + n$ (see (20)) is $y_{m+1} = (n-1)(n + n^3 + ... + n^{m-1})$. Consequently, $x_{m+1} = (n-1)(n + n^3 + ... + n^{m-1}) + 1$, i.e.

$$x_{m+1} = \frac{n^{m+1} + 1}{n+1}, \quad y_{m+1} = \frac{n^{m+1} - n}{n+1}.$$

2. m is odd. Then $x_m > y_m$, $x_{m+1} = ny_m + n - 1$ and $y_{m+1} = ny_m + n$. In this case $y_m = nx_{m-1}$ and consequently, $x_{m+1} = n^2 x_{m-1} + n - 1$. Solving the last recurrent relations with the initial conditions $x_1 = 1$ (see (17)) and $x_3 = n^2 - n + 1$ (see (19)) we obtain $x_{m+1} = (n-1)(1+n^2+n^4+...+n^{m-1})$ and, consequently, $y_{m+1} = (n-1)(1+n^2+n^4+...+n^{m-1})+1$, i.e.

$$x_{m+1} = \frac{n^{m+1}-1}{n+1}, \quad y_{m+1} = \frac{n^{m+1}+n}{n+1}. \qquad \square$$

Let us consider the structure of the polynomials involved in (21) and (23).

Theorem 2. *For $m \geq 4$ the polynomials $P_N^{(m)}(z)$ and $P_D^{(m)}(z)$ are of the form*

$$P_N^{(m)}(z) = A_m^n(z) B_m^{n-1}(z) C_m(z), \tag{30}$$

$$P_D^{(m)}(z) = A_m^n(z) B_m^n(z), \tag{31}$$

where $A_m(z)$, $B_m(z)$, $C_m(z)$ are polynomials of z which satisfy the following recurrent relations

$$A_{m+1}(z) = A_m^n(z) B_m^{n-1}(z), \tag{32}$$

$$B_{m+1}(z) = C_m(z) + z^2 B_m(z), \tag{33}$$

$$C_{m+1}(z) = C_m(z) + (z^2-1) B_m(z), \tag{34}$$

with the initial conditions

$$A_4(z) = (4z^2-1)^{n-1}, \quad A_5 = (4z^2-1)^{n(n-1)}(8z^2-4)^{n-1}, \tag{35}$$

$$B_4(z) = 8z^2 - 4, \quad B_5(z) = 16z^4 - 12z^2 + 1, \tag{36}$$

$$C_4(z) = 8z^4 - 8z^2 + 1, \quad C_5(z) = 16z^4 - 20z^2 + 5. \tag{37}$$

Proof. Substituting (26) and (27) into (24) and (25) and using (21), (23) and (14) we obtain $P_N^{(m)}$ and $P_D^{(m)}$ of the forms (30) and (31) where A_m, B_m and C_m fulfil (32)–(34).

The initial conditions (35)–(37) follow from (19)–(20). $\square$

Corollary 2. *The explicit forms of the polynomials A_m, B_m and C_m for $m \geq 6$ are:*

$$B_m(z) = F(z)(\alpha_1^{m-4}(z) - \alpha_2^{m-4}(z)) - G(z)(\alpha_1^{m-5}(z) - \alpha_2^{m-5}(z)), \tag{38}$$

$$C_m(z) = F(z)(\alpha_1^{m-4}(z) - \alpha_2^{m-4}(z)) - (F(z)+G(z))(\alpha_1^{m-5}(z) - \alpha_2^{m-5}(z)) + \\ +G(z)(\alpha_1^{n-6}(z) - \alpha_2^{n-6}(z)), \tag{39}$$

$$A_m(z) = A_4^{n^{m-4}}(z) \prod_{k=4}^{m-1} B_k^{n^{m-k-1}(n-1)}(z), \tag{40}$$

where

$$F(z) = \frac{B_5(z)}{\alpha_1(z) - \alpha_2(z)}, \quad G(z) = \frac{B_4(z)}{\alpha_1(z) - \alpha_2(z)},$$

$$\alpha_1(z) = \frac{1 + z^2 + \sqrt{z^4 + 2z^2 - 3}}{2}, \quad \alpha_2(z) = \frac{1 + z^2 - \sqrt{z^4 + 2z^2 - 3}}{2}.$$

Proof. From (33) and (34) we obtain

$$C_{m+1}(z) = B_{m+1}(z) - B_m(z) \tag{41}$$

$$B_{m+1}(z) - (1+z^2)B_m(z) + B_{m-1}(z) = 0. \tag{42}$$

Solving recurrence relations (42) with initial conditions (36) we obtain (38). Using (38) and (41) we arrive at (39). Using (32) we obtain (40). □

4. Description of the spectrum. Let us describe the spectrum of the operator $\mathcal{L}$ on S_m^n with a real potential $q(x) \in L_2(0,a)$ such that $q(a-x) \overset{a.e.}{=} q(x)$.

The spectrum of $\mathcal{L}$ is the union of the set of zeros of $s(\lambda, a)$, the set of zeros of $c(\lambda, a)$ with account of multiplicities, the set of zeros of $A_m(c^2(\lambda, a))$, zeros of $B_m(c^2(\lambda, a))$ and zeros of $C_m(c^2(\lambda, a))$. The following theorem is a consequence of (21) and (30).

Theorem 3. *λ_0 belongs to the spectrum of $\mathcal{L}$ if and only if one of the following conditions is true:*

1) $s(\lambda_0, a) = 0$;
2) $c(\lambda_0, a) = 0$;
3) $c^2(\lambda_0, a) = z_p$, *where* z_p ($p \in \{1, 2, ..., p_0(m)\}$, $p_0(m) = \frac{n^{m+1}-1}{2(n-1)} - \frac{1}{2}(z_m + x_m)$) *is a zero of* $A_m(z)$ *or of* $B_m(z)$ *or of* $C_m(z)$.

Corollary 3. *The zeros z_p of $A_m(z)$, of $B_m(z)$ and of $C_m(z)$ are all real and $0 < z_p < 1$.*

Proof. Our proof is indirect. First of all let us notice that $B_m(0) \neq 0$, $C_m(0) \neq 0$ and $A_m(0) \neq 0$. Consider problem (1)–(6) with $q(x) \equiv 0$. In this case $s(\lambda, a) = \frac{\sin \lambda a}{\lambda}$, $c(\lambda, a) = \cos \lambda a$. Since the operator $\mathcal{L}$ is self-adjoint, the spectrum of L is contained in the real line. On the other hand, in this case according to Theorem 1 if λ_0 is an eigenvalue and $\frac{\sin \lambda_0 a}{\lambda_0} \neq 0$ and $\cos \lambda_0 a \neq 0$ then $\cos^2 \lambda_0 a = z_p$, where z_p is a zero of $A_m(z)$ or $B_m(z)$ or $C_m(z)$. Since $0 < \cos^2 \lambda_0 a < 1$, the proof is completed. □

Theorem 4. *The spectrum of $\mathcal{L}$ on S_m^n with a real $q(x) \in L_2(0,a)$ such that $q(a-x) \overset{a.e.}{=} q(x)$ can be represented as the union of subsequences (branches)*

$$\{\lambda_k^{(1)}\}_{-\infty,\ k\neq 0}^{\infty} \cup \{\lambda_k^{(2)}\}_{-\infty,\ k\neq 0}^{\infty} \bigcup_{j=1}^{p_0(m)} \{\lambda_k^{(j,+)}\}_{-\infty,\ k\neq 0}^{\infty} \bigcup_{j=1}^{p_0(m)} \{\lambda_k^{(j,-)}\}_{-\infty,\ k\neq 0}^{\infty}$$

($\lambda_{-k}^{(j)} = -\lambda_k^{(j)}$) *which satisfy the following conditions.*

1. The eigenvalues $\{\lambda_k^{(1)}\}_{-\infty,\ k\neq 0}^{\infty}$ each of multiplicity $z_m - 1$ behave asymptotically as

$$\lambda_k^{(1)} = \frac{\pi k}{a} - \frac{K}{\pi k} + \frac{\beta_k^{(1)}}{k},\ \lambda_{-k}^{(1)} = -\lambda_k^{(1)}, \qquad k = 1, 2, ...,$$

with $\{\beta_k^{(1)}\}_{-\infty,\ k\neq 0}^{\infty} \in l_2$.

2. The eigenvalues $\{\lambda_k^{(2)}\}_{-\infty,\ k\neq 0}^{\infty}$ each of multiplicity x_m behave asymptotically as

$$\lambda_k^{(2)} = \frac{\pi(k-1/2)}{a} + \frac{K}{\pi k} + \frac{\beta_k^{(2)}}{k},\ \lambda_{-k}^{(2)} = -\lambda_k^{(2)}, \qquad k = 1, 2, ..., \tag{43}$$

with $\{\beta_k^{(2)}\}_{-\infty,\ k\neq 0}^{\infty} \in l_2$.

3. The eigenvalues of $\{\lambda_k^{(1)}\}_{-\infty,\ k\neq 0}^{\infty}$ *are interlaced with the eigenvalues of* $\{\lambda_k^{(2)}\}_{-\infty,\ k\neq 0}^{\infty}$*:*

$$...\lambda_{-k}^{(1)} < \lambda_{-k}^{(2)} < ... < \lambda_{-1}^{(1)} < \lambda_{-1}^{(2)} < 0 < \lambda_1^{(2)} < \lambda_1^{(1)} ... < \lambda_k^{(2)} < \lambda_k^{(1)} < ...\ (k > 0)$$

4. The eigenvalues of the other branches behave asymptotically as

$$\lambda_{k,+}^{(j,+)} = \frac{\pi(2k-1)}{a} - \frac{1}{a}\arccos(-\sqrt{z_j}) + \frac{K}{2\pi(2k-1)} + \frac{\beta_k^{(j,+)}}{2k-1},\ \lambda_{-k,+}^{(j,+)} = -\lambda_{k,+}^{(j,+)},\ k > 0, \quad (44)$$

$$\lambda_{k,-}^{(j,+)} = \frac{\pi(2k-1)}{a} + \frac{1}{a}\arccos(-\sqrt{z_j}) + \frac{K}{4\pi k} + \frac{\beta_k^{(j,+)}}{2k},\ \lambda_{-k,-}^{(j,+)} = -\lambda_{k,-}^{(j,+)},\ k > 0, \quad (45)$$

$$\lambda_k^{(j,+)} = \lambda_{k,\pm}^{(j,+)}, \quad (46)$$

$$\lambda_{k,+}^{(j,-)} = \frac{\pi(2k-1)}{a} - \frac{1}{a}\arccos\sqrt{z_j} + \frac{K}{2\pi(2k-1)} + \frac{\beta_k^{(j,+)}}{2k-1},\ \lambda_{-k,+}^{(j,+)} = -\lambda_{k,+}^{(j,+)},\ k > 0, \quad (47)$$

$$\lambda_{k,-}^{(j,-)} = \frac{\pi(2k-1)}{a} + \frac{1}{a}\arccos\sqrt{z_j} + \frac{K}{4\pi k} + \frac{\beta_k^{(j,+)}}{2k},\ \lambda_{-k,-}^{(j,+)} = -\lambda_{k,-}^{(j,+)},\ k > 0, \quad (48)$$

$$\lambda_k^{(j,-)} = \lambda_{k,\pm}^{(j,-)}, \quad (49)$$

where z_j *are the zeros of the polynomial* $P_N^{(m)}(z)$ *and* $\{\beta_k^{(j,\pm)}\}_{-\infty,\ k\neq 0}^{\infty} \in l_2$.

Proof. The eigenvalues of $\mathcal{L}$ are nothing but the zeros of $\phi_N^{(m)}(\lambda)$. According to (21) the set of zeros of $\phi_N^{(m)}(\lambda)$ is composed by

1) the set of zeros of $s(\lambda, a)$ each of multiplicity $z_m - 1$;
2) the set of zeros of $c(\lambda, a)$ each of multiplicity x_m;
3) the set of zeros of $(c(\lambda, a) - \sqrt{z_p})$ (for each $p = 1, 2, ..., p_0(m)$);
4) the set of zeros $(c(\lambda, a) + \sqrt{z_p})$ (for each $p = 1, 2, ..., p_0(m)$).

For the set $\{\lambda_k^{(1)}\}_{-\infty,\ k\neq 0}^{\infty}$ of zeros of $s(\lambda, a)$ and the set $\{\lambda_k^{(2)}\}_{-\infty,\ k\neq 0}^{\infty}$ of zeros of $c(\lambda, a)$ the statements 1–3 are known from [17]. Using representations (9), (43) and standard methods involving Rouché's theorem (see [17]) we obtain (44)–(49). □

Conditions 1–4 are necessary for a union of sequences of real numbers

$$\{\lambda_k^{(1)}\}_{-\infty,\ k\neq 0}^{\infty} \bigcup \{\lambda_k^{(2)}\}_{-\infty,\ k\neq 0}^{\infty} \bigcup_{j=1}^{p_0(m)} \{\lambda_k^{(j,+)}\}_{-\infty,\ k\neq 0}^{\infty} \bigcup_{j=1}^{p_0(m)} \{\lambda_k^{(j,-)}\}_{-\infty,\ k\neq 0}^{\infty}$$

to be the spectrum of $\mathcal{L}$ on S_m^n with a real $q(x) \in L_2(0, a)$ such that $q(a-x) \overset{a.e.}{=} q(x)$.

Acknowledgements The author was partly supported by the Swiss National Science Foundation through grant IZ73Z0-128135 and by grant UBD/GSR/S&T/19.

REFERENCES

1. G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe,* Acta Math., **78** (1946), 1–96.

2. R. Carlson, *Adjoint and self-adjoint differential operators on graphs,* Electron. J. Diff. Eqns., **6** (1998), 10 p.
3. R. Carlson, *Hill's equation on a homogeneous tree,* Electron. J. Diff. Eqns., **23** (1997), 1–30.
4. C. Cattaneo, *The spectrum of the continuous Laplacian on a graph,* Mh. Math., **124** (1997), 215–235.
5. S. Currie, B. Watson, *Eigenvalue asymptotics for differential operators on graphs,* J. Comp. Appl. Math., **182** (2005), 13–31.
6. J. Friedman, J.-P. Tillich, *Wave equations for graphs and the edge-based Laplacian,* Pacific J. Math., **216** (2004), №2, 229–266.
7. P. Exner, *Weakly coupled states on branching graphs,* Lett. Math. Phys., **38** (1996), №3, 313–320.
8. P. Exner, *A duality between Schrodinger operators on graphs and certain Jacobi matrices,* Annales de l'I. H. P., section A, **66** (1997), №4, 359–371.
9. I. Kac, V. Pivovarchik, *On multiplicity of a quantum graph spectrum,* J. Phys. A: Math. Theor., **44** (2011), 14 p.
10. T. Kottos, U. Smilansky, *Quantum chaos on graphs,* Phys. Rev. Lett., **79** (1997), 4794–4797.
11. P. Kuchment, *Quantum graphs: an introduction and a breif survey,* 'Analysis on Graphs and its Applications' Proc. Symp. Pure Math. AMS, (2008), 291–314.
12. P. Kuchment, *Quantum graphs II: Some spectral properties of quantum and combinatorial graphs,* J. Phys. A: Math., Gen., **38** (2005), 4887–4900.
13. V. Kostrikin, R. Schrader, *Kirchhoff's rule for quantum wires,* J. Phys. A: Math. Gen., **32** (1999), 595–630.
14. C.-K. Law, V. Pivovarchik, *Characteristic functions of quantum graphs,* J. Phys. A: Math. Theor., **42** (2009), 11 p.
15. B. Levitan, Inverse Sturm-Liouville Problems, VSP, Zeist Nauka Moscow, 1987. (in Russian)
16. B. Levitan, M. Gasymov, *Determination of a differential equation by two of its spectra,* Uspekhi Mat. Nauk, **19** (1964), №2(116), 3–63. (in Russian)
17. V. Marchenko, Sturm-Liouville operators and applications, Naukova Dumka, Kiev (1977), English translation: Oper. Theory Adv. Appl., V.22, Birkhäuser Verlag, Basel, 1986.
18. K. Pankrashkin, *Spectra of Schrödinger operators on equilateral quantum graphs,* Lett. Math. Phys., **77** (2006), №2, 139–154.
19. V. Pivovarchik, *Inverse problem for the Sturm-Liouville equation on a simple graph,* SIAM J. Math. Anal., **32** (2000), 801–819.
20. V. Pivovarchik, *Inverse problem for the Sturm-Liouville equation on a star-shaped graph,* Math. Nachr., **13–14** (2007), 1595–1619.
21. V. Pivovarchik, N. Rozhenko, *Inverse Sturm-Liouville problem on equilateral regular tree,* (2011) to apear in Appl. Analysis.
22. Yu. Pokornyi, V. Pryadiev, *The qualitative Sturm-Liouville theory on spatial networks,* J. Mathematical Sciences, **119** (2004), №6, 788–835.
23. S. Seshu, M. Reed, Linear graphs and electrical networks, Addison-Wesley Pub. Co., 1961.
24. M. Solomyak, *On the spectrum of the Laplacian on regular metric trees,* Waves Random Media, **14** (2004), 155–171.
25. C. Texier, G. Montambaux, *Scattering theory on graphs,* J. Phys. A: Math. Gen., **34** (2001), 10307–10326.
26. C. Texier, *On the spectrum of the Laplace operator of metric graphs attached at a vertex – spectral determinant approach,* J. Phys. A: Math. Theor., **41** (2008), 085207.

Odessa National I.I.Mechnikov University,
Dvoryanskaya 2, 65082
Odessa, Ukraine

University of Brunei Darussalam,
BE1410 Bandar Seri Begawan,Brunei
mainatasha@mail.ru

Received 26.09.2011