УДК 517.537

M. M. Sheremeta

ON THE MAXIMAL TERMS OF SUCCESSIVE GELFOND-LEONT'EV-SĂLĂGEAN AND GELFOND-LEONT'EV-RUSCHEWEYH DERIVATIVES OF A FUNCTION ANALYTIC IN THE UNIT DISC

M. M. Sheremeta. *On the maximal terms of succesive Gelfond-Leont'ev-Sălăgen and Gelfond-Leont'ev-Ruscheweyh derivatives of a function analytic in the unit disc*, Mat. Stud. **37** (2012), 58–64.

For a function analytic in the unit disc the concepts of Gelfond-Leont'ev-Sălăgen and Gelfond-Leont'ev-Ruscheweyh derivatives of n-th order are introduced and the asymptotic behaviour of the maximal terms of their power development as $n \to \infty$ is investigated.

М. Н. Шеремета. *О максимальных членах последовательных производных Гельфонда-Леонтьева-Салагена и Гельфонда-Леонтьева-Рушевая аналитических в единичном круге функций* // Мат. Студiї. – 2012. – Т.37, №1. – C.58–64.

Для аналитической в единичном круге функции введены понятия производных Гельфонда-Леонтьева-Салагена и Гельфонда-Леонтьева-Рушевая n-го порядка и исследовано асимптотическое поведение максимальных членов их степенных розложений при $n \to \infty$.

1. Introduction. For formal power series $f(z) = \sum\limits_{k=0}^{\infty} f_k z^k$ and $l(z) = \sum\limits_{k=0}^{\infty} l_k z^k$ $(l_k > 0)$ the formal power series

$$D_l^n f(z) = \sum_{k=0}^{\infty} \frac{l_k}{l_{k+n}} f_{k+n} z^k$$

is called the Gelfond-Leont'ev derivative ([1]). If $l(z) = e^z$ (i.e. $l_k = 1/k!$) then $D_l^n f = f^{(n)}$ is a usual derivative. Further we assume that $l_0 = 1$.

Let H be a class of analytic in the disk $\{z\colon |z| < 1\}$ functions given by power series

$$f(z) = z + \sum_{k=2}^{\infty} f_k z^k \qquad (1)$$

with the radius of convergence $R[f] = 1$ and the operator $D_{[S]}^n f$ $(n \geq 0)$ be defined by $D_{[S]}^0 f(z) = f(z)$, $D_{[S]}^1 f(z) = D_{[S]} f(z) = z f'(z)$ and

$$D_{[S]}^n f(z) = D_{[S]}(D_{[S]}^{n-1} f(z)) = z + \sum_{k=2}^{\infty} k^n f_k z^k.$$

2010 *Mathematics Subject Classification:* 30D99.
Keywords: analytic function, derivatives of Gelfond-Leont'ev, Sălăgen and Rusheweyh, maximal term of power series.

© M. M. Sheremeta, 2012

The operator $D^n_{[S]}f$ is known as the Sălăgean derivative ([2]). For $f \in H$

$$D^n_{[R]}f(z) = \frac{z}{n!}\frac{d^n}{dz^n}\{z^{n-1}f(z)\} = z + \sum_{k=2}^{\infty}\frac{(k+n-1)!}{n!(k-1)!}f_k z^k$$

is called the Ruscheweyh derivative ([3]).

Combining the definitions of Gelfond-Leont'ev derivative with Sălăgean derivative and Ruscheweyh derivative we obtain for $f \in H$

$$D^n_{[GLS],l}f(z) = l_1 z D^1_l(D^{n-1}_{[GLS],l}f(z)) = z + \sum_{k=2}^{\infty}\left(\frac{l_1 l_{k-1}}{l_k}\right)^n f_k z^k \tag{2}$$

and

$$D^n_{[GLR],l}f(z) = z l_n D^n_l\{z^{n-1}f(z)\} = z + \sum_{k=2}^{\infty}\frac{l_{k-1}l_n}{l_{n+k-1}}f_k z^k. \tag{3}$$

The operator $D^n_{[GLS],l}$ will be called the *Gelfond-Leont'ev-Sălăgean derivative* and the operator $D^n_{[GLR],l}$ will be called the *Gelfond-Leont'ev-Ruscheweyh derivative.*

We denote $\varkappa_k = l_k/l_{k+1}\,(k \geq 0)$ and remark that $D^n_{[GLR],l}f \in H$ *for every* $f \in H$ *and all* $n \geq 1$ *if and only if* $\sqrt[k]{\varkappa_k} \to 1\,(k \to \infty)$. Indeed, $\sqrt[k]{\varkappa_k} \to 1\,(k \to \infty)$ if and only if $\sqrt[k]{l_{k-1}/l_k} \to 1\,(k \to \infty)$. If $\sqrt[k]{l_{k-1}/l_k} \to 1\,(k \to \infty)$ then $\sqrt[k]{l_{k-1}/l_{k+n-1}} \to 1\,(k \to \infty)$ for every $n \geq 1$ and, thus, $\varlimsup\limits_{k\to\infty}\sqrt[k]{(l_n l_{k-1}/l_{k+n-1})|f_k|} = \varlimsup\limits_{k\to\infty}\sqrt[k]{|f_k|} = 1$, that is $D^n_{[GLR],l}f \in H$. On the other hand, if $\sqrt[k_j]{l_{k_j-1}/l_{k_j}} \to \alpha \neq 1\,(j \to \infty)$ for some sequence $(k_j) \uparrow \infty$ then we put $f_{k_j} = 1$ and $f_k = 0$ for $k \neq k_j$. Hence $f \in H$ and for $n = 1$ we have $\varlimsup\limits_{k\to\infty}\sqrt[k]{(l_1 l_{k-1}/l_k)|f_k|} = \lim\limits_{j\to\infty}\sqrt[k_j]{l_{k_j-1}/l_{k_j}} = \alpha$, that is $D^1_{[GLR],l}f \notin H$.

By analogy we can prove that $D^n_{[GLS],l}f \in H$ *for every* $f \in H$ *and all* $n \geq 1$ *if and only if* $\sqrt[k]{\varkappa_k} \to 1\,(k \to \infty)$.

Let $\mu(r,f) = \max\{|f_n|r^n\colon n \geq 1\}$ be the maximal term of series (1) and $\nu(r,f) = \max\{n\colon |f_n|r^n = \mu(r,f)\}$ be its central index. Then $\nu(r,f) \geq 1$ for all $r \in [0,1)$ and $\mu(r,f) = |f_{\nu(r,f)}|r^{\nu(r,f)}$.

Further we investigate asymptotic behaviour of the sequences $(\nu(r, D^n_{[GLS],l}f))$, $(\mu(r, D^n_{[GLS],l}f))$, $(\nu(r, D^n_{[GLR],l}f))$ and $(\mu(r, D^n_{[GLR],l}f))$ as $n \to \infty$.

2. Growth of the sequences of maximal terms and central indices. Here we prove the following theorem.

Theorem 1. *Let* $\sqrt[k]{\varkappa_k} \to 1\,(k \to \infty)$. *If the sequence* $(\varkappa_k)$ *is nondecreasing then for every* $r \in [0,1)$ *the sequences* $(\nu(r, D^n_{[GLS],l}f))$, $(\mu(r, D^n_{[GLS],l}f))$, $(\nu(r, D^n_{[GLR],l}f))$ *and* $(\mu(r, D^n_{[GLR],l}f))$ *are nondecreasing. In particular, if* $\varkappa_k \nearrow \infty\,(k \to \infty)$ *then* $\nu(r, D^n_{[GLS],l}f) \to \infty$, $\mu(r, D^n_{[GLS],l}f) \to \infty$, $\nu(r, D^n_{[GLR],l}f) \to \infty$ *and* $\mu(r, D^n_{[GLR],l}f) \to \infty$ *as* $n \to \infty$ *for every* $r \in [0,1)$.

Proof. If we denote $D^n = D^n_{[GLS],l}f$ then in view of (2) we have

$$\mu(r, D^{n+1}) = \left(\frac{l_{\nu(r,D^{n+1})-1}l_1}{l_{\nu(r,D^{n+1})}}\right)^{n+1}|f_{\nu(r,D^{n+1})}|r^{\nu(r,D^{n+1})} =$$

$$= \frac{l_{\nu(r,D^{n+1})-1}l_1}{l_{\nu(r,D^{n+1})}} \left(\frac{l_{\nu(r,D^{n+1})-1}l_1}{l_{\nu(r,D^{n+1})}}\right)^n |f_{\nu(r,D^{n+1})}| r^{\nu(r,D^{n+1})} \le \frac{l_{\nu(r,D^{n+1})-1}l_1}{l_{\nu(r,D^{n+1})}}\mu(r,D^n).$$

On the other hand,

$$\mu(r,D^n) = \left(\frac{l_{\nu(r,D^n)-1}l_1}{l_{\nu(r,D^n)}}\right)^n |f_{\nu(r,D^n)}| r^{\nu(r,D^n)} =$$

$$= \frac{l_{\nu(r,D^n)}}{l_{\nu(r,D^n)-1}l_1} \left(\frac{l_{\nu(r,D^n)-1}l_1}{l_{\nu(r,D^n)-1}}\right)^{n+1} |f_{\nu(r,D^n)}| r^{\nu(r,D^n)} \le \frac{l_{\nu(r,D^n)}}{l_{\nu(r,D^n)-1}l_1}\mu(r,D^{n+1}).$$

Thus, for all $n \ge 0$ and $r \in [0,1)$

$$\frac{l_{\nu(r,D^n_{[GLS],l}f)-1}l_1}{l_{\nu(r,D^n_{[GLS],l}f)}} \le \frac{\mu(r,D^{n+1}_{[GLS],l}f)}{\mu(r,D^n_{[GLS],l}f)} \le \frac{l_{\nu(r,D^{n+1}_{[GLS],l}f)-1}l_1}{l_{\nu(r,D^{n+1}_{[GLS],l}f)}}. \tag{4}$$

Using (3) by analogy we obtain for all $n \ge 0$ and $r \in [0,1)$

$$\frac{l_{n+1}}{l_n}\frac{l_{\nu(r,D^n_{[GLR],l}f)+n-1}}{l_{\nu(r,D^n_{[GLR],l}f)+n}} \le \frac{\mu(r,D^{n+1}_{[GLR],l}f)}{\mu(r,D^n_{[GLR],l}f)} \le \frac{l_{n+1}}{l_n}\frac{l_{\nu(r,D^{n+1}_{[GLR],l}f)+n-1}}{l_{\nu(r,D^{n+1}_{[GLR],l}f)+n}}. \tag{5}$$

Since the sequence $(\varkappa_k)$ is nondecreasing from (4) and (5) it follows that

$$\varkappa_{\nu(r,D^n_{[GLS],l}f)-1} \le \varkappa_{\nu(r,D^{n+1}_{[GLS],l}f)-1}, \quad \varkappa_{\nu(r,D^n_{[GLR],l}f)+n-1} \le \varkappa_{\nu(r,D^{n+1}_{[GLR],l}f)+n-1}$$

and, thus,

$$\nu(r,D^n_{[GLS],l}f) \le \nu(r,D^{n+1}_{[GLS],l}f), \quad \nu(r,D^n_{[GLR],l}f) \le \nu(r,D^{n+1}_{[GLR],l}f)$$

for all $n \ge 0$ and $r \in [0,1)$. But $\nu(r,D^n_{[GLS],l}f) \ge 1$ and $\nu(r,D^n_{[GLR],l}f) \ge 1$ for all $n \ge 0$ and $r \in [0,1)$. Therefore, in view of the nondecrease of $(\varkappa_k)$ we have

$$\frac{l_{\nu(r,D^n_{[GLS],l}f)-1}l_1}{l_{\nu(r,D^n_{[GLS],l}f)}} \ge 1, \quad \frac{l_{n+1}}{l_n}\frac{l_{\nu(r,D^n_{[GLR],l}f)+n-1}}{l_{\nu(r,D^n_{[GLR],l}f)+n}} \ge 1$$

and from (4) and (5) it follows also that

$$\mu(r,D^n_{[GLS],l}f) \le \mu(r,D^{n+1}_{[GLS],l}f), \quad \mu(r,D^n_{[GLR],l}f) \le \mu(r,D^{n+1}_{[GLR],l}f)$$

for all $n \ge 0$ and $r \in [0,1)$. Thus, the first part of Theorem 1 is proved.

Now let $\varkappa_k \nearrow \infty\, (k \to \infty)$. We assume on the contrary that $\nu(r_0,D^n_{[GLS],l}f) \le K < +\infty$ for some $r_0 \in [0,1)$ and all $n \ge 1$. Since $\nu(r,D^n_{[GLS],l}f)$ takes positive integer values, one has that $\nu(r_0,D^n_{[GLS],l}f) = p_0 = \text{const}$ for all $n \ge n_0$ and $\mu(r_0,D^n_{[GLS],l}f) = \left(\frac{l_1 l_{p_0-1}}{l_{p_0}}\right)^n |f_{p_0}| r_0^{p_0}$. Let $k_0 = \min\{k > p_0\colon f_k \ne 0\}$. Then

$$1 \ge \frac{(l_1 l_{k_0-1}/l_{k_0})^n |f_{k_0}| r_0^{k_0}}{(l_1 l_{p_0-1}/l_{p_0})^n |f_{p_0}| r_0^{p_0}} = \left(\frac{\varkappa_{k_0-1}}{\varkappa_{p_0-1}}\right)^n \frac{|f_{k_0}|}{|f_{p_0}|} r_0^{k_0-p_0} \to \infty, \quad n \to \infty,$$

and it is impossible.

By analogy, if $\nu(r_0, D^n_{[GLR],l}f) \le K < +\infty$ for some $r_0 \in [0,1)$ and all $n \ge 1$ we obtain as above

$$1 \ge \frac{(l_{k_0-1}l_n/l_{n+k_0-1})|f_{k_0}|r_0^{k_0}}{(l_{p_0-1}l_n/l_{n+p_0-1})|f_{p_0}|r_0^{p_0}} = \frac{l_{n+p_0-1}}{l_{n+k_0-1}}\frac{l_{k_0-1}}{l_{p_0-1}}\frac{|f_{k_0}|}{|f_{p_0}|}r_0^{k_0-p_0} = \\ = \frac{l_{k_0-1}}{l_{p_0-1}}\frac{|f_{k_0}|}{|f_{p_0}|}r_0^{k_0-p_0}\prod_{j=n+p_0-1}^{n+k_0-2}\varkappa_j \to \infty, \quad n \to \infty,$$

and it is impossible.

Thus, $\nu(r, D^n_{[GLS],l}f) \nearrow \infty$ and $\nu(r, D^n_{[GLR],l}f) \nearrow \infty$ as $n \to \infty$ and from (4) and (5) it follows that $\mu(r, D^n_{[GLS],l}f) \nearrow \infty$ and $\mu(r, D^n_{[GLR],l}f) \nearrow \infty$ as $n \to \infty$ for every $r \in [0,1)$. The proof of Theorem 1 is complete. □

3. Some estimates. It is clear that $\mu(r, D^n_{[GLS],l}f) = \max\{(l_1\varkappa_{k-1})^n|f_k|r^k\colon k \ge 1\}$. Since $l_0 = 1$ by definition of $\varkappa_n$ we have $l_k = \prod_{j=0}^{k-1}\frac{1}{\varkappa_j}$. Therefore,

$$\frac{l_{k-1}l_n}{l_{n+k-1}} = \prod_{j=0}^{k-2}\frac{1}{\varkappa_j}\prod_{j=0}^{n-1}\frac{1}{\varkappa_j}\prod_{j=0}^{n+k-2}\varkappa_j = \prod_{j=0}^{n-1}\frac{1}{\varkappa_j}\prod_{j=k-1}^{n+k-2}\varkappa_j = \prod_{j=0}^{n-1}\frac{\varkappa_{j+k-1}}{\varkappa_j}$$

that is

$$\mu(r, D^n_{[GLR],l}f) = \max\left\{|f_k|r^k\prod_{j=0}^{n-1}\frac{\varkappa_{j+k-1}}{\varkappa_j}\colon k \ge 1\right\}. \tag{6}$$

Now we put $\eta_k = \dfrac{l_k^2}{l_{k-1}l_{k+1}}$ $(k \ge 1)$ and suppose that the sequence (η_k) is nonincreasing. Then $\varkappa_n = \eta_n\varkappa_{n-1} = \eta_n\eta_{n-1}\varkappa_{n-2} = \cdots = \varkappa_0\prod_{m=1}^{n}\eta_m = \frac{1}{l_1}\prod_{m=1}^{n}\eta_m$ and

$$\varkappa_{j+k-1} = \frac{1}{l_1}\prod_{m=1}^{j+k-1}\eta_m = \varkappa_j\prod_{m=j+1}^{j+k-1}\eta_m = \varkappa_j\prod_{m=1}^{k-1}\eta_{m+j} \le \varkappa_j\prod_{m=1}^{k-1}\eta_m \le l_1\varkappa_j\varkappa_{k-1}.$$

Therefore, from (6) we obtain

$$\mu(r, D^n_{[GLR],l}f) \le \max\left\{|f_k|r^k\prod_{j=0}^{n-1}(l_1\varkappa_{k-1})\colon k \ge 1\right\} = \max\left\{(l_1\varkappa_{k-1})^n|f_k|r^k\colon k \ge 1\right\}.$$

Thus, the following proposition is true.

Proposition 1. *If* $\sqrt[k]{\varkappa_k} \to 1\,(k \to \infty)$ *and the sequence* (η_k) *is nonincreasing then*

$$\mu(r, D^n_{[GLR],l}f) \le \mu(r, D^n_{[GLS],l}f).$$

We remark that if $\varkappa_k \nearrow \varkappa < \infty\,(k \to \infty)$ and the sequence (η_k) is nonincreasing then $\mu(r, D^n_{[GLR],l}f) \le \mu(r, D^n_{[GLS],l}f) \le (l_1\varkappa)^n\mu(r,f)$ for every $r \in [0,1)$ and all $n \ge 1$. The estimates are sharp because for $l(z) = 1/(1-z)$ we have $l_k = 1$, $\varkappa_k = 1$ for all $k \ge 0$ and $\mu(r, D^n_{[GLR],l}f) = \mu(r, D^n_{[GLS],l}f) = \mu(r,f)$.

Therefore, further we will investigate an asymptotic behaviour only of $\mu(r, D^n_{[GLS],l}f)$ provided $\varkappa_k \uparrow +\infty\,(k \to \infty)$. Putting $\lambda_k = \ln(l_1\varkappa_{k-1})$ and $a_k = |f_k|r^k$ we consider the functional sequence $F_r(\sigma) = (a_k e^{\sigma\lambda_k})$. If $\sqrt[k]{\varkappa_k} \to 1\,(k \to \infty)$ then the maximal term $\hat\mu(\sigma) = \hat\mu(\sigma, F_r) = \max\{a_k e^{\sigma\lambda_k} \colon k \geq 1\}$ of $F_r(\sigma)$ exsists for all $\sigma \in \mathbb{R}$ and $\mu(r, D^n_{[GLS],l}f) = \hat\mu(n, F_r)$ for every $r \in [0,1)$ and all $n \geq 1$. We can interpret $\hat\mu(\sigma)$ as the maximal term of an entire Dirichlet series. The relations between the growth of the maximal term of entire Dirichlet series and its coefficients are well studied. Using such relations we can obtain various results on behaviour of $\mu(r, D^n_{[GLS],l}f)$. Here we dwell on the following well known formula

$$\varlimsup_{\sigma\to+\infty} \frac{\ln\ln\hat\mu(\sigma, F_r)}{\sigma} = \varlimsup_{k\to\infty} \frac{\lambda_k \ln\lambda_k}{-\ln a_k}, \tag{7}$$

which correspondes to functions of finite R-order in the theory of Dirichlet series.

Theorem 2. *Let $f \in H$, $\varkappa_k \uparrow \infty\,(k \to \infty)$ and $\rho \colon = \varlimsup\limits_{k\to\infty} \dfrac{\ln\varkappa_k \ln\ln\varkappa_k}{k} < +\infty$. Then for every $r \in (0,1)$*

$$\varlimsup_{n\to\infty} \frac{\ln\ln\mu(r, D^n_{[GLS],l}f)}{n} \leq \frac{\rho}{|\ln r|}, \tag{8}$$

and if $\sqrt[k]{|f_k|} \to 1$ $(k \to \infty)$ then in (8) the sign $(\leq)$ can be replaced with the sign $(=)$.

Proof. Since $\rho < \infty$ we have $\dfrac{\ln\varkappa_k}{k} \to 0$ and, therefore, $\sqrt[k]{\varkappa_k} \to 1$ $(k \to \infty)$. From the equality $\varlimsup\limits_{k\to\infty} \sqrt[k]{|f_k|} = 1$ we obtain $\ln|f_k| \leq \varepsilon k$ for every $\varepsilon > 0$ and $k \geq k_0(\varepsilon)$ that is $\ln(|f_k|r^k) \leq -(1+o(1))k|\ln r|\,(k \to \infty)$. Therefore, from (7) in view of the equalities $\lambda_k = \ln(l_1\varkappa_{k-1})$ and $a_k = |f_k|r^k$ we have

$$\varlimsup_{\sigma\to+\infty} \frac{\ln\ln\hat\mu(\sigma, F_r)}{\sigma} \leq \varlimsup_{k\to\infty} \frac{\ln\varkappa_k \ln\ln\varkappa_k}{k|\ln r|} = \frac{\rho}{|\ln r|}. \tag{9}$$

Since $\ln\hat\mu(\sigma, F_r) \nearrow +\infty\,(\sigma \to +\infty)$ and $\mu(r, D^n_{[GLS],l}f) = \hat\mu(n, F_r)$, we obtain that (8) is valid.

If $\sqrt[k]{|f_k|} \to 1$ $(k \to \infty)$ then $\ln(|f_k|r^k) = -(1+o(1))k|\ln r|(k \to \infty)$ and we can replace $(\leq)$ with $(=)$ in (9) and in (8) respectively. □

Now we consider the case, when $l_k = 1/k!$. Then $\mu(r, D^n_{[GLS],l}f) = \mu(r, D^n_{[S]}f)$, $\mu(r, D^n_{[GLR],l}f) = \mu(r, D^n_{[R]}f)$ and the following theorem is true.

Theorem 3. *For every $r \in (0,1)$*

$$\varlimsup_{n\to\infty} \frac{\ln\mu(r, D^n_{[R]}f)}{n\ln n} \leq \varlimsup_{n\to\infty} \frac{\ln\mu(r, D^n_{[S]}f)}{n\ln n} \leq 1 \tag{10}$$

and if $\lim\limits_{k\to\infty} \sqrt[k]{|f_k|} = \gamma \in (0,1]$ then $\ln\mu(r, D^n_{[S]}f) \sim n\ln n$ as $n \to \infty$.

Proof. Since $l_k = 1/k!$ we have $\varkappa_k = k+1$ and $\mu(r, D^n_{[S]}f) = \hat\mu(n)$, where $\hat\mu(\sigma) = \max\{|f_k|r^k e^{\sigma\ln k} \colon k \geq 1\}$. We remark that if $\mu(\sigma)$ is the maximal term of the functional sequence $(e^{-Ak+\sigma\ln k})$, $A>0$, and $\nu(\sigma)$ is its central index then $\nu(\sigma)=\sigma/A+\alpha(\sigma)$, $|\alpha(\sigma)|\le 1$, and

$$\ln\mu(\sigma)=-A\left(\frac{\sigma}{A}+\alpha(\sigma)\right)+\sigma\ln\left(\frac{\sigma}{A}+\alpha(\sigma)\right)=(1+o(1))\sigma\ln\sigma$$

as $\sigma\to+\infty$. Therefore, since $\ln(|f_k|r^k)\le-(1+o(1))k|\ln r|$ as $k\to\infty$, we obtain $\ln\hat\mu(\sigma)\le(1+o(1))\sigma\ln\sigma,\ \sigma\to+\infty$, and, thus, the last inequality in (10) holds. The first inequality in (10) follows from Proposition 1.

If $\varliminf\limits_{k\to\infty}\sqrt[k]{|f_k|}=\gamma\in(0,1]$ then $\ln\hat\mu(\sigma)\ge(1+o(1))\sigma\ln\sigma$, $\sigma\to+\infty$, and, therefore, $\ln\mu(r,D^n_{[S]}f)=\ln\hat\mu(n)\sim n\ln n$ as $n\to\infty$. The proof of Theorem 3 is complete. □

The condition $\varliminf\limits_{k\to\infty}\sqrt[k]{|f_k|}=\gamma\in(0,1]$ is not necessary for $\ln\mu(r,D^n_{[S]}f)\sim n\ln n$ as $n\to\infty$. In order to find such a condition we use one result from [4].

Let Ω be the class of positive unbounded on $(-\infty,+\infty)$ functions Φ such that the derivative Φ' is positive, continuous and increasing to $+\infty$ on $(-\infty,+\infty)$. We denote by φ the inverse function to Φ', and let $\Psi(\sigma)=\sigma-\Phi(\sigma)/\Phi'(\sigma)$ be the function associated with Φ in the sense of Newton.

Lemma 1 ([4]). *Let $\Phi\in\Omega$ and let the function Φ'/Φ be nonincreasing. As above, let $\hat\mu(\sigma)=\max\{a_ke^{\sigma\lambda_k}\colon k\ge1\}$. In order that $\ln\hat\mu(\sigma)\sim\Phi(\sigma)$ as $\sigma\to+\infty$, it is necessary and sufficient that for every $\varepsilon>0$:*

1) *there exists $k_0=k_0(\varepsilon)$ such that $\ln a_k\le-\lambda_k\Psi(\varphi(\lambda_k/(1+\varepsilon)))$ for all $k\ge k_0$;*
2) *there exists an increasing sequence (k_j) of positive integers such that $\ln a_{k_j}\ge-\lambda_{k_j}\times$ $\times\Psi(\varphi(\lambda_{k_j}/(1-\varepsilon)))$ for all $j\ge1$ and $\lambda_{k_j}/\lambda_{k_{j+1}}\to1$ as $j\to\infty$.*

If we choose $\Phi\in\Omega$ such that $\Phi(\sigma)=\sigma\ln\sigma$ for $\sigma\ge\sigma_0$ then $x\Psi(\varphi(x))=e^{x-1}$ for $x\ge x_0$. Therefore, using Lemma 1 with $a_k=|f_k|r^k$ and $\lambda_k=\ln k$ we obtain the following statement.

In order that $\ln\hat\mu(\sigma)\sim\sigma\ln\sigma$ as $\sigma\to+\infty$, it is necessary and sufficient that for every $\varepsilon>0$:

1) *there exists $k_0=k_0(\varepsilon)$ such that $\ln(|f_k|r^k)\le-\dfrac{1+\varepsilon}{e}k^{1/(1+\varepsilon)}$ for all $k\ge k_0$;*
2) *there exists an increasing sequence (k_j) of positive integers such that $\ln(|f_{k_j}|r^{k_j})\ge-\dfrac{1-\varepsilon}{e}k_j^{1/(1-\varepsilon)}$ for all $j\ge1$ and $\ln k_j/\ln k_{j+1}\to1$ as $j\to\infty$.*

The condition $\ln(|f_k|r^k)\le-\frac{1+\varepsilon}{e}k^{1/(1+\varepsilon)}$ is equivalent to the condition

$$|\ln r|-\frac{\ln|f_k|}{k}\ge\frac{1+\varepsilon}{e}k^{-\varepsilon/(1+\varepsilon)}$$

and holds for $k\ge k_0$ because of $\varlimsup\limits_{k\to\infty}\frac{\ln|f_k|}{k}=0$.

The condition $\ln(|f_{k_j}|r^{k_j})\ge-\frac{1-\varepsilon}{e}k_j^{1/(1-\varepsilon)}$ holds if and only if

$$\sqrt[k_j]{|f_{k_j}|}\ge\exp\left\{-\frac{1-\varepsilon}{e}k_j^{\varepsilon/(1-\varepsilon)}-\ln r\right\}.$$

In view of arbitrariness of ε, the last condition is equivalent to the condition $\sqrt[k_j]{|f_{k_j}|}\ge\exp\left\{-k_j^{\varepsilon}\right\}$. Thus, the following proposition is proved.

Proposition 2. *In order that* $\ln \mu(r, D^n_{[S]} f) \sim n \ln n \, (n \to \infty)$ *for every* $r \in (0, 1)$ *it is necessary and sufficient that for every* $\varepsilon > 0$ *there exists an increasing sequence* (k_j) *of positive integers such that* $\sqrt[k_j]{|f_{k_j}|} \geq \exp\left\{-k_j^{\varepsilon}\right\}$ *and* $\ln k_{j+1} \sim \ln k_j$ *as* $j \to \infty$.

REFERENCES

1. Gelfond A.O., Leont'ev A.F. *On a generalisation of Fourier series*// Matem. Sb. – 1951. – V.29, №3. – P. 477–500. (in Russian)
2. Sălăgean G.St. *Subclasses of univalent functions*// Lecture Notes in Math. – 1983. – V.1013. – P. 362–372.
3. Ruscheweyh St. *New criteria for univalent functions*// Proc. Amer. Math. Soc. – 1975. – V.49. – P. 109–115.
4. Zabolotskyi M.V., Sheremeta M.M. *A generalisation of Lindelöff theorem*// Ukr. Mat. Zh. – 1998. – V.50, №9. – P. 1177–1192. (in Ukrainian)

Department of Mechanics and Mathematics,
Ivan Franko National University of Lviv,
m_m_sheremeta@list.ru

Received 28.12.2011