УДК 517.53

A. Ya. Khrystiyanyn, A. A. Kondratyuk, N. B. Sokul's'ka

GROWTH CHARACTERISTICS OF LOXODROMIC AND ELLIPTIC FUNCTIONS

A. Ya. Khrystiyanyn, A. A. Kondratyuk, N. B. Sokul's'ka. *Growth characteristics of loxodromic and elliptic functions*, Mat. Stud. **37** (2012), 52–57.

The asymptotic behaviour of the Nevanlinna characteristic for loxodromic functions as well as the Nevanlinna type characteristic for elliptic functions are investigated.

А. Я. Християнин, А. А. Кондратюк, Н. Б. Сокульская. *Характеристики роста локсодромных и эллиптических функций* // Мат. Студiї. – 2012. – Т.37, №1. – C.52–57.

Исследовано асимптотическое поведение характеристики Неванлинны локсодромных функций, а также характеристики роста типа Неванлинны эллиптических функций.

1. Introduction. Let $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ be the punctured plane. The function $z = e^{2\pi s}$ maps the complex plane $\mathbb{C}$ onto $\mathbb{C}^*$. That is $\mathbb{C}^* = \exp\{2\pi\mathbb{C}\}$. If we denote $z = re^{i\varphi}$, $s = \sigma + it$, then

$$\sigma = \frac{\log r}{2\pi}, \quad t = \frac{\varphi}{2\pi}.$$

We will call the couples (σ, t) *log-polar coordinates* in $\mathbb{C}^*$, $z = e^{2\pi(\sigma+it)}$. These coordinates are local. However $e^{2\pi it} = e^{2\pi i(t-[t])}$, where $[t]$ denotes the integer part of t. The function $t \mapsto t-[t]$ maps one-to-one the one-dimentional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ onto $[0, 1)$. Hence $\mathbb{C}^* = \mathbb{R}\times\mathbb{T}$. Thus the global coordinates in $\mathbb{C}^*$ are (σ, t) where $\sigma \in \mathbb{R}$, $t \in \mathbb{T}$. This means that each function f on $\mathbb{C}^*$ may be considered as a periodic in log-polar coordinates, that is in $\mathbb{C}$, with the period i, $f(e^{2\pi s}) = g(s)$, $g(s + i) = f(e^{2\pi(s+i)}) = f(z)$, and vice versa. If moreover $g(s)$ has another period ω_1, say 1, then $g(s + 1) = g(s)$ implies $f(e^{2\pi}e^{2\pi s}) = f(e^{2\pi s})$, i. e. $f(e^{2\pi}z) = f(z)$. This means that f is *multiplicatively periodic of multiplicator* $e^{2\pi}$ and g is a function on two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

More generally, after the homotety $s \mapsto \frac{s}{\omega}$, taking an arbitrary period $\omega_1 > 0$ we obtain a double-periodic function g with the period lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_2 = i\omega$, $\omega > 0$, and a multiplicatively periodic function f of multiplicator $\frac{1}{q} = e^{2\pi\frac{\omega_1}{\omega}} = e^{2\pi i\frac{\omega_1}{\omega_2}}$, $0 < q < 1$. In the general case the connection between a multiplicatively periodic function $f(z)$ of multiplicator q and the associated double periodic function $g(s)$ which admits the period lattice Λ with complex numbers ω_1, ω_2 is the following

$$f(e^{\frac{2\pi i}{\omega_2}s}) = g(s), \quad \frac{1}{q} = e^{2\pi i\frac{\omega_1}{\omega_2}}, \quad \operatorname{Im}\frac{\omega_1}{\omega_2} < 0.$$

2010 *Mathematics Subject Classification:* 30D35.
Keywords: meromorphic function, loxodromic function, elliptic function, double-periodic function, Nevanlinna characteristics.
doi:10.30970/ms.37.1.52-57

The theory of meromorphic multiplicatively periodic functions was elaborated by O. Rausenberger ([1]). G. Valiron ([2]) called these functions loxodromic because the points in which such a function in the case of non-real q acquires the same value lay on logarithmic spirals. The images of these last on the Riemann sphere intersect each meridian under the same angle, and are called loxodromic curves ($\lambda o\xi o\zeta$ — oblique, $\delta\rho o\mu o\zeta$ — way). In log-polar coordinates they are straight lines.

Double-periodic meromorphic functions are elliptic functions and more known due to the works of K. Jacobi, N. Abel, K. Weierstrass.

Summarizing we can conclude that loxodromic meromorphic functions give a simple construction of elliptic functions. Furthermore, the recent research give its various applications ([3]–[6]), and show that interest to these objects increases. We study here the growth characteristics of both loxodromic and elliptic functions.

2. Growth characteristics of loxodromic functions.

Definition 1 ([1], [2])**.** A meromorphic function f in the punctured plane $\mathbb{C}^*$ is called *loxodromic of multiplicator* q if it satisfies the condition

$$f(qz) = f(z), \quad 0 < |q| < 1, \quad z \in \mathbb{C}^*. \tag{1}$$

It is clear that the loxodromic functions of multiplicator q form a field which is denoted by $\mathcal{L}_q$ ([6]).

The Nevanlinna type characteristics of meromorphic in $\mathbb{C}^*$ functions were introduced and studied in [7], [8] (see also [9]).

Namely, Nevanlinna characteristics of f is defined by the relation

$$T_0(r,f) = N_0(r,f) + m_0(r,f), \quad 1 \le r,$$

where

$$m_0(r,f) = m(r,f) + m\left(\frac{1}{r}, f\right) - 2m(1,f),\ m(r,f) = \frac{1}{2\pi}\int_0^{\pi} \log^+ |f(re^{i\varphi})| d\varphi,$$

$$a^+ = \max(a,0),\ N_0(r,f) = \int_1^r \frac{n_0(r,f)}{r} dr,$$

and $n_0(r,f)$ is the counting function of poles of f in the annulus $\{z : 1/r < |z| \le r\}$.

Theorem A ([7], [9])**.** *The characteristic* $T_0(r,f)$ *is non-negative, continuous, non-decreasing and convex with respect to* $\log r$ *on* $[1;+\infty)$, $T(1,f) = 0$.

Recall some properties of functions from $\mathcal{L}_q$ ([1], [2], [6]). Denote $A_r = \{z : |q|r < |z| \le r\}$. Each non-constant loxodromic meromorphic function of multiplicator q has at least two poles in A_r. The number of poles of f is the same in each A_r. Denote this number by m. It is called *the order of* f. It follows from (1) that $f(q^n z) = f(z)$, $n \in \mathbb{Z}$.

Theorem 1. *Let* f *belong to* $\mathcal{L}_q$ *and have order* m. *Then*

$$T_0(r,f) = \frac{m}{\log\frac{1}{|q|}} \log^2 r + O(\log r), \quad r > 1, \tag{2}$$

where

$$|O(\log r)| \le 2m \log r + C, \tag{3}$$

$$C = \max\left\{T_0\left(\frac{1}{|q|}, f\right), 2m(1,f)\right\}.$$

Proof. If $|q|^{1-n} < r \le |q|^{-n}$, $n \in \mathbb{N}$, then

$$2m\left(\frac{\log r}{\log\frac{1}{|q|}} - 1\right) \le 2m(n-1) \le n_0(r,f) \le 2mn \le 2m\left(\frac{\log r}{\log\frac{1}{|q|}} + 1\right).$$

Therefore,

$$\frac{m\log^2 r}{\log\frac{1}{|q|}} - 2m\log r \le N_0(r,f) \le \frac{m\log^2 r}{\log\frac{1}{|q|}} + 2m\log r. \tag{4}$$

The function $f \in \mathcal{L}_q$ is determined by its values in $A_{\frac{1}{|q|}}$. Thus,

$$-2m(1,f) \le m_0(r,f) \le T_0\left(\frac{1}{|q|}, f\right). \tag{5}$$

The relations (2) and (3) follow from (4) and (5) that finishes the proof. □

Theorem 1 and Theorem 10.1 from [9] give another way to represent a loxodromic function as follows.

The number of zeroes of $f \in \mathcal{L}_q$ in A_1 coincides with the number of its poles in A_r ([2]). Denote the zeroes of f in A_r by $a_1, a_2, ..., a_m$ and its poles by $b_1, b_2, ..., b_m$. Then the relation (1) implies $z_j = a_k q^n, w_j = b_k q^n$, $n \in \mathbb{Z}$, where $\{z_j\}$ are zeroes and $\{\omega_j\}$ are poles of f.

Let

$$\widetilde{z_j} = \begin{cases} z_j, & \text{if } |z_j| > 1, \\ \frac{1}{z_j}, & \text{if } |z_j| \le 1, \end{cases} \quad \widetilde{w_j} = \begin{cases} w_j, & \text{if } |w_j| > 1, \\ \frac{1}{w_j}, & \text{if } |w_j| \le 1. \end{cases}$$

The genus of the sequences $\{\widetilde{z_j}\}$ is defined as the lowest non-negative integer ν such that $\sum_j |\widetilde{z_j}|^{-\nu-1} < +\infty$. It is easy to see that the genus of $\widetilde{z_j}$ and $\widetilde{w_j}$ is zero, and the representation (10.2) from [9] acquires the form

$$f(z) = Cz^p \frac{\prod_{|z_j|\le 1}(1-\frac{z_j}{z})\prod_{|z_j|>1}(1-\frac{z}{z_j})}{\prod_{|w_j|\le 1}(1-\frac{w_j}{w})\prod_{|w_j|>1}(1-\frac{w}{w_j})}, \tag{6}$$

where $p \in \mathbb{Z}$, C is a constant.

Since the products in the relation (6) converge absolutely it can be rewritten in the form

$$f(z) = Cz^p \frac{\prod_{k=1}^m P(\frac{z}{a_k})}{\prod_{k=1}^m P(\frac{z}{b_k})}, \tag{7}$$

where P is the Schottky–Klein prime function ([4], [12], [13])

$$P(z) = (1-z)\prod_{n=1}^{\infty}(1-q^n z)\left(1-\frac{q^n}{z}\right).$$

There is $p \in \mathbb{Z}$ ([2], [6]) such that

$$\frac{a_1 a_2 ... a_m}{b_1 b_2 ... b_m} = q^p. \tag{8}$$

The relation (1) implies ([2], [6]) that the integer p in (7) must be equal to this one in (8).

We have obtained a known representation (7) of $f \in \mathcal{L}_q$ ([2], [6]) with p satisfying (8). It is similar to the representation of a rational function in which $P(\frac{z}{a_k})$ and $P(\frac{z}{b_k})$ is replaced by $(1-\frac{z}{a_k})$ and $(1-\frac{z}{b_k})$ respectively. A rational function is meromorphic on the Riemann sphere, which is a compact Riemann surface of genus zero. The Schottky–Klein prime function $P(\frac{z}{c})$ generalizes ([3]) $(1-\frac{z}{c})$ on the genus-one Riemann surfase which is a torus.

Thus, we can consider $f \in \mathcal{L}_q$ as a rational function on a torus.

3. Growth characteristics of elliptic functions. Since any elliptic function is meromorphic in $\mathbb{C}$, its classical Nevanlinna characteristic can be used. But the connection of elliptic functions with loxodrmic allows to consider and study another more intrinsic growth characteristic of elliptic functions.

As we noted in the introduction, if ω and ω_1 are positive numbers, $q = \exp(-2\pi\frac{\omega_1}{\omega})$, and $f(z)$ is a loxodromic meromorphic functions in $\mathbb{C}^*$, then the function

$$g(s) = f(e^{\frac{2\pi}{\omega}s}), \qquad s = \sigma + it = \frac{\log r}{2\pi}\omega + i\frac{\arg z}{2\pi}\omega$$

is elliptic with the period lattice Λ where $\omega_2 = i\omega$. Its Nevanlinna type characteristic acquires the form

$$T_0(\sigma, g) = N_0(\sigma, g) + m_0(\sigma, g), \tag{9}$$

where

$$m_0(\sigma, g) = m(\sigma, g) + m(-\sigma, g) - 2m(0, g),$$
$$m(\sigma, g) = \frac{1}{\omega}\int_0^{\omega} \log^+ |g(\sigma + it)|dt, N_0(\sigma, g) = \frac{2\pi}{\omega}\int_0^{\sigma} n_0(\eta, g)d\eta,$$

and $n_0(\eta, g)$ is the counting function of poles of g in the rectangle $\{\sigma + it : -\eta < \sigma \le \eta,\ 0 \le t < \omega\}$. If under the above assumptions f has order m, then the number of poles of g are also m in each rectangle $P_\sigma = \{\eta + it : \sigma - \omega_1 < \eta \le \sigma,\ 0 \le t < \omega\}$ what follows from the property of f mentioned above. We will call this number m the *order of* g.

Let $T_0(\sigma, g)$ be the *Nevanlinna characteristic of* g defined by relation (9). Theorem A and Theorem 1 yield the following result.

Theorem 2. *The Nevanlinna characteristic $T_0(\sigma, g)$ of an elliptic function g of order m with the period lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_1 > 0$, $\omega_2 = i\omega$, $\omega > 0$, is non-negative, non-decreasing, convex function on $\mathbb{R}_+$, and*

$$T_0(\sigma, g) = \frac{m\omega}{2\pi\omega_1}\sigma^2 + O(\sigma), \quad \sigma > 0,$$

where $|O(\sigma)| \le \frac{4m\pi}{\omega}\sigma + C$, $C = \max(T_0(\omega_1, g), 2m(0, g))$.

Note that for a meromorphic function f in $\mathbb{C}^*$ a counterpart of Jensen's Theorem is true ([7], [9]).

$$N_0\left(r, \frac{1}{f}\right) - N_0(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi})|d\varphi + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\frac{e^{i\varphi}}{r}\right)\right|d\varphi -$$
$$-\frac{1}{\pi}\int_0^{2\pi} \log|f(e^{i\varphi})|d\varphi, \quad 1 \le r. \tag{10}$$

For the elliptic function g associated with a loxodromic function f relation (10) acquires the form

$$N_0\left(\sigma, \frac{1}{g}\right) - N_0(\sigma, g) = \frac{1}{\omega}\int_0^{\omega} \log|g(\sigma+it)|dt + \frac{1}{\omega}\int_0^{\omega} \log|g(-\sigma+it)|dt - \\ -\frac{2}{\omega}\int_0^{\omega} \log|g(it)|dt, \quad 0 \le \sigma.$$

This is a version of Littlewood's Theorem ([10], [11]) for elliptic functions.

In the general case of two periods ω_1, ω_2, $\operatorname{Im}\frac{\omega_1}{\omega_2} < 0$, we have

$$z = e^{\frac{2\pi i}{\omega_2}s}, \quad z = re^{i\varphi}, \quad s = \sigma + it, \quad g(s) = f\left(e^{\frac{2\pi i}{\omega_2}s}\right).$$

Hence, $z = \exp\{2\pi|\omega_2|^{-2}[(\sigma \operatorname{Im}\omega_2 - t\operatorname{Re}\omega_2) + i(\sigma\operatorname{Re}\omega_2 + t\operatorname{Im}\omega_2)]\}$. If $|z| = r$, then

$$\sigma = \frac{\operatorname{Re}\omega_2}{\operatorname{Im}\omega_2}t + \frac{|\omega_2|^2}{2\pi\operatorname{Im}\omega_2}\log r, \quad \varphi = \frac{2\pi}{\operatorname{Im}\omega_2}t - \frac{\operatorname{Re}\omega_2}{\operatorname{Im}\omega_2}, \quad d\varphi = \frac{2\pi}{\operatorname{Im}\omega_2}dt.$$

The value $\varphi = 0$ corresponds to $t = \frac{\operatorname{Re}\omega_2}{2\pi}$, and $\varphi = 2\pi$ corresponds to $t = \operatorname{Im}\omega_2 + \frac{\operatorname{Re}\omega_2}{2\pi}$. We can assume $\operatorname{Im}\omega_2 > 0$. In the opposite case we replace ω_1 and ω_2 by $-\omega_1$ and $-\omega_2$ respectively.

Thus,

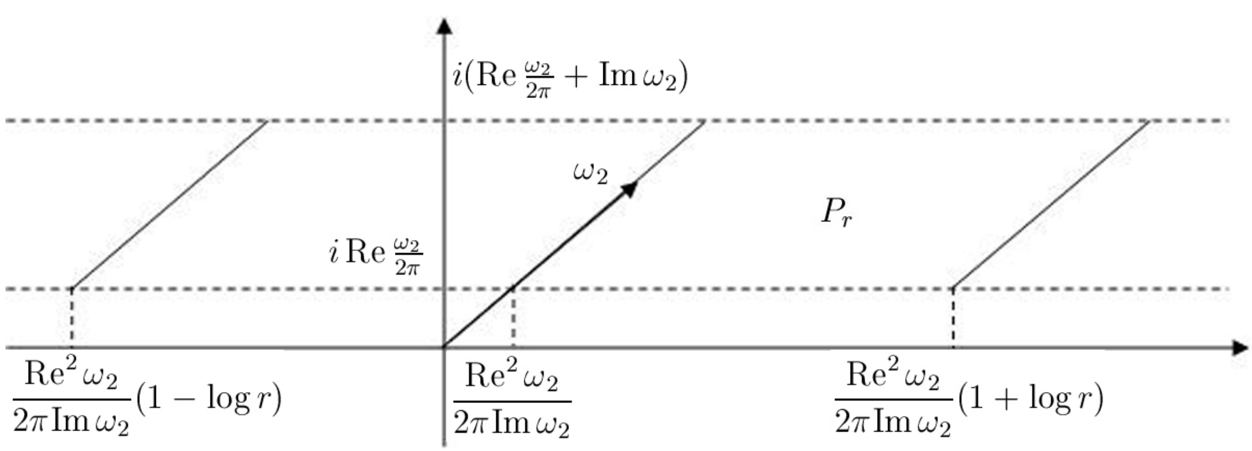
$$m(r,g) = \frac{1}{\operatorname{Im}\omega_2}\int_{\frac{\operatorname{Re}\omega_2}{2\pi}}^{\frac{\operatorname{Re}\omega_2}{2\pi}+\operatorname{Im}\omega_2} \log^+\left|g\left(\frac{\operatorname{Re}\omega_2}{\operatorname{Im}\omega_2}t + it + \frac{|\omega_2|^2}{2\pi\operatorname{Im}\omega_2}\log r\right)\right|dt,$$

$$m_0(r,g) = m(r,g) + m\left(\frac{1}{r}, g\right) - 2m(1,g), \quad r \ge 1. \tag{11}$$

The counting function $n_0(r,f)$ coincides with the counting function $n_0(r,g)$ of poles of g on the set

$$P_r = \left\{\sigma + it : \frac{\operatorname{Re}\omega_2}{\operatorname{Im}\omega_2}t - \frac{|\omega_2|^2}{2\pi\operatorname{Im}\omega_2}\log r \le \sigma < \frac{\operatorname{Re}\omega_2}{\operatorname{Im}\omega_2}t + \frac{|\omega_2|^2}{2\pi\operatorname{Im}\omega_2}\log r, \right. \\ \left. \frac{\operatorname{Re}\omega_2}{2\pi} \le t < \frac{\operatorname{Re}\omega_2}{2\pi} + \operatorname{Im}\omega_2\right\}.$$

$$N_0(r,g) = \frac{2\pi\operatorname{Im}\omega_2}{|\omega_2|^2}\int_0^{\log r} n_0(u,g)du, \quad r \ge 1. \tag{12}$$



Now the Nevanlinna characteristic of g can be written in the form $T_0(r,g) = m_0(r,g) + N_0(r,g),\ r \geq 1$, where $m_0(r,g)$ and $N_0(r,g)$ are given by the relations (11) and (12), respectively.

If f is loxodromic of multiplicator $q = e^{-2\pi i \frac{\omega_1}{\omega_2}}$, $g(s) = f(e^{\frac{2\pi i}{\omega_2} s})$, then g is elliptic with the period lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and Theorem 1 yields

$$T_0(r,g) = \frac{m}{\log \frac{1}{|q|}} \log^2 r + O(\log r), \quad r > 1.$$

REFERENCES

1. O. Rausenberger, Lehrbuch der Theorie der periodischen functionen einer Variabeln, Leipzig, Druck und Ferlag von B.G.Teubner, 1884, 470 p.
2. G. Valiron, Cours d'Analyse Mathematique, Theorie des fonctions, 2nd Edition, Masson et.Cie., Paris, 1947, 522 p.
3. D.G. Crowdy, *Geometric function theory: a modern view of a classical subject*, IOP Pablishing Ltd and London Mathematical Society, Nonlinearity, **21** (2008), T205–T219.
4. D.G. Crowdy, *Exact solutions on Steady Capillary Waves on a Fluid Annulus*, J.Nonlinear Sci., **9** (1999), 615–640.
5. S. Richardson, *Hele-Shaw flaws with time dependent free boundaries involving a concentric annulus*, Philosophical Transactions: Mathemetical, Physical and Engeneering Sciences, **354** (1996), №1718, 2513–2553.
6. Y. Hellegouarch, Invitation to the Mathematics of Fermat-Wiles, Academic Press, 2002, 381 p.
7. A.Ya. Khrystiyanyn, A.A. Kondratyuk, *On the Nevanlinna Theory for meromorphic functions on annuli. I*, Mat. Stud., **23** (2005), №1, 19–30.
8. A.Ya. Khrystiyanyn, A.A. Kondratyuk, *On the Nevanlinna Theory for meromorphic functions on annuli. II*, Mat. Stud., **24** (2005), №1, 57–68.
9. A. Kondratyuk, I. Laine, *Meromorphic functions in multiply connected domains*, Fourier series method in complex analysis (Merkrijärvi, 2005), Univ. Joensuu Dept. Math. Rep. Ser., **10** (2006), 9–111.
10. J.E. Littlewood, *On the zeros of the Riemann zeta-function*, Proc. Camb. Philos. Soc., **22** (1924), 295–318.
11. J.C. Titchmarsh, The theory of the Riemann zeta-function, Second edition, Revised by D.R. Heath-Brown. Oxford, 1986, 412 p.
12. F. Klein, *Zur Theorie der Abel'schen Functionen*, Math. Ann., **36** (1890), 1–83.
13. F. Schottky, *Über eine specielle Function welche bei einer bestimmten linearen Transformationihres Arguments unverändert bleibt*, J. Reine Angew. Math., **101** (1887), 227–272.

Faculty of Mechanics and Mathematics
Ivan Franko National University of Lviv
khrystiyanyn@ukr.net
kond@franko.lviv.ua
natalia_sokulska@yahoo.com

Received 12.12.2011