УДК 517.53

M. P. Mahola, P. V. Filevych

# THE ANGULAR VALUE DISTRIBUTION OF RANDOM ANALYTIC FUNCTIONS

M. P. Mahola, P. V. Filevych. *The angular value distribution of random analytic functions*, Mat. Stud. **37** (2012), 34–51.

Let $\mathcal{R}\in(0,+\infty]$, $f(z)=\sum c_nz^n$ be an analytic function in the disk $\{z\colon |z|<\mathcal{R}\}$, $T_f(r)$ be the Nevanlinna characteristic, $N_f(r,\alpha,\beta,a)$ be the integrated counting function of $a$-points of $f$ in the sector $0<|z|\le r$, $\alpha\le\arg_\alpha z<\beta$, and $(\omega_n(\omega))$ be a sequence of independent equidistributed on $[0,1]$ random variables. Under some conditions on the growth of $f$ it is proved that for random analytic function $f_\omega(z)=\sum e^{2\pi i\omega_n(\omega)}a_nz^n$ almost surely for every $a\in\mathbb{C}$ and all $\alpha<\beta\le\alpha+2\pi$ the relation $N_{f_\omega}(r,\alpha,\beta,a)\sim\frac{\beta-\alpha}{2\pi}T_{f_\omega}(r)$, $r\to\mathcal{R}$, holds outside some exceptional set $E\subset(0,\mathcal{R})$.

М. П. Магола, П. В. Филевич. *Угловое распределение значений случайных аналитических функций* // Мат. Студії. – 2012. – Т.37, №1. – С.34–51.

Пусть $\mathcal{R}\in(0,+\infty]$, $f(z)=\sum c_nz^n$ — аналитическая в круге $\{z\colon |z|<\mathcal{R}\}$ функция, $T_f(r)$ — характеристика Неванлинны, $N_f(r,\alpha,\beta,a)$ — усредненная считающая функция $a$-точек функции $f$ в секторе $0<|z|\le r$, $\alpha\le\arg_\alpha z<\beta$, а $(\omega_n(\omega))$ — последовательность независимых равномерно распределенных на $[0,1]$ случайных величин. При некоторых условиях на рост $f$ доказано, что для случайной аналитической функции $f_\omega(z)=\sum e^{2\pi i\omega_n(\omega)}a_nz^n$ почти наверное для всех $a\in\mathbb{C}$ и любых $\alpha<\beta\le\alpha+2\pi$ вне некоторого исключительного множества $E\subset(0,\mathcal{R})$ выполняется соотношение $N_{f_\omega}(r,\alpha,\beta,a)\sim\frac{\beta-\alpha}{2\pi}T_{f_\omega}(r)$, $r\to\mathcal{R}$.

**1. Introduction.** Let $\mathcal{D}(r)=\{z\in\mathbb{C}\colon |z|<r\}$ for all $r\in(0,+\infty]$, $\ln^+x=\ln\max\{x,1\}$ for each $x\in[0,+\infty)$, and $\mathcal{S}(r,\alpha,\beta)=\{z\in\mathbb{C}\colon 0<|z|\le r,\ \alpha\le\arg_\alpha z<\beta\}$ for any $\alpha,\beta\in\mathbb{R}$ such that $\alpha<\beta\le\alpha+2\pi$ (here, for a complex number $z\ne0$, $\arg_\alpha z$ is the value of its argument, which belongs to the interval $[\alpha,\alpha+2\pi)$). By $L$ we denote the class of positive unbounded nondecreasing functions on $[0,+\infty)$.

We consider a measurable set $E\subset\mathbb{R}$, and let $\mathcal{R}\in(0,+\infty]$. As usual, if $\mathcal{R}=+\infty$ ($\mathcal{R}<+\infty$), then the integral

$$\int_{E\cap(1,+\infty)}\frac{dr}{r}\quad\left(\int_{E\cap(0,\mathcal{R})}\frac{dr}{\mathcal{R}-r}\right)$$

is called the logarithmic measure of the set $E$ on $(0,\mathcal{R})$. The limits

$$\varlimsup_{r\to+\infty}\int_{E\cap(0,r)}\frac{dt}{r},\ \varliminf_{r\to+\infty}\int_{E\cap(0,r)}\frac{dt}{r}\quad\left(\varlimsup_{r\to\mathcal{R}}\int_{E\cap(0,r)}\frac{(\mathcal{R}-r)dt}{(\mathcal{R}-t)^2},\ \varliminf_{r\to\mathcal{R}}\int_{E\cap(0,r)}\frac{(\mathcal{R}-r)dt}{(\mathcal{R}-t)^2}\right)$$

2010 *Mathematics Subject Classification:* 30B20, 30D20, 30D35.
*Keywords:* meromorphic function, value distribution, random analytic function, Nevanlinna characteristic, counting function, integrated counting function.

© M. P. Mahola, P. V. Filevych, 2012

are called the upper density and the lower density of the set $E$ on $(0,\mathcal{R})$, respectively.

We say that a set $E$ has density $d$ on $(0,\mathcal{R})$, if its upper density and its lower density on $(0,\mathcal{R})$ are equal to $d$. It is easy to prove that every set $E$ of finite logarithmic measure on $(0,\mathcal{R})$ has density 0 on $(0,\mathcal{R})$.

All functions meromorphic (in particular, analytic) in a disk considered below are assumed to be different from constants.

We use the standard notations from the value distribution theory of meromorphic functions ([1, 2]). In particular, if $\mathcal{R}\in(0,+\infty]$, $r\in(0,\mathcal{R})$, $\alpha<\beta\le\alpha+2\pi$, and $f$ is a function meromorphic in $\mathcal{D}(\mathcal{R})$, then let $n_f(r)$ be the counting functions of poles of the function $f$, $\widetilde{n}_f(r)=n_f(r)-n_f(0)$, and $\widetilde{n}_f(r,\alpha,\beta)$ be the counting functions of poles of the function $f$ in the sector $\mathcal{S}(r,\alpha,\beta)$. We define the integrated counting functions of poles, integrated counting functions of poles in the sector $\mathcal{S}(r,\alpha,\beta)$, proximity function, Nevanlinna characteristic, and maximum modulus of the function $f$ by

$$N_f(r)=\int_0^r\widetilde{n}_f(t)\frac{dt}{t}+n_f(0)\ln r,\quad N_f(r,\alpha,\beta)=\int_0^r\widetilde{n}_f(t,\alpha,\beta)\frac{dt}{t}+\frac{\beta-\alpha}{2\pi}n_f(0)\ln r,$$

$$m_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\ln^+|f(re^{i\theta})|d\theta,\quad T_f(r)=N_f(r)+m_f(r),\quad M_f(r)=\sup\{|f(z)|\colon |z|=r\},$$

respectively. For every $a\in\mathbb{C}$ we put $X_f(r,a):=X_{\frac{1}{f-a}}(r)$, where $X$ is some of the characteristics $n$, $\widetilde{n}$, $N$, $m$ or $T$, $\widetilde{n}_f(r,\alpha,\beta,a)=\widetilde{n}_{\frac{1}{f-a}}(r,\alpha,\beta)$, $N_f(r,\alpha,\beta,a)=N_{\frac{1}{f-a}}(r,\alpha,\beta)$, and let $c_f(a)$ be the first non-zero coefficient in the Laurent series of the function $f(z)-a$ in a neighborhood of the point $z=0$.

Denote by $\mathcal{H}(\mathcal{R})$ the class of all functions analytic in the disk $\mathcal{D}(\mathcal{R})$ of the form

$$f(z)=\sum_{n=0}^{\infty}c_nz^n \tag{1}$$

such that $S_f(r):=\left(\sum_{n=0}^{\infty}|c_n|^2r^{2n}\right)^{\frac{1}{2}}\to+\infty$ $(r\to\mathcal{R})$.

Consider a probability space $(\Omega,\mathcal{A},P)$, where $\Omega$ is some set, $\mathcal{A}$ is a $\sigma$-algebra of subset of $\Omega$, $P$ is a complete probability measure on $(\Omega,\mathcal{A})$, and suppose that on this space there exists a Steinhaus sequence $(\omega_n(\omega))$, i. e. a sequence of independent uniformly distributed on $[0,1]$ random variables (see [3]). From now on we assume that such a probabilistic space and a corresponding Steinhaus sequence are given and fixed.

Along with an analytic function $f\in\mathcal{H}(\mathcal{R})$ of the form (1) we consider the random analytic function

$$f_\omega(z)=\sum_{n=0}^{\infty}e^{2\pi i\omega_n(\omega)}c_nz^n. \tag{2}$$

The value distribution of random analytic functions of form (2) were studied in the papers [4] (for $\mathcal{R}=1$) and [5] (for $\mathcal{R}=+\infty$). In particular, in [5] it is proved the following theorems (Theorem A is proved for $\mathcal{R}=+\infty$).

**Theorem A.** *Let $\mathcal{R}\in(0,+\infty]$, and $f\in\mathcal{H}(\mathcal{R})$ be an analytic function of form (1). Then for the random analytic function defined by (2) almost surely (a. s.) the inequality*

$$\ln S_f(r)\le N_{f_\omega}(r,0)+C_0\ln N_{f_\omega}(r,0)\quad (r_0(\omega)\le r<\mathcal{R})$$

*holds, where $C_0>0$ is an absolute constant.*

**Theorem B.** *Let $f$ be an entire function of form (1), $\varphi \in L$, and $\int_0^{+\infty} \frac{dx}{\varphi(x)} < +\infty$. Then there exists a set $E$ of finite logarithmic measure on $(0, +\infty)$ such that for the random entire function defined by (2) a. s. for every $a \in \mathbb{C}$ we have*

$$\ln S_f(r) \le N_{f_\omega}(r, a) + \ln^2 N_{f_\omega}(r, a)\varphi(\ln N_{f_\omega}(r, a)) \quad (r \ge r_0(\omega, a),\ r \notin E).$$

The proof of Theorem A for the case $\mathcal{R} \in (0, +\infty)$ is analogous to that for the case $\mathcal{R} = +\infty$ given in [5]. So, we assume that Theorem A is proved for all $\mathcal{R} \in (0, +\infty]$.

In this paper we consider some problems concerning the angular value distribution of random analytic functions of form (2). We also use some refinements to make Theorem B more precise.

Note that questions about the angular value distribution of analytic functions in the terms of characteristic $N_f(r, \alpha, \beta, a)$ were investigated in [6]–[8]. Mainly these papers deal with entire functions (in particular, entire functions presented by lacunary power series), satisfying the condition

$$\ln M_f(r) \sim T_f(r) \quad (E_1 \ni r \to +\infty), \tag{3}$$

where $E_1$ is a set, that is large in some sense. The following result of W. K. Hayman and J. F. Rossi [8] is one of the most general in this direction.

**Theorem C.** *Let $f$ be an entire function of the order*

$$\rho_f := \varlimsup_{r\to+\infty} \frac{\ln\ln M_f(r)}{\ln r} > 0$$

*such that relation (3) holds on a set $E_1$ of density $1$ on $(0, +\infty)$. Then there exists a set $E_2$ of upper density $1$ on $(0, +\infty)$ such that for every $a \in \mathbb{C}$ and all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \le \alpha + 2\pi$, we have*

$$N_f(r, \alpha, \beta, a) \sim \frac{\beta - \alpha}{2\pi} T_f(r) \quad (E_2 \ni r \to +\infty).$$

The next assertion follows from Theorems A and C.

**Corollary A.** *Let $f$ be an entire function of the order $\rho_f > 0$ and form (1). Then for the random entire function defined by (2) a. s. there exists a set $E_\omega$ of upper density $1$ on $(0, +\infty)$ such that*

$$N_{f_\omega}(r, \alpha, \beta, a) \sim \frac{\beta - \alpha}{2\pi} \ln S_f(r) \tag{4}$$

*as $E_\omega \ni r \to +\infty$ for every $a \in \mathbb{C}$ and all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \le \alpha + 2\pi$.*

We omit a justification of Corollary A, since below we shall prove a stronger statement.

The following theorems are the main results of our paper.

**Theorem 1.** *Let $\mathcal{R} \in (0, +\infty]$, and let $f \in \mathcal{H}(\mathcal{R})$ be an analytic function of form (1). Then there exists a function $h \in L$ such that for the random analytic function defined by (2) a. s. for every $a \in \mathbb{C}$ the inequality*

$$\ln S_f(r) \le N_{f_\omega}(r, a) + C_1 \ln\ln S_f(R) + \ln \frac{R}{R - r} + h(|a|) \quad (r_1(\omega) \le r < R < \mathcal{R}), \tag{5}$$

*holds, where $C_1 > 0$ is an absolute constant.*

**Theorem 2.** *Let $\mathcal{R} \in (0, +\infty]$, let $f \in \mathcal{H}(\mathcal{R})$ be an analytic function of form (1), let $r_0 \in (0, \mathcal{R})$ be an arbitrary fixed number such that $S_f(r_0) \geq \max\{e, \sqrt{1+|c_0|^2}\}$, and*

$$l_f(r) = \min\left\{\ln\ln S_f(R) + \ln\frac{R}{R-r} \colon R \in [r, \mathcal{R})\right\} \quad (r_0 < r < \mathcal{R}).$$

*Then for the random analytic function defined by (2) a. s. for every $a \in \mathbb{C}$ there exists a constant $C = C(\omega, a) > 0$ such that for all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \leq \alpha + 2\pi$, we have*

$$N_{f_\omega}(r, \alpha, \beta, a) \leq \frac{\beta-\alpha}{2\pi}\ln S_f(r) + C_2\left(\ln^2(S_f(r)+C)\int_{r_0}^{r}(l_f(t)+C)\ln\frac{r}{t}\frac{dt}{t}\right)^{\frac{1}{3}} + C$$

*for each $r \in (r_0, \mathcal{R})$, where $C_2 > 0$ is an absolute constant.*

Next, we formulate some corollaries from Theorems 1 and 2.

**Corollary 1.** *Let $f$ be an entire function of form (1). Then there exist a function $h \in L$ and a set $E_3$ of finite logarithmic measure on $(0, +\infty)$ such that for the random entire function defined by (2) a. s. for every $a \in \mathbb{C}$ the inequality*

$$\ln S_f(r) \leq N_{f_\omega}(r, a) + C_3\ln\ln S_f(r) + h(|a|) \quad (r \geq r_2(\omega),\ r \notin E_3) \tag{6}$$

*holds, where $C_3 > 0$ is an absolute constant.*

**Corollary 2.** *Let $f$ be an entire function of form (1) such that*

$$\varlimsup_{r\to+\infty}\frac{\ln S_f(r)}{\ln^2 r\ln\ln r} = +\infty. \tag{7}$$

*Then there exists a set $E_4$ of upper density 1 on $(0, +\infty)$ such that for the random entire function defined by (2) a. s. for every $a \in \mathbb{C}$ and all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \leq \alpha + 2\pi$, relation (4) holds as $E_4 \ni r \to +\infty$.*

**Corollary 3.** *Let $\rho \in (0, +\infty)$, and let $f$ be an entire function of the order $\rho_f \geq \rho$ and form (1). Then there exists a set $E_5$ of upper density 1 on $(0, +\infty)$ such that for the random entire function defined by (2) a. s. for every $a \in \mathbb{C}$ and all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \leq \alpha + 2\pi$, the inequality*

$$\left|N_{f_\omega}(r, \alpha, \beta, a) - \frac{\beta-\alpha}{2\pi}\ln S_f(r)\right| \leq \frac{C_4}{\sqrt[3]{\rho^2}}\ln^{\frac{2}{3}} S_f(r)\ln\ln S_f(r) \quad (r \geq r_3(\omega, a),\ r \in E_5) \tag{8}$$

*holds, where $C_4 > 0$ is an absolute constant.*

**Corollary 4.** *Let $f$ be an entire function of finite order and form (1). Then for the random entire function defined by (2) a. s. for every $a \in \mathbb{C}$ we have*

$$N_{f_\omega}(r, a) \sim \ln S_f(r) \quad (r \to +\infty). \tag{9}$$

**Corollary 5.** *Let $f$ be an entire function of finite order and form (1) such that*

$$\lim_{r\to+\infty}\frac{\ln S_f(r)}{\ln^3 r} = +\infty. \tag{10}$$

*Then for the random entire function defined by (2) a. s. for every $a \in \mathbb{C}$ and all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \leq \alpha + 2\pi$, relation (4) holds as $r \to +\infty$.*

**Corollary 6.** *Let $\mathcal{R}\in(0,+\infty)$, and let $f\in\mathcal{H}(\mathcal{R})$ be an analytic function of form (1) such that*
$$\varlimsup_{r\to\mathcal{R}}\frac{\ln S_f(r)}{\ln\frac{1}{\mathcal{R}-r}}=+\infty. \tag{11}$$
*Then there exists a set $E_6$ of upper density 1 on $(0,\mathcal{R})$ such that for the random analytic function defined by (2) a. s. for every $a\in\mathbb{C}$ and all $\alpha,\beta\in\mathbb{R}$, $\alpha<\beta\le\alpha+2\pi$, relation (4) holds as $E_6\ni r\to\mathcal{R}$.*

Concluding the Introduction, we note that the value distribution and other properties of some classes of random analytic functions were investigated also in [9]–[24].

**2. Auxiliary results.** Let $x_1,\dots,x_n\in[0,+\infty)$. The following inequalities
$$\ln^+\left|\prod_{\nu=1}^n x_\nu\right|\le\sum_{\nu=1}^n\ln^+|x_n|,\quad \ln^+\left|\sum_{\nu=1}^n x_\nu\right|\le\sum_{\nu=1}^n\ln^+|x_n|+\ln n$$
are well known (see, for example, [2], p. 14). Below we will use these inequalities without additional explanations.

The following lemma is proved in [25].

**Lemma A.** *Let $\mathcal{R}\in(0,+\infty]$, and let $g$ be a meromorphic function in the disk $\mathcal{D}(\mathcal{R})$ such that $g(0)=1$. Then for arbitrary $\alpha,\beta\in(0,1)$ the inequality*
$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{g'(re^{i\theta})}{g(re^{i\theta})}\right|^\alpha d\theta\le C(\alpha,\beta)\left(\frac{T_g(R)}{r}\frac{R}{R-r}\right)^\alpha\quad(0<r<R<\mathcal{R}) \tag{12}$$
*is true, where*
$$C(\alpha,\beta)=\left(\frac{2}{1-\beta}\right)^\alpha+\left(\frac{4+\left(2^{\frac{1+\alpha}{1-\alpha}}+2^{\frac{2+\alpha}{1-\alpha}}\right)^{1-\alpha}}{\beta^\alpha}\right)\sec\frac{\alpha\pi}{2}.$$

For a function $f$ meromorphic in $\mathcal{D}(\mathcal{R})$ and every $z\in\mathcal{D}(\mathcal{R})$ we put $g^*(z)=zg'(z)$. Then inequality (12) is equivalent to the inequality
$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{g^*(re^{i\theta})}{g(re^{i\theta})}\right|^\alpha d\theta\le C(\alpha,\beta)\left(T_g(R)\frac{R}{R-r}\right)^\alpha\quad(0<r<R<\mathcal{R}). \tag{13}$$

Arguing as in the paper [26] in the proof of its main result, and using inequality (13) instead of inequality (12), it is easy to prove the following statement.

**Lemma B.** *Let $\mathcal{R}\in(0,+\infty]$, and let $g$ be a function meromorphic in the disk $\mathcal{D}(\mathcal{R})$ such that $g(0)=1$. Then*
$$m_{\frac{g^*}{g}}(r)\le\ln^+\left(T_g(R)\frac{R}{R-r}\right)+4,8517\quad(0<r<R<\mathcal{R}).$$

**Lemma 1.** *Let $\mathcal{F}\subset[0,2\pi]$ be a measurable set, $\mathcal{R}\in(0,+\infty]$, $r\in(0,\mathcal{R})$, $f$ be a function analytic in the disk $\mathcal{D}(\mathcal{R})$ of form (1). Then*
$$\frac{1}{2\pi}\int_{\mathcal{F}}\ln^+|f(re^{i\theta})|d\theta\le\frac{1}{2e}+\frac{\mu(\mathcal{F})}{2\pi}\ln^+S_f(r), \tag{14}$$
*where $\mu(\mathcal{F})$ is the Lebesgue measure of the set $\mathcal{F}$.*

*Proof.* Let $\mathcal{E} = \{\theta \in \mathcal{F} \colon |f(re^{i\theta})| > 1\}$. If $\mu(\mathcal{E}) = 0$ then inequality (14) is trivial. If $\mu(\mathcal{E}) > 0$, then, using the Jensen inequality (see, for example, [27], p. 42)

$$\frac{1}{\mu(\mathcal{E})} \int_{\mathcal{E}} \ln |f(re^{i\theta})|^2 d\theta \le \ln \left( \frac{1}{\mu(\mathcal{E})} \int_{\mathcal{E}} |f(re^{i\theta})|^2 d\theta \right)$$

and the Parseval equality

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = 2\pi S_f^2(r),$$

we obtain

$$\frac{1}{2\pi} \int_{\mathcal{F}} \ln^+ |f(re^{i\theta})| d\theta = \frac{1}{4\pi} \int_{\mathcal{E}} \ln |f(re^{i\theta})|^2 d\theta \le \frac{\mu(\mathcal{E})}{4\pi} \ln \left( \frac{1}{\mu(\mathcal{E})} \int_{\mathcal{E}} |f(re^{i\theta})|^2 d\theta \right) \le$$
$$\le \frac{\mu(\mathcal{E})}{4\pi} \ln \left( \frac{1}{\mu(\mathcal{E})} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right) = \frac{\mu(\mathcal{E})}{4\pi} \ln \frac{2\pi}{\mu(\mathcal{E})} + \frac{\mu(\mathcal{E})}{2\pi} \ln S_f(r).$$

Since the most value of the function $y(x) = \frac{x}{2} \ln \frac{1}{x}$ on the interval $(0, +\infty)$ is equal to $\frac{1}{2e}$, Lemma 1 is proved. □

**Lemma 2.** *Let $\mathcal{R} \in (0, +\infty]$, and let $f$ be a function analytic in the disk $\mathcal{D}(\mathcal{R})$. Then for every $a \in \mathbb{C}$ and all $r, R \in (0, \mathcal{R})$, $r < R$, we have*

$$m_{\frac{f^*}{f-a}}(r) \le \ln^+ \ln^+ S_f(R) + \ln \frac{R}{R-r} + \ln^+ \frac{1}{|c_f(a)|} +$$
$$+ n_f(0,a) \ln^+ \frac{1}{R} + \ln^+ n_f(0,a) + \ln^+ |a| + 7. \tag{15}$$

*Proof.* We fix arbitrary $a \in \mathbb{C}$ and $r, R \in (0, \mathcal{R})$, $r < R$. Put

$$g(z) = \frac{f(z) - a}{c_f(a) z^{n_f(0,a)}} \quad (z \in \mathcal{D}(\mathcal{R})).$$

It is easily verified that

$$\frac{f^*(z)}{f(z) - a} = \frac{g^*(z)}{g(z)} + n_f(0,a) \quad (z \in \mathcal{D}(\mathcal{R})).$$

Consequently,

$$m_{\frac{f^*}{f-a}}(r) \le m_{\frac{g^*}{g}}(r) + \ln^+ n_f(0,a) + \ln 2. \tag{16}$$

In addition, $g(0) = 1$. Therefore, by Lemma B, we have

$$m_{\frac{g^*}{g}}(r) \le \ln^+ T_g(R) + \ln \frac{R}{R-r} + 4,8517. \tag{17}$$

Next note that Lemma 1 implies the inequality

$$T_f(r) \le \frac{1}{2e} + \ln^+ S_f(r) \quad (r \in (0, \mathcal{R})). \tag{18}$$

Using this inequality with $R$ instead of $r$, we obtain

$$\ln^+ T_g(R) \le \ln^+ T_f(R) + \ln^+ |a| + \ln 2 + \ln^+ \frac{1}{|c_f(a)|} + n_f(0,a) \ln^+ \frac{1}{R} \le$$
$$\le \ln^+ \ln^+ S_f(R) + 2\ln 2 + \ln^+ |a| + \ln^+ \frac{1}{|c_f(a)|} + n_f(0,a) \ln^+ \frac{1}{R}. \tag{19}$$

Then inequality (15) is an obvious consequence from inequalities (16), (17), and (19). □

Let $\mathcal{R} \in (0, +\infty]$, $r \in (0, \mathcal{R})$, and let $g$ be a function analytic in the disk $\mathcal{D}(\mathcal{R})$. We set

$$\mathcal{E}_g(r) = \{\theta \in \mathbb{R} \colon g(te^{i\theta}) \neq 0 \text{ for all } t \in (0, r]\}.$$

Note, that $\mathcal{E}_g(r_2) \subset \mathcal{E}_g(r_1)$ if $0 < r_1 < r_2 < \mathcal{R}$. The set $\mathcal{E}_g(r)$ is periodic in the sense that $\theta \in \mathcal{E}_g(r)$ if and only if $(\theta + 2\pi) \in \mathcal{E}_g(r)$. In addition, $[0, 2\pi) \backslash \mathcal{E}_g(r)$ is a finite set for all $r \in (0, \mathcal{R})$.

Suppose that $g(0) = 1$, and fix an arbitrary $\theta \in \mathcal{E}_g(r)$. Then $g(te^{i\theta}) \neq 0$ for each $t \in [0, r]$. In view of this, by $v_g(t, \theta)$ we denote the continuous branch of the argument of the function $g(te^{i\theta})$ such that $v_g(0, \theta) = 0$, and put

$$V_g(r, \theta) = \frac{1}{2\pi} \int_0^r v_g(t, \theta) \frac{dt}{t}.$$

The following statement is well known (see [8], [6], and [28], p. 126).

**Lemma C.** *Let $\mathcal{R} \in (0, +\infty]$, $r \in (0, \mathcal{R})$, and let $g$ be a function analytic in the disk $\mathcal{D}(\mathcal{R})$ such that $g(0) = 1$. Then:*

*(i) for all $\alpha, \beta \in \mathcal{E}_g(r)$ such that $\alpha < \beta \leq \alpha + 2\pi$ we have*

$$N_g(r, \alpha, \beta, 0) = \frac{1}{2\pi} \int_\alpha^\beta \ln |g(re^{i\theta})| d\theta + V_g(r, \alpha) - V_g(r, \beta);$$

*(ii) for all $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta \leq \alpha + 2\pi$ we have*

$$\int_\alpha^\beta V_g(r, \theta) d\theta = \frac{1}{2\pi} \int_0^r (\ln |g(te^{i\alpha})| - \ln |g(te^{i\beta})|) \ln \frac{r}{t} \frac{dt}{t}.$$

Let $g$ be a function analytic in the disk $\mathcal{D}(\mathcal{R})$ such that $g(0) = 1$. Consider an arbitrary interval $(\varphi, \psi) \subset \mathcal{E}_g(r)$, fix some point $\alpha$ in this interval, and let $\beta \neq \alpha$ be an arbitrary point of this interval. Then the function $g$ has no zeros in the sector $\mathcal{S}(r, \min\{\alpha, \beta\}, \max\{\alpha, \beta\})$. Therefore, $N_g(r, \min\{\alpha, \beta\}, \max\{\alpha, \beta\}, 0) = 0$. According to (i) of Lemma C we have

$$V_g(r, \beta) = V_g(r, \alpha) + \frac{1}{2\pi} \int_\alpha^\beta \ln |g(re^{i\theta})| d\theta.$$

Since for a fixed $\alpha$ the function $y(\beta) = \int_\alpha^\beta \ln |g(re^{i\theta})| d\theta$ is continuous and bounded on every finite interval of the real axis, then $V_g(r, \beta)$, as a function of $\beta$, is continuous and bounded on the interval $(\varphi, \psi)$. From the above considerations, as well as from the periodicity of the set $\mathcal{E}_g(r)$ and the finiteness of the set $[0, 2\pi) \backslash \mathcal{E}_g(r)$, we obtain that the function $V_g(r, \beta)$ is continuous and bounded on $\mathcal{E}_g(r)$.

Now let $f$ be an arbitrary function analytic in the disk $\mathcal{D}(\mathcal{R})$. Put

$$g(z) = \frac{f(z)}{c_f(0) z^{n_f(0,0)}}.$$

Then $g(0) = 1$, $\mathcal{E}_f(r) = \mathcal{E}_g(r)$, and $\widetilde{n}_f(r, \alpha, \beta, 0) = \widetilde{n}_g(r, \alpha, \beta, 0)$. Therefore, setting $V_f(r, \theta) = V_g(r, \theta)$ for all $\theta \in \mathcal{E}_f(r)$, from Lemma C, as a consequence, we obtain the following statement.

**Lemma D.** *Let $\mathcal{R} \in (0, +\infty]$, $r \in (0, \mathcal{R})$, and let $f$ be a function analytic in the disk $\mathcal{D}(\mathcal{R})$. Then there exists a function $V_f(r, \theta)$ continuous and bounded on $\mathcal{E}_f(r)$ such that:*

*(i) for all* $\alpha, \beta \in \mathcal{E}_f(r)$ *such that* $\alpha < \beta \le \alpha + 2\pi$ *we have*

$$N_f(r, \alpha, \beta, 0) = \frac{1}{2\pi}\int_\alpha^\beta \ln|f(re^{i\theta})|d\theta - \frac{\beta-\alpha}{2\pi}\ln|c_f(0)| + V_f(r,\alpha) - V_f(r,\beta);$$

*(ii) for all* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha < \beta \le \alpha + 2\pi$ *we have*

$$\int_\alpha^\beta V_f(r,\theta)d\theta = \frac{1}{2\pi}\int_0^r (\ln|f(te^{i\alpha})| - \ln|f(te^{i\beta})|)\ln\frac{r}{t}\frac{dt}{t}.$$

Note that the equality from assertion (i) of Lemma D is a generalization of the following classical Jensen equality

$$N_f(r, 0) = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta - \ln|c_f(0)| \quad (r \in (0, \mathcal{R})). \tag{20}$$

Also note that equality (20) implies the inequality

$$N_f(r, 0) \le T_f(r) - \ln|c_f(0)| \quad (r \in (0, \mathcal{R})). \tag{21}$$

In fact, the following statement [29] is an immediate consequence of the classical Borel-Nevanlinna lemma (see [2], p. 90).

**Lemma E.** *Let* $u(r)$ *be a nondecreasing function unbounded on* $[r_0, +\infty)$, $x_0 = u(r_0)$, *and let* $\varphi(x)$ *be a continuous positive function increasing to* $+\infty$ *on* $[x_0, +\infty)$ *such that* $\int_{x_0}^{+\infty}\frac{dx}{\varphi(x)} < +\infty$. *Then for all* $r \ge r_0$ *outside a set* $E$ *of finite logarithmic measure on* $(0, +\infty)$ *we have*

$$u\left(r\exp\left\{\frac{1}{\varphi(\ln u(r))}\right\}\right) < eu(r).$$

**Lemma 3.** *Let* $0 \le r_0 < \mathcal{R} < +\infty$, $u(r)$ *be a nondecreasing function unbounded on* $[r_0, \mathcal{R})$, $x_0 = u(r_0)$, *and let* $\varphi(x)$ *be a continuous positive function increasing to* $+\infty$ *on* $[x_0, +\infty)$ *such that* $\int_{x_0}^{+\infty}\frac{dx}{\varphi(x)} < +\infty$. *Then for all* $r \in [r_0, \mathcal{R})$ *outside a set* $E$ *of finite logarithmic measure on* $(0, \mathcal{R})$ *we have*

$$u\left(\mathcal{R} - (\mathcal{R} - r)\exp\left\{-\frac{1}{\varphi(\ln u(r))}\right\}\right) < eu(r).$$

*Proof.* It suffices to show that the set

$$F = \left\{r \in [r_0, \mathcal{R}) \colon u\left(\mathcal{R} - (\mathcal{R} - r)\exp\left\{-\frac{1}{\varphi(\ln u(r))}\right\}\right) \ge eu(r)\right\}$$

has finite logarithmic measure on $(0, \mathcal{R})$.

Put

$$r' = \frac{\mathcal{R}}{\mathcal{R} - r}, \quad r_0' = \frac{\mathcal{R}}{\mathcal{R} - r_0}, \quad v(r') = u\left(\mathcal{R}\frac{r'-1}{r'}\right).$$

Then $r = \mathcal{R}\left(1 - \frac{1}{r'}\right)$. It is easy to verify that the set $F$ is the image of the set

$$F' = \left\{r' \ge r_0' \colon v\left(r'\exp\left\{\frac{1}{\varphi(\ln v(r'))}\right\}\right) \ge ev(r')\right\}$$

under the mapping $r = \mathcal{R}\left(1-\frac{1}{r'}\right)$. By Lemma E, the set $F'$ has finite logarithmic measure on $(0,+\infty)$. Therefore,

$$\int_F \frac{dr}{\mathcal{R}-r} = \int_{F'} \frac{r'}{\mathcal{R}} d\mathcal{R}\left(1-\frac{1}{r'}\right) = \int_{F'} \frac{dr'}{r'} < +\infty,$$

i. e. the set $F$ has finite logarithmic measure on $(0,\mathcal{R})$. □

**3. Proofs of theorems and corollaries.**

*Proof of Theorem 1.* Let $\mathcal{R} \in (0,+\infty]$, $f \in \mathcal{H}(\mathcal{R})$ be an analytic function of form (1), and $m = \min\{n \in \mathbb{N} \colon c_n \neq 0\}$. Fix an arbitrary $r_0 \in (0,\mathcal{R})$ such that $S_{f^*}(r) \geq S_f(r) \geq e$ $(r \geq r_0)$, and for each $x \in [0,+\infty)$ put

$$h_0(x) = m\ln^+\frac{1}{r_0} + \ln^+ m + \ln^+ x + 7, \quad h(x) = h_0(x) + \ln^+\max\{|c_0|+x, |c_m|\} + \ln^+\frac{1}{m|c_m|}.$$

It is clear that $h \in L$.

Consider the random analytic function defined by (2). Then, as easily seen, for all $\omega \in \Omega$ and each $a \in \mathbb{C}$ the relations

$$|c_{f^*_\omega}(0)| = m|c_m|, \quad |c_{f_\omega}(a)| \leq \max\{|c_0|+|a|, |c_m|\}, \quad |n_{f_\omega}(0,a)| \leq m$$

are true.

Let $C_0$ be the constant from Theorem A, and $A$ is the following event: there exists $r_0(\omega) \in (0,\mathcal{R})$ such that

$$\ln S_{f^*}(r) \leq N_{f^*_\omega}(r,0) + (C_0+1)\ln\ln S_{f^*}(r) \quad (r_0(\omega) \leq r < \mathcal{R}). \tag{22}$$

By Theorem A we have $P(A) = 1$. Furthermore, since $S_{f^*}(r) \geq S_f(r)$ $(r \geq r_0)$ and the function $y(x) = x - (C_0+1)\ln x$ is increasing on $[x_0,+\infty)$, for every $\omega \in A$ satisfying (22) we obtain

$$\ln S_f(r) \leq N_{f^*_\omega}(r,0) + (C_0+1)\ln\ln S_f(r) \quad (r_1(\omega) \leq r < \mathcal{R}), \tag{23}$$

where $r_1(\omega) \geq r_0$.

Fix an arbitrary $\omega \in \Omega$. Using Jensen's formula (20), written for the functions $f^*_\omega$ and $f_\omega - a$, and Lemma 2 for the function $f_\omega$ instead of $f$, for each $a \in \mathbb{C}$ and all $r, R \in [r_0,\mathcal{R})$, $r < R$, we have

$$N_{f^*_\omega}(r,0) - N_{f_\omega}(r,a) = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|\frac{f^*_\omega(re^{i\theta})}{f_\omega(re^{i\theta})-a}\right| d\theta - \ln\frac{|c_{f^*_\omega}(0)|}{|c_{f_\omega}(a)|} \leq$$
$$\leq m_{\frac{f^*_\omega}{f_\omega - a}}(r) - \ln\frac{|c_{f^*_\omega}(0)|}{|c_{f_\omega}(a)|} \leq \ln\ln S_f(R) + \ln\frac{R}{R-r} + \ln^+\frac{1}{|c_{f_\omega}(a)|} + h_0(|a|) - \ln\frac{|c_{f^*_\omega}(0)|}{|c_{f_\omega}(a)|} =$$
$$= \ln\ln S_f(R) + \ln\frac{R}{R-r} + h_0(|a|) + \ln^+|c_{f_\omega}(a)| - \ln|c_{f^*_\omega}(0)| \leq$$
$$\leq \ln\ln S_f(R) + \ln\frac{R}{R-r} + h(|a|).$$

From this and from (23) for arbitrary $\omega \in A$ and $a \in \mathbb{C}$ we obtain

$$\ln S_f(r) \leq N_{f_\omega}(r,a) + (C_0+2)\ln\ln S_f(R) + \ln\frac{R}{R-r} + h(|a|) \quad (r_1(\omega) \leq r < R < \mathcal{R}).$$ □

Theorem 2 is obtained from Theorem 1 and the following statement.

**Theorem 3.** *Let $\mathcal{R} \in (0, +\infty]$, $g \in \mathcal{H}(\mathcal{R})$, and let $r_0 \in (0, \mathcal{R})$ be an arbitrary fixed number such that $S_g(r_0) \geq 1$, and*

$$h_g(r) = \int_{r_0}^{r} \left( \ln S_g(t) - N_g(t, 0) - \ln |c_g(0)| + \frac{1}{e} \right) \ln \frac{r}{t} \frac{dt}{t} \quad (r_0 < r < \mathcal{R}). \tag{24}$$

*Then there exists a constant $C_7 > 0$ such that for all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \leq \alpha + 2\pi$, we have*

$$N_g(r, \alpha, \beta, 0) \leq \frac{\beta - \alpha}{2\pi} \ln S_g(r) + 3 \sqrt[3]{\frac{3}{\pi^2} \ln^2 S_g(r) h_g(r)} + C_7 \quad (r_0 < r < \mathcal{R}). \tag{25}$$

*Proof.* Let $C_7 = \frac{1}{e} + 2C_8 + 2|\ln |c_g(0)||$, where $C_8$ is a constant such that $|V_g(r_0, \theta)| \leq C_8$ for every $\theta \in \mathcal{E}_g(r_0)$ (see Lemma D).

Fix arbitrary $r > r_0$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta \leq \alpha + 2\pi$. If $8h_g(r) \geq \ln S_g(r)$, then inequality (25) holds. Indeed, using inequalities (21) and (18) with the function $g$ instead of $f$, we have

$$N_g(r, \alpha, \beta, 0) \leq N_g(r, 0) \leq \frac{\beta - \alpha}{2\pi} \ln S_g(r) + N_g(r, 0) \leq$$
$$\leq \frac{\beta - \alpha}{2\pi} \ln S_g(r) + \ln S_g(r) + \frac{1}{2e} - \ln |c_g(0)| \leq \frac{\beta - \alpha}{2\pi} \ln S_g(r) + 2 \sqrt[3]{\ln^2 S_g(r) h_g(r)} + C_7.$$

Now let $8h_g(r) < \ln S_g(r)$ and

$$\varepsilon = \sqrt[3]{\frac{8\pi}{9} \frac{h_g(r)}{\ln S_g(r)}}.$$

Inequalities (21) and (18) imply that $\ln S_g(t) - N_g(t, 0) - \ln |c_g(0)| + \frac{1}{e} \geq \frac{1}{2e}$ for all $t \in (r_0, r)$. Thus, $h_g(r) > 0$. Moreover, $\ln S_g(r) > 0$. Therefore, $\varepsilon > 0$. On the other hand,

$$\varepsilon < \sqrt[3]{\frac{8\pi}{9} \frac{1}{8}} < \frac{\pi}{4}.$$

Put

$$\varphi(\theta) = \frac{1}{2\pi} \int_{r_0}^{r} \ln |g(te^{i\theta})| \ln \frac{r}{t} \frac{dt}{t}.$$

Then, applying Lemma 1 to the function $g$, we obtain

$$I_1 := \int_{\alpha - 3\varepsilon}^{\alpha - 2\varepsilon} \varphi(\theta) d\theta = \int_{r_0}^{r} \left( \frac{1}{2\pi} \int_{\alpha - 3\varepsilon}^{\alpha - 2\varepsilon} \ln |g(te^{i\theta})| d\theta \right) \ln \frac{r}{t} \frac{dt}{t} \leq \int_{r_0}^{r} \left( \frac{1}{2e} + \frac{\varepsilon}{2\pi} \ln S_g(t) \right) \ln \frac{r}{t} \frac{dt}{t}.$$

From this and from the mean value theorem, applied to the integral $I_1$, it follows the existence of a number $\zeta_1 \in [\alpha - 3\varepsilon, \alpha - 2\varepsilon]$ such that

$$\varphi(\zeta_1) \leq \frac{1}{\varepsilon} \int_{r_0}^{r} \left( \frac{1}{2e} + \frac{\varepsilon}{2\pi} \ln S_g(t) \right) \ln \frac{r}{t} \frac{dt}{t}. \tag{26}$$

Since for any $x, y \in \mathbb{R}$ the Jensen formula implies the equality

$$\frac{1}{2\pi} \int_{x}^{y} \ln |g(re^{i\theta})| d\theta = N_g(r, 0) + \ln |c_g(0)| - \frac{1}{2\pi} \int_{y}^{x + 2\pi} \ln |g(re^{i\theta})| d\theta, \quad x, y \in \mathbb{R},$$

using Lemma 1, in the case $x < y \le x + 2\pi$ we have

$$\frac{1}{2\pi}\int_x^y \ln|g(re^{i\theta})|d\theta \ge N_g(r,0) + \ln|c_g(0)| - \frac{1}{2e} - \frac{x+2\pi-y}{2\pi}\ln S_g(r).$$

Then

$$I_2 := \int_{\alpha-\varepsilon}^{\alpha}\varphi(\theta)d\theta = \int_{r_0}^{r}\left(\frac{1}{2\pi}\int_{\alpha-\varepsilon}^{\alpha}\ln|g(te^{i\theta})|d\theta\right)\ln\frac{r}{t}\frac{dt}{t} \ge$$
$$\ge \int_{r_0}^{r}\left(N_g(t,0) + \ln|c_g(0)| - \frac{1}{2e} - \frac{2\pi-\varepsilon}{2\pi}\ln S_g(t)\right)\ln\frac{r}{t}\frac{dt}{t}.$$

Therefore, the mean value theorem applied to the integral $I_2$, yields the existence of a number $\zeta_2 \in [\alpha-\varepsilon, \alpha]$ such that

$$\varphi(\zeta_2) \ge \frac{1}{\varepsilon}\int_{r_0}^{r}\left(N_g(t,0) + \ln|c_g(0)| - \frac{1}{2e} - \frac{2\pi-\varepsilon}{2\pi}\ln S_g(t)\right)\ln\frac{r}{t}\frac{dt}{t}. \qquad (27)$$

Then, using Lemma D and inequalities (26) and (27), we obtain

$$I_3 := \int_{\zeta_1}^{\zeta_2}(V_g(r,\theta) - V_g(r_0,\theta))d\theta = \frac{1}{2\pi}\int_{r_0}^{r}(\ln|g(te^{i\zeta_1})| - \ln|g(te^{i\zeta_2})|)\ln\frac{r}{t}\frac{dt}{t} =$$
$$= \varphi(\zeta_1) - \varphi(\zeta_2) \le \frac{1}{\varepsilon}\int_{r_0}^{r}(\ln S_g(t) - N_g(t,0) - \ln|c_g(0)| + \frac{1}{e})\ln\frac{r}{t}\frac{dt}{t} = \frac{1}{\varepsilon}h_g(r).$$

From the inequality $\zeta_2 - \zeta_1 \ge \varepsilon$ and from the mean value theorem, applied to the integral $I_3$, it follows the existence of a number $\zeta \in [\zeta_1, \zeta_2] \cap \mathcal{E}_g(r)$ such that

$$V_g(r,\zeta) - V_g(r_0,\zeta) \le \frac{1}{\varepsilon^2}h_g(r). \qquad (28)$$

Similarly we can prove that there exist numbers $\eta_1 \in [\beta, \beta+\varepsilon]$ and $\eta_2 \in [\beta+2\varepsilon, \beta+3\varepsilon]$ such that

$$\varphi(\eta_1) \ge \frac{1}{\varepsilon}\int_{r_0}^{r}\left(N_g(t,0) + \ln|c_g(0)| - \frac{1}{2e} - \frac{2\pi-\varepsilon}{2\pi}\ln S_g(t)\right)\ln\frac{r}{t}\frac{dt}{t},$$
$$\varphi(\eta_2) \le \frac{1}{\varepsilon}\int_{r_0}^{r}\left(\frac{1}{2e} + \frac{\varepsilon}{2\pi}\ln S_g(t)\right)\ln\frac{r}{t}\frac{dt}{t}.$$

Then
$$I_4 := \int_{\eta_1}^{\eta_2}(V_g(r,\theta) - V_g(r_0,\theta))d\theta = \varphi(\eta_1) - \varphi(\eta_2) \ge -\frac{1}{\varepsilon}h_g(r).$$

The inequality $\eta_2 - \eta_1 \le 3\varepsilon$ together with the mean value theorem applied to the integral $I_4$ implies the existence of a number $\eta \in [\eta_1, \eta_2] \cap \mathcal{E}_g(r)$ such that

$$V_g(r,\eta) - V_g(r_0,\eta) \ge -\frac{1}{3\varepsilon^2}h_g(r). \qquad (29)$$

Using Lemmas D and 1, as well as inequalities (28) and (29), and taking into account that $\eta - \zeta \le \beta - \alpha + 6\varepsilon < 4\pi$, we obtain

$$N_g(r,\alpha,\beta,0) \le N_g(r,\zeta,\eta,0) = \frac{1}{2\pi}\int_\zeta^\eta \ln|g(re^{i\theta})|d\theta - \frac{\eta-\zeta}{2\pi}\ln|c_g(0)| + V_g(r,\zeta) - V_g(r,\eta) \le$$
$$\le \frac{\eta-\zeta}{2\pi}\ln S_g(r) + \frac{1}{e} - \frac{\eta-\zeta}{2\pi}\ln|c_g(0)| + V_g(r,\zeta) - V_g(r,\eta) \le$$
$$\le \frac{\beta-\alpha+6\varepsilon}{2\pi}\ln S_g(r) + \frac{1}{e} + 2|\ln|c_g(0)|| + \frac{1}{\varepsilon^2}h_g(r) + C_8 + \frac{1}{3\varepsilon^2}h_g(r) + C_8 =$$
$$= \frac{\beta-\alpha}{2\pi}\ln S_g(r) + 3\sqrt[3]{\frac{3}{\pi^2}\ln^2 S_g(r) h_g(r)} + C_7. \qquad \square$$

*Proof of Theorem 2.* Let $f \in \mathcal{H}(\mathcal{R})$ be an analytic function of form (1), and let $r_0 \in (0,\mathcal{R})$ be an arbitrary fixed number such that $S_f(r_0) \ge \max\{e, \sqrt{1+|c_0|^2}\}$.

Consider the random analytic function defined by (2). For arbitrary $\omega \in \Omega$, $a \in \mathbb{C}$, and $r \in (0,\mathcal{R})$ we have $S^2_{f_\omega - a}(r) = |c_0 e^{2\pi i \omega_0(\omega)} - a|^2 + S_f^2(r) - |c_0|^2$. This implies the inequalities

$$S_f^2(r) - |c_0|^2 \le S^2_{f_\omega-a}(r) \le S_f^2(r) + (|c_0| + |a|)^2.$$

Then $S^2_{f_\omega-a}(r) \ge 1$ by the first of these inequalities. By the second of these inequalities, there exists a constant $C_9 = C_9(a) > 0$ such that

$$\ln S_{f_\omega-a}(r) \le \ln S_f(r) + C_9 \quad (r_0 < r < \mathcal{R}). \tag{30}$$

Let $C_1 > 0$ is the absolute constant and $h \in L$ is the function, the existence of which follows from Theorem 1. Let $B$ be the next event: for every $a \in \mathbb{C}$ inequality (5) holds. Then, by Theorem 1, $P(B) = 1$.

Fix arbitrary $\omega \in B$ and $a \in \mathbb{C}$, and let $g(z) = f_\omega(z) - a$. Then (5) and (30) implies the existence of a constant $C_{10} = C_{10}(\omega,a) > 0$ such that

$$\ln S_g(r) - N_g(r,0) - \ln|c_g(0)| + \frac{1}{e} \le C_1(l_f(r) + C_{10}) \quad (r_0 < r < \mathcal{R}).$$

So, if $h_g$ is the function defined by the equality (24), then

$$h_g(r) \le C_1 \int_{r_0}^r (l_f(t) + C_{10}) \ln\frac{r}{t}\frac{dt}{t} \quad (r_0 < r < \mathcal{R}). \tag{31}$$

By Theorem 3, there exists a constant $C_7 = C_7(\omega,a) > 0$ such that for arbitrary $\alpha,\beta \in \mathbb{R}$, $\alpha < \beta \le \alpha + 2\pi$, inequality (25) holds. Using this inequality and also inequalities (30) and (31), for arbitrary $\alpha,\beta \in \mathbb{R}$, $\alpha < \beta \le \alpha + 2\pi$, and all $r \in (r_0,\mathcal{R})$ we obtain

$$N_{f_\omega}(r,\alpha,\beta,a) = N_g(r,\alpha,\beta,0) \le \frac{\beta-\alpha}{2\pi}\ln S_g(r) + 3\sqrt[3]{\frac{3}{\pi^2}\ln^2 S_g(r) h_g(r)} + C_7 \le$$
$$\le \frac{\beta-\alpha}{2\pi}\ln S_f(r) + C_9 + 3\left(\frac{3}{\pi^2}C_1 \ln^2(S_f(r) + C_9)\int_{r_0}^r (l_f(t) + C_{10})\ln\frac{r}{t}\frac{dt}{t}\right)^{\frac{1}{3}} + C_7.$$

Finally, putting $C(\omega,a) = \max\{C_9(a) + C_7(\omega,a), C_{10}(\omega,a)\}$ and $C_2 = 3\sqrt[3]{\frac{3C_1}{\pi^2}}$, we complete the proof of Theorem 2. $\square$

*Proof of Corollary 1.* Let $f$ be an entire function of form (1), $r_0=\min\{r\ge 0\colon\ \ln S_f(r)\ge e\}$, and $x_0=\ln S_f(r_0)$. Put

$$R(r)=r\exp\left\{\frac{1}{\ln^2\ln S_f(r)}\right\}\quad (r\ge r_0).$$

Applying Lemma E to the functions $u(r)=\ln S_f(r)$ $(r\ge r_0)$ and $\varphi(x)=x^2$ $(x\ge x_0)$, we have

$$\ln S_f(R(r))<e\ln S_f(r)\quad (r\ge r_0,\ r\notin E_3),\tag{32}$$

where $E_3$ is a set of finite logarithmic measure on $(0,+\infty)$. Furthermore, obviously,

$$\frac{R(r)}{R(r)-r}\sim\ln^2\ln S_f(r)\quad (r\to+\infty).\tag{33}$$

Let $C_3=C_1+1$, where $C_1$ is the constant from Theorem 1. Using this theorem with $R=R(r)$ and taking into account (32) and (33), we see that for the random entire function defined by (2) a. s. for each $a\in\mathbb{C}$ the inequality (6) holds. □

*Proof of Corollary 2.* Let $f$ be an entire function of form (1), for which condition (7) holds, and let $r_0\in(0,+\infty)$ be a fixed number such that $S_f(r_0)\ge\max\{e,\sqrt{1+|c_0|^2}\}$. Put

$$y(r)=\frac{\ln S_f(r)}{\ln^2 r\ln\ln S_f(r)}\quad (r>r_0).\tag{34}$$

Then, obviously, condition (7) is equivalent to the condition

$$\overline{\lim_{r\to+\infty}}\, y(r)=+\infty.\tag{35}$$

First we prove that (35) implies the existence of a set $E$ of upper density 1 on $(0,+\infty)$ such that

$$\lim_{E\ni r\to+\infty} y(r)=+\infty.\tag{36}$$

There is nothing to prove if the limit $\lambda:=\underline{\lim}_{r\to+\infty}\, y(r)$ is equal to $+\infty$. Let $\lambda<+\infty$, and let $(\lambda_n)$ be an arbitrary sequence from the interval $(\lambda,+\infty)$ increasing to $+\infty$. Taking into account that the function $y(r)$ is continuous on $(r_0,+\infty)$ and using (35), it is easy to justify the existence of sequences $(s_n)$ and $(t_n)$ increasing to $+\infty$ such that $r_0<s_0<t_0<s_1<t_1<\ldots$, $y(s_n)=4\lambda_n$, $y(t_n)=\lambda_n$, and $\lambda_n\le y(r)\le 4\lambda_n$ for $r\in[s_n,t_n]$ and all $n\ge 0$. Put $E=\bigcup_{n=0}^{\infty}[s_n,t_n]$. Then obviously (36) holds. We show that $E$ is a set of upper density 1 on $(0,+\infty)$. Indeed, since the function

$$h(r)=\frac{\ln S_f(r)}{\ln\ln S_f(r)}$$

is increasing on $(r_1,+\infty)$, we have that

$$\ln^2 t_n-\ln^2 s_n=\frac{h(t_n)}{\lambda_n}-\frac{h(s_n)}{4\lambda_n}>\frac{3h(s_n)}{4\lambda_n}=3\ln^2 s_n\quad (n\ge n_0).$$

This implies that $s_n<\sqrt{t_n}$ $(n\ge n_0)$. Therefore,

$$\overline{\lim_{r\to+\infty}}\int_{E\cap(0,r)}\frac{dt}{r}\ge\overline{\lim_{n\to\infty}}\int_{E\cap(s_n,t_n)}\frac{dt}{t_n}=\overline{\lim_{n\to\infty}}\,\frac{t_n-s_n}{t_n}=1,\tag{37}$$

i. e. the set $E$ has upper density 1 on $(0,+\infty)$.

Consider the function $l_f$, introduced in Theorem 2. It is clear that this function is increasing on $(r_0,+\infty)$, and therefore, by Theorem 2, for the random entire function defined by (2) a. s. for each $a\in\mathbb{C}$ and arbitrary $\alpha,\beta\in\mathbb{R}$, $\alpha<\beta\le\alpha+2\pi$, we obtain

$$N_{f_\omega}(r,\alpha,\beta,a)\le\frac{\beta-\alpha}{2\pi}\ln S_f(r)+C_2\sqrt[3]{\ln^2 S_f(r)l_f(r)\ln^2 r}\quad(r\ge r_4(\omega,a)).\tag{38}$$

Let $E_3$ be a set of finite logarithmic measure on $(0,+\infty)$ for which (32) holds. Then $l_f(r)\le 2\ln\ln S_f(r)$ for all $r\ge r_2$, $r\notin E_3$, and from (38) we have

$$\begin{aligned}N_{f_\omega}(r,\alpha,\beta,a)&\le\frac{\beta-\alpha}{2\pi}\ln S_f(r)+C_2\sqrt[3]{2\ln^2 S_f(r)\ln\ln S_f(r)\ln^2 r}=\\&=\ln S_f(r)\left(\frac{\beta-\alpha}{2\pi}+C_2\sqrt[3]{\frac{2}{y(r)}}\right)\quad(r\ge r_5(\omega,a),\ r\notin E_3).\end{aligned}\tag{39}$$

Put $E_4=E\backslash E_3$. It is clear that the set $E_4$ has upper density 1 on $(0,+\infty)$. Using (39) and (36), a. s. for each $a\in\mathbb{C}$ and arbitrary $\alpha,\beta\in\mathbb{R}$, $\alpha<\beta\le\alpha+2\pi$, we obtain

$$\varlimsup_{E_4\ni r\to+\infty}\frac{N_{f_\omega}(r,\alpha,\beta,a)}{\ln S_f(r)}\le\frac{\beta-\alpha}{2\pi}.\tag{40}$$

Then, obviously, the validity of relation (4) as $E_4\ni r\to+\infty$ follows from the equality

$$N_{f_\omega}(r,\alpha,\beta,a)+N_{f_\omega}(r,\beta,\alpha+2\pi,a)=N_{f_\omega}(r,a),\tag{41}$$

inequality (6) and inequality (40), applied to the angles $\beta$ and $\alpha+2\pi$ instead of the angles $\alpha$ and $\beta$, respectively. □

*Proof of Corollary 3.* Let $\rho\in(0,+\infty)$, and $f$ be an entire function of the order $\rho_f\ge\rho$ and form (1). It is well known that in the definition of $\rho_f$ the characteristic $M_f(r)$ can be replaced with the characteristic $S_f(r)$, i. e.

$$\varlimsup_{r\to+\infty}\frac{\ln\ln S_f(r)}{\ln r}=\rho_f\ge\rho.$$

We consider the set $E=\{r>r_0\colon\ \ln\ln S_f(r)>\frac{\rho}{2}\ln r\}$ and prove that its upper density on $(0,+\infty)$ is equal to 1. There is nothing to prove if there exists $r_1>0$ such that $\ln\ln S_f(r)>\frac{\rho}{2}\ln r$ $(r\ge r_1)$. Otherwise, $E$, as an open set, we can represent in the form of a countable union of intervals. From this union one can choose a sequence of intervals $((s_n,t_n))$ such that for every $n\ge0$ we have $s_n<t_n<s_{n+1}$, $\ln\ln S_f(s_n)=\frac{\rho}{2}\ln s_n$, $\ln\ln S_f(t_n)=\frac{\rho}{2}\ln t_n$, and there exists $x_n\in(s_n,t_n)$ such that $\ln\ln S_f(x_n)=\frac{2\rho}{3}\ln x_n$. Then

$$\ln t_n=\frac{2}{\rho}\ln\ln S_f(t_n)>\frac{2}{\rho}\ln\ln S_f(x_n)=\frac{4}{3}\ln x_n,$$

from which we obtain the relation $s_n=o(t_n)$, $n\to+\infty$. This relation implies (43), i. e. the set $E$ has upper density 1 on $(0,+\infty)$.

Put $E_5=E\backslash E_3$, where $E_3$ is a set of finite logarithmic measure on $(0,+\infty)$ for which (32) is satisfied. The set $E_5$ has upper density 1 on $(0,+\infty)$ and, according to (39), for the random entire function defined by (2) a. s. for each $a \in \mathbb{C}$ and arbitrary $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \le \alpha + 2\pi$, we have

$$N_{f_\omega}(r, \alpha, \beta, a) \le \frac{\beta - \alpha}{2\pi} \ln S_f(r) + C_2 \sqrt[3]{2 \ln^2 S_f(r) \ln\ln S_f(r) \frac{4}{\rho^2} \ln^2 \ln S_f(r)} \le$$
$$\le \frac{\beta - \alpha}{2\pi} \ln S_f(r) + \frac{2C_2}{\sqrt[3]{\rho^2}} \ln^{\frac{2}{3}} S_f(r) \ln\ln S_f(r) \quad (r \ge r_5(\omega, a),\ r \in E_5).$$

Finally, using equality (41) and inequality (6), and putting $C_4 = 2C_2$, we complete the proof of Corollary 3. □

*Proof of Corollary 4.* Let $f$ be an entire function of finite order and form (1). Corollary 4 is obvious if $f$ is a polynomial.

Let the function $f$ be transcendental. Then $\ln r = o(\ln S_f(r))$ $(r \to +\infty)$. In addition, $\ln\ln S_f(r) \le 2\rho \ln r$ $(r \ge r_6)$. Therefore, using Theorem 1, for the random entire function defined by (2) a. s. for each $a \in \mathbb{C}$ we obtain

$$\ln S_f(r) \le N_{f_\omega}(r, a) + C_1 \ln\ln S_f(2r) + \ln 2 + h(|a|) \quad (r \ge r_1(\omega)).$$

This implies that

$$\varlimsup_{r \to +\infty} \frac{\ln S_f(r)}{N_{f_\omega}(r, a)} \le 1.$$

On the other hand, using (21) with $f_\omega - a$ instead of $f$ and (18) with $f_\omega$ instead of $f$, for arbitrary $\omega \in \Omega$ and $a \in \mathbb{C}$ we have

$$N_{f_\omega}(r, a) \le \ln S_f(r) + \frac{1}{2e} + \ln^+ |a| + \ln 2 - \ln |c_{f_\omega}(a)|.$$

From this it follows that

$$\varliminf_{r \to +\infty} \frac{\ln S_f(r)}{N_{f_\omega}(r, a)} \ge 1.$$

Therefore, a. s. for every $a \in \mathbb{C}$ relation (9) holds. □

*Proof of Corollary 5.* Let $f$ be an entire function of finite order $\rho$ and form (1) such that relation (10) holds. For the function $l_f$, introduced in Theorem 2, we have $l_f(r) \le 2\rho \ln r$ $(r \ge r_7)$. Therefore, using (38), for the random entire function defined by (2) a. s. for each $a \in \mathbb{C}$ and arbitrary $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \le \alpha + 2\pi$, we obtain

$$N_{f_\omega}(r, \alpha, \beta, a) \le \frac{\beta - \alpha}{2\pi} \ln S_f(r) + C_2 \ln r \sqrt[3]{2\rho \ln^2 S_f(r)} \quad (r \ge r_6(\omega, a)).$$

This together with (10) implies that

$$\varlimsup_{r \to +\infty} \frac{N_{f_\omega}(r, \alpha, \beta, a)}{\ln S_f(r)} \le \frac{\beta - \alpha}{2\pi}.$$

Using equality (41) and Corollary 4, we complete the proof of Corollary 5. □

*Proof of Corollary 6.* Let $\mathcal{R}\in(0,+\infty)$, and let $f\in\mathcal{H}(\mathcal{R})$ be an analytic function of form (1) such that relation (11) holds. Put $y(r)=\ln S_f(r)/\ln\frac{1}{\mathcal{R}-r}$ $(r\in(0,\mathcal{R}))$ and prove that (11) implies the existence of a set $E$ of upper density 1 on $(0,\mathcal{R})$ such that

$$\lim_{E\ni r\to+\infty} y(r)=+\infty. \tag{42}$$

Then, obviously, (36) holds. We show that $E$ is a set of upper density 1 on $(0,+\infty)$. Indeed, since the function

$$h(r)=\frac{\ln S_f(r)}{\ln\ln S_f(r)}$$

is increasing on $(r_1,+\infty)$, one has that

$$\ln^2 t_n-\ln^2 s_n=\frac{h(t_n)}{\lambda_n}-\frac{h(s_n)}{4\lambda_n}>\frac{3h(s_n)}{4\lambda_n}=3\ln^2 s_n \quad (n\geq n_0).$$

This implies that $s_n<\sqrt{t_n}$ $(n\geq n_0)$. Therefore,

$$\overline{\lim_{r\to+\infty}}\int_{E\cap(0,r)}\frac{dt}{r}\geq\overline{\lim_{n\to\infty}}\int_{E\cap(s_n,t_n)}\frac{dt}{t_n}=\overline{\lim_{n\to\infty}}\frac{t_n-s_n}{t_n}=1, \tag{43}$$

i. e. the set $E$ has upper density 1 on $(0,+\infty)$.

There is nothing to prove if the limit $\lambda:=\underline{\lim}_{r\to+\infty}y(r)$ is equal to $+\infty$. Let $\lambda<+\infty$, and $(\lambda_n)$ be an arbitrary sequence from the interval $(\lambda,+\infty)$ increasing to $+\infty$. Taking into account that the function $y(r)$ is continuous on $(0,\mathcal{R})$, and using (11), it is easy to justify the existence of sequences $(s_n)$ and $(t_n)$ increasing to $\mathcal{R}$ such that $0<s_0<t_0<s_1<t_1<\dots$, $y(s_n)=2\lambda_n$, $y(t_n)=\lambda_n$, and $\lambda_n\leq y(r)\leq 2\lambda_n$ for $r\in[s_n,t_n]$ and all $n\geq 0$. Put $E=\bigcup_{n=0}^{\infty}[s_n,t_n]$. Then, obviously, (42) holds. In addition,

$$\ln\frac{1}{\mathcal{R}-t_n}=\frac{\ln S_f(t_n)}{\lambda_n}>\frac{\ln S_f(s_n)}{\lambda_n}=2\ln\frac{1}{\mathcal{R}-s_n}.$$

This implies that $\mathcal{R}-t_n<(\mathcal{R}-s_n)^2$ $(n\geq 0)$. Therefore,

$$\overline{\lim_{r\to\mathcal{R}}}\int_{E\cap(0,r)}\frac{(\mathcal{R}-r)dt}{(\mathcal{R}-t)^2}\geq\overline{\lim_{n\to\infty}}\int_{E\cap(s_n,t_n)}\frac{(\mathcal{R}-t_n)dt}{(\mathcal{R}-t)^2}=\overline{\lim_{n\to\infty}}\left(1-\frac{\mathcal{R}-t_n}{\mathcal{R}-s_n}\right)=1,$$

i. e. the set $E$ has upper density 1 on $(0,\mathcal{R})$.

Let $r_0=\min\{r\in[0,\mathcal{R})\colon S_f(r)\geq\max\{e^e,\sqrt{1+|c_0|^2}\}\}$, and $x_0=\ln S_f(r_0)$. Put

$$R(r)=\mathcal{R}-(\mathcal{R}-r)\exp\left\{-\frac{1}{\ln^2\ln S_f(r)}\right\} \quad (r\in[r_0,\mathcal{R})).$$

Applying Lemma 3 for the functions $u(r)=\ln S_f(r)$ $(r\in[r_0,\mathcal{R}))$ and $\varphi(x)=x^2$ $(x\geq x_0)$, we have

$$\ln S_f(R(r))<e\ln S_f(r) \quad (r\in[r_0,\mathcal{R}),\ r\notin E_7), \tag{44}$$

where $E_7$ is a set of finite logarithmic measure on $(0,\mathcal{R})$. Furthermore, obviously,

$$\frac{R(r)}{R(r)-r}\sim\frac{\mathcal{R}}{\mathcal{R}-r}\ln^2\ln S_f(r) \quad (r\to\mathcal{R}). \tag{45}$$

Let $C_3 = C_1 + 1$, where $C_1$ is the constant from Theorem 1. Using this theorem with $R = R(r)$ and taking into account (44) and (45), we see that for random analytic function (2) a. s. for each $a \in \mathbb{C}$ the inequality

$$\ln S_f(r) \leq N_{f_\omega}(r, a) + C_3 \ln\ln S_f(r) + \ln\frac{1}{\mathcal{R} - r} + h(|a|) \quad (r_7(\omega) \leq r < \mathcal{R}) \tag{46}$$

holds.

We consider the function $l_f$, introduced in Theorem 2. Putting $C_{11} = C_2 \ln^2 \frac{\mathcal{R}}{r_0}$, by Theorem 2 for the random analytic function defined by (2) a. s. for each $a \in \mathbb{C}$ and arbitrary $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \leq \alpha + 2\pi$, we obtain

$$N_{f_\omega}(r, \alpha, \beta, a) \leq \frac{\beta - \alpha}{2\pi} \ln S_f(r) + C_{11} \sqrt[3]{\ln^2 S_f(r) l_f(r)} \quad (r_8(\omega, a) \leq r < \mathcal{R}). \tag{47}$$

Next we note that (44) and (45) implies the inequality

$$l_f(r) \leq 2(\ln\ln S_f(r) - \ln(\mathcal{R} - r)) \quad (r \in [r_9, \mathcal{R}),\ r \notin E_7).$$

Using this inequality together with (47), we have

$$N_{f_\omega}(r, \alpha, \beta, a) \leq \frac{\beta - \alpha}{2\pi} \ln S_f(r) + C_{11} \sqrt[3]{2\ln^2 S_f(r) \left(\ln\ln S_f(r) + \ln\frac{1}{\mathcal{R} - r}\right)} =$$

$$= \ln S_f(r) \left( \frac{\beta - \alpha}{2\pi} + C_2 \sqrt[3]{\frac{2\ln\ln S_f(r)}{\ln S_f(r)} + \frac{2}{y(r)}} \right) \quad (r_{10}(\omega, a) \leq r < \mathcal{R},\ r \notin E_7). \tag{48}$$

Put $E_6 = E \backslash E_7$. It is clear that the set $E_6$ has upper density 1 on $(0, \mathcal{R})$. Using (42) and (48), a. s. for every $a \in \mathbb{C}$ and all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta \leq \alpha + 2\pi$, we obtain

$$\varlimsup_{E_6 \ni r \to \mathcal{R}} \frac{N_{f_\omega}(r, \alpha, \beta, a)}{\ln S_f(r)} \leq \frac{\beta - \alpha}{2\pi}. \tag{49}$$

Then, obviously, the validity of the relation (4) as $E_6 \ni r \to \mathcal{R}$ follows from equality (41), inequality (46) and inequality (49), applied to the angles $\beta$ and $\alpha + 2\pi$ instead of the angles $\alpha$ and $\beta$, respectively. □

## REFERENCES

1. Hayman W.K. Meromorphic functions. – Clarendon Press (Oxford), 1964.
2. Goldberg A.A., Ostrovskii I.V. Value distribution of meromorphic functions. – Translations of Mathematical Monographs, V.236. American Mathematical Society, Providence, R.I., 2008.
3. Kahane J.-P. Some Random Series of Functions. – Cambridge University Press, 1994.
4. Offord A.C. *The distribution of the values of a random function in the unit disk*// Studia Math. – 1972. – V.41. – P. 71–106.
5. Mahola M.P., Filevych P.V. *The value distribution of a random entire function*// Mat. Stud. – 2010. – V.34, №2. – P. 120–128.
6. Hayman W.K. *Angular value distribution of power series with gaps*// Proc. London Math. Soc. (3). – 1972. – V.24. – P. 590–624.
7. Murai T. *The deficiency of entire functions with Fejér gaps*// Ann. Inst. Fourier (Grenoble). – 1983. – V.33. – P. 39–58.

8. Hayman W.K., Rossi J.F. *Characteristic, maximum modulus and value distribution*// Trans. Amer. Math. Soc. – 1984. – V.284, №2. – P. 651–664.
9. Offord A.C. *The distribution of the values of an entire function whose coefficients are independent random variables (I)*// Proc. London Math. Soc. – 1965. – V.14A. – P. 199–238.
10. Offord A.C. *The distribution of the values of an entire function whose coefficients are independent random variables (II)*// Math. Proc. Cambridge Phil. Soc. – 1995. – V.118, №3. – P. 527–542.
11. Filevych P.V. *Some classes of entire functions in which the Wiman-Valiron inequality can be almost certainly improved*// Mat. Stud. – 1996. – V.6. – P. 59–66. (in Ukrainian)
12. Filevych P.V. *Correlations between the maximum modulus and maximum term of random entire functions*// Mat. Stud. – 1997. – V.7, №2. – P. 157–166. (in Ukrainian)
13. Filevych P.V. *Wiman–Valiron type inequalities for entire and random entire functions of finite logarithmic order*// Sib. Mat. Zhurn. – 2003. – V.42, №3. – P. 683–694. (in Russian). English translation in: Siberian Math. J. – 2003. – V.42, №3. – P. 579–586.
14. Filevych P.V. *On relations between the abscissa of convergence and the abscissa of absolute convergence of random Dirichlet series*// Mat. Stud. – 2003. – V.20, №1. – P. 33–39.
15. Filevych P.V. *The Baire categories and Wiman's inequality for entire functions*// Mat. Stud. – 2003. – V.20, №2. – P. 215–221.
16. Skaskiv O.B. *Random gap series and Wiman's inequality*// Mat. Stud. – 2008. – V.30, №1. – P. 101–106. (in Ukrainian)
17. Skaskiv O.B., Zrum O.V. *On an exeptional set in the Wiman inequalities for entire functions*// Mat. Stud. – 2004. – V.21, №1. – P. 13–24. (in Ukrainian)
18. Zrum O.V., Skaskiv O.B. *On Wiman's inequality for entire functions of two variables*// Mat. Stud. – 2005. – V.23, №2. – P. 149–160. (in Ukrainian)
19. Zrum O.V., Skaskiv O.B. *Wiman's inequalities for entire functions of two complex variables with rapidly oscillating coefficients*// Math. Meth. Fhys.-Mech. Filds. – 2005. – V.48, №4. – P. 78–87. (in Ukrainian)
20. Skaskiv O.B., Zrum O.V. *Wiman's inequalities for entire functions of two complex variables with rapidly oscillating coefficients*// Math. Bull. Shevchenko Sci. Soc. – 2006. – V.3. – P. 56–68. (in Ukrainian)
21. Zadorozhna O.Yu., Skaskiv O.B. *On the domains of convergence of the double random Dirichlet series*// Mat. Stud. – 2009. – V.32, №1. – P. 81–85. (in Ukrainian)
22. Skaskiv O.B., Zadorozhna O.Yu. *On domains of convergence of multiple random Dirichlet series*// Mat. Stud. – 2011. – V.36, №1. – P. 51–57.
23. Skaskiv O.B., Kuryliak A.O. *Direct analogues of Wiman's inequality for analytic functions in the unit disc*// Carpathian Math. Publications. – 2010. – V.2, №1. – P. 109–118. (in Ukrainian)
24. Skaskiv O.B., Kuryliak A.O. *The probability of absence zeros in the disc for some random analytic functions*// Math. Bull. Shevchenko Sci. Soc. – 2011. – V.8. – P. 335–352.
25. Benbourenane D., Korhonen R. *On the growth of the logarithmic derivative*// Comput. Meth. Func. Theory. – 2001. – V.1, №2. – P. 301–310.
26. Kondratyuk A.A., Kshanovskyy I.P. *On the logarithmic derivative of a meromorphic function*// Mat. Stud. – 2004. – V.21, №1. – P. 98–100.
27. Hayman W.K., Kennedy P.B. Subharmonic functions. V.1. – London Mathematical Society Monographs, №9, Academic Press, London, 1976.
28. Levin B.Ja. Lectures on entire functions. – Translations of Mathematical Monographs, vol. 150, American Mathematical Society, Providence, R.I., 1996.
29. Andrusyak I.V., Filevych P.V. *Minimal growth of an entire function with given zeros*// Sci. Bull. Chernivtsi Univ., Ser. Math. – 2008. – V.421. – P. 13–19. (in Ukrainian)

Pidstryhach Institute for Applied Problems of
Mechanics and Mathematics of NASU,
marichka_stanko@ukr.net

S. Z. Gzhytskyy Lviv National University of
Veterinary Medicine and Biotechnologies,
filevych@mail.ru

*Received 5.09.2011*
*Revised 18.01.2012*