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O. R. Nykyforchyn

CONTINUOUS AND DUALLY CONTINUOUS IDEMPOTENT L-SEMIMODULES

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We introduce L-idempotent analogues of topological vector spaces by means of domain theory, study their basic properties, and prove the existence of free (dually) continuous L-semimodules over domains, (dually) continuous lattices and semilattices.

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Средствами теории областей введены L-идемпотентные аналоги топологических векторных пространств, изучены их основные свойства, и доказано существование свободных (дуально) непрерывных L-полумодулей над областями и (дуально) непрерывными (полу)решетками.

1. Introduction. The goal of this paper is to develop foundations for lattice-valued idempotent functional analysis. Although the latter branch of mathematics exposes rapid growth ([8]), it is mostly focused at analogues of "conventional" objects of analysis obtained by replacing the field of reals with idempotent semirings of "extended reals". The most important of them is the *max-plus semiring* $(\bar{\mathbb{R}}, \oplus, \odot)$, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, $a \oplus b = \max\{a, b\}$, $a \odot b = a + b$. The real vector spaces are replaced respectively with idempotent semimodules, and meaningful analogues of convexity, separation etc. are introduced ([2]). Nevertheless, despite rather general definitions, they are applied almost exclusively to finite or infinite powers of the chosen semiring ([19, 20]). The situation looks like if the entire real functional analysis was developed only in $\mathbb{R}^n$ and $\mathbb{R}^\omega$.

We are going to show that idempotent analogues of notions and statements of the theory of topological vector spaces are most naturally defined and investigated in rather general settings, which are not restricted to reals or "almost reals", but use "scalars" from completely distributive lattices and "vectors" from continuous posets. An example of "lattice-valued" approach to measure theory, which is intrinsically linked to our topic, can be found in [12]. We refer the reader to the survey article [9] by J. D. Lawson on connections between idempotent analysis and continuous semilattices. Our task is more simple and narrow: to define categories for objects and morphisms, which correspond to topological vector spaces and continuous linear and affine operators, to study their elementary properties and to establish useful technical results, and to describe *free objects* (in the sense of category theory) in the introduced categories over the objects of underlying categories of continuous posets.

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This paper extends and mostly supercedes [13], in which similar free object also have been constructed, but under more restrictive conditions and in less convenient terms of hyperspaces.

2. Preliminaries. Elementary concepts of category theory can be found in [11]. Reader is referred to [1] or [16] for the definitions of idempotent semiring and semimodule. An excellent exposition of the most important topics that concern partially ordered sets, including directed complete partial orders, continuous (semi)lattices, Scott/lower/Lawson topologies etc. can be found in [4]. We recall only main definitions, facts and notation.

The bottom and the top elements of a poset (if they exist) are usually denoted resp. by 0 and 1. A subset A in a poset $(X, \leqslant)$ is *directed* (*filtered*) if for all $a_1, a_2 \in A$ there is $a_3 \in A$ such that $a_1 \leqslant a_3$ and $a_2 \leqslant a_3$ (respectively $a_3 \leqslant a_1$ and $a_3 \leqslant a_2$). A poset is *directed complete* (*dcpo* for short) if each non-empty directed subset has the least upper bound. An element a is *way below* an element b in a poset $(X, \leqslant)$ (which is denoted $a \ll b$) provided if a set $D \subset X$ is directed and $\sup D \geqslant b$, then there is $d \in D$ such that $d \geqslant a$. A poset $(X, \leqslant)$ is *continuous* if, for each $b \in X$, the set $\{a \in X \mid a \ll b\}$ is directed, and b is its least upper bound. A *domain* is a continuous dcpo. A domain which is also a semilattice (a complete lattice) is called a *continuous semilattice* (respectively a *continuous lattice*). For a set A in a poset $(X, \leqslant)$ we denote $A{\uparrow} = \{x \in X \mid a \leqslant x \text{ for some } a \in A\}$, $A{\downarrow} = \{x \in X \mid x \leqslant a \text{ for some } a \in A\}$. If $A{\uparrow} = A$ (or $A{\downarrow} = A$), then the set A is called *upper* (resp. *lower*). A filtered upper set is called a *filter*.

On a poset $(X, \leqslant)$, the *Scott topology* $\sigma(X)$ is the least topology such that all directed complete lower sets are closed. The *lower topology* $\omega(X)$ is the least topology such that all the sets $\{x\}{\uparrow}$, for $x \in X$, are closed. The *Lawson topology* $\lambda(X)$ is the join of $\sigma(X)$ and $\omega(X)$, i.e., the least topology that contains the both these topologies.

For a partial order $\leqslant$ on X, the *reverse* partial order $\tilde{\leqslant}$ is defined as $a \tilde{\leqslant} b \iff b \leqslant a$. The poset X with this order is denoted by either $\tilde{X}$ or X^{op}. Then $(\tilde{-})$ or $(-)^{op}$ is a *functor* ([11]) in the category of partially ordered sets and isotone mappings. We apply $\tilde{}$ also to all denotations to mark *dual* notions and constructions that are obtained when the order on a set is reversed. For example, $\tilde{0} = 1$, $\tilde{\sup} A = \inf A$ etc. The topologies $\sigma(\tilde{X})$, $\omega(\tilde{X})$, and $\lambda(\tilde{X})$ are called respectively the *dual Scott topology*, the *upper topology*, and the *dual Lawson topology* on X. A poset X is called a *dually continuous semilattice (lattice)* if $\tilde{X}$ is a continuous semilattice (lattice).

Recall that an isotone mapping between *dcpo*s is Scott continuous (i.e., continuous w.r.t. the Scott topologies on both sets) if and only if it preserves all suprema of directed sets.

We use the following notation for the most used categories of (dually) continuous posets and their isotone mappings.

The category of all domains and their Scott continuous mappings is denoted by $\mathcal{D}om$ (note that we do not require that the preimages of the open filters are open filters, compare with the definition of $\mathcal{D}om\mathcal{F}ilt$, which will appear later). Its full subcategory with the objects being all domains with bottom elements is denoted $\mathcal{D}om_\perp$. If it is also required that bottom elements are preserved by the morphisms, the subcategory $\mathcal{D}om_0$ is obtained. This notation style is applied also to the following categories.

The category $\mathcal{CS}em$ consists of all continuous (meet) semailttices and their Scott continuous meet-preserving mappings. $\mathcal{CS}em_\perp$ is its full subcategory that contain only the semilattices with bottom elements, and $\mathcal{CS}em_0$ is obtained when also bottom *preservation* is required. Similarly $\mathcal{CS}em_1$ is the category of all continuous semilattices with top elements and Scott continuous meet-preserving top-preserving maps.

If a continuous semilattice is complete, then it a compact Hausdorff topological lower semilattice in its Lawson topology. Moreover, it is a *Lawson semilattice*, i.e., in each point it possesses a local base consisting of subsemilattices. Hence the category of such semilattices and their meet-preserving mappings that are respectively Lawson continuous, Scott continuous and lower continuous, are denoted by $\mathcal{LL}aws$, $\mathcal{LL}aws_\uparrow$, and $\mathcal{LL}aws_\downarrow$. Let also $\mathcal{CL}$, $\mathcal{CL}_\uparrow$, and $\mathcal{CL}_\downarrow$ be the full subcategories resp. of $\mathcal{LL}aws$, $\mathcal{LL}aws_\uparrow$, and $\mathcal{LL}aws_\downarrow$, generated by the continuous lattices.

If $\tilde{}$ (order reversing) is applied to all objects and morphisms of the categories $\mathcal{CS}em$, $\mathcal{LL}aws$, $\mathcal{LL}aws_\uparrow$, $\mathcal{LL}aws_\downarrow$, $\mathcal{CL}$, $\mathcal{CL}_\uparrow$, and $\mathcal{CL}_\downarrow$, then the obtained categories are denoted $\mathcal{DS}em$, $\mathcal{UL}aws$, $\mathcal{UL}aws_\downarrow$, $\mathcal{UL}aws_\uparrow$, $\mathcal{DL}$, $\mathcal{DL}_\downarrow$, and $\mathcal{DL}_\uparrow$ (note that $\uparrow$ is changed to $\downarrow$ and vice versa). E.g., $\mathcal{DL}_\downarrow$ is the category of dually continuous lattices and dually Scott continuous join-preserving mappings.

Again, we add 0 or 1 to subscripts to require also that the bottom or the top elements are preserved by the morphisms.

3. Idempotent semirings and idempotent semimodules. Linear and affine mappings. In the sequel $(L, \oplus, \otimes)$ will be the lattice with the bottom and a top elements 0 and 1 respectively, and a binary operation $*\colon L\times L\to L$ such that 1 is a two-sided unit and $*$ is associative and distributive w.r.t. $\oplus$ in both variables. Then $(L, \oplus, \otimes)$ is a *semiring.*

Recall that a (left idempotent) $(L, \oplus, *)$-semimodule ([1]) is a set X with operations $\bar\oplus\colon X\times X\to X$ and $\bar *\colon L\times X\to X$ such that for all $x, y, z\in X$, $\alpha,\beta\in L$:

(1) $x\mathbin{\bar\oplus} y = y\mathbin{\bar\oplus} x$;

(2) $(x\mathbin{\bar\oplus} y)\mathbin{\bar\oplus} z = x\mathbin{\bar\oplus}(y\mathbin{\bar\oplus} z)$;

(3) there is an (obviously unique) element $\bar 0\in X$ such that $x\mathbin{\bar\oplus}\bar 0 = x$ for all x;

(4) $\alpha\mathbin{\bar *}(x\mathbin{\bar\oplus} y) = (\alpha\mathbin{\bar *} x)\mathbin{\bar\oplus}(\alpha\mathbin{\bar *} y)$, $(\alpha\oplus\beta)\mathbin{\bar *} x = (\alpha\mathbin{\bar *} x)\mathbin{\bar\oplus}(\beta\mathbin{\bar *} x)$;

(5) $(\alpha * \beta)\mathbin{\bar *} x = \alpha\mathbin{\bar *}(\beta\mathbin{\bar *} x)$;

(6) $1\mathbin{\bar *} x = x$;

(7) $0\mathbin{\bar *} x = \bar 0$.

Observe that these axioms imply that $(X, \bar\oplus)$ is an upper semilattice with the bottom element $\bar 0$, the order is defined as $x\leqslant y \iff x\mathbin{\bar\oplus} y = y$, and $\alpha\mathbin{\bar *}\bar 0 = \bar 0$ for all $\alpha\in L$. The operation $\bar *$ is isotone in both variables.

Since an $(L, \oplus, *)$-semimodule is an analogue of a vector space, for all $x_1,\dots,x_n\in X$ and $\alpha_1,\dots,\alpha_n\in L$, it is natural to call the expression $\alpha_1\mathbin{\bar *} x_1\mathbin{\bar\oplus}\dots\mathbin{\bar\oplus}\alpha_n\mathbin{\bar *} x_n$ a *linear combination* of the elements x_i with the coefficients α_i. If $\alpha_1\oplus\dots\oplus\alpha_n = 1$, then the latter combination is called *convex.*

Obviously, a closed under linear combinations subset of an L-semimodule is an L-semimodule itself, i.e. a *subsemimodule* of the previous one. A subset that is closed under *convex* combinations is, as usual, called *convex.* Observe that, similarly to L-semimodules, a convex set is an upper semilattice with the operation $x\vee y = 1x\oplus 1y$, but need not contain a bottom element.

Analogues exist also for linear and affine mappings. A mapping $f\colon X\to Y$ between $(L, \oplus, *)$-semimodules is called *linear* if it preserves the linear combinations, i.e., for all $x_1,\dots,x_n\in X$ and $\alpha_1,\dots,\alpha_n\in L$, the equality

$$f(\alpha_1\mathbin{\bar *} x_1\mathbin{\bar\oplus}\dots\mathbin{\bar\oplus}\alpha_n\mathbin{\bar *} x_n) = \alpha_1\mathbin{\bar *} f(x_1)\mathbin{\bar\oplus}\dots\mathbin{\bar\oplus}\alpha_n\mathbin{\bar *} f(x_n)$$

is valid. If the latter equality is ensured only for convex combinations, i.e. whenever $\alpha_1\oplus\dots\oplus\alpha_n = 1$, then f is called *affine.* Observe that an affine mapping f preserves joins, i.e.

$f(x_1 \bar{\oplus} x_2) = f(x_1) \bar{\oplus} f(x_2)$ for all $x_1, x_2 \in X$, therefore it is isotone. An affine mapping is linear if and only if it preserves the least element.

4. Completeness and (order) continuity. From now on we will require that $(L, \oplus, *)$ is a *unital quantale* ([18]), i.e. a complete lattice such that, for $*\colon L \times L \to L$, the top element 1 is a two-sided unit and $*$ is associative and infinitely distributive w.r.t. supremum in both variables. Recall that we can treat $\oplus$ as a disjunction, and $*$ will be a (possibly noncommutative) conjunction in an L-valued fuzzy logic ([5]). The Boolean case is obtained for $L = \{0, 1\}$, $\oplus = \vee$ and $* = \wedge$.

Respectively, an L-semimodule X is called *complete* if it is a complete lattice and the multiplication $\bar{*}\colon L \times X \to X$ is infinitely distributive w.r.t. supremum in both variables as well. In a complete L-semimodule X, it is possible to consistently define the linear combination of an arbitrary number of points: $\bigoplus_{i \in \mathcal{I}} \alpha_i \mathbin{\bar{*}} x_i = \sup\{\alpha_i \mathbin{\bar{*}} x_i \mid i \in \mathcal{I}\}$. See [2] for introduction to complete L-semimodules and their morphisms in a particular but important case when L is a *max-plus idempotent semiring*.

From now on, we additionally demand that L is a completely distributive lattice ([4]), hence a compact Hausdorff distributive Lawson lattice. Therefore L and L^{op} are continuous lattices, and the elements of L are well approximated both from above and from below. We impose similar requirements at idempotent semimodules, and it will be shown in the subsequent paper that the obtained properties are proper analogues of local convexity for the theory of compact closed sets in locally convex topological spaces.

Note that $*\colon L \times L \to L$, being infinitely distributive over suprema in both variables, is Scott continuous. We call an L-semimodule $(X, \bar{\oplus}, \bar{*})$ *continuous* if X is a domain, and $\bar{*}\colon L \times X \to X$ is Scott continuous.

Observe that a continuous L-semimodule is a continuous, hence a complete latttice and $\bar{*}$ is infinitely distributive in each variable w.r.t. the suprema. Recall that such a lattice is a compact Hausdorff Lawson lower semilattice in its Lawson topology. We consider the categories $(L, \oplus, *)$-$\mathcal{CSM}od_{\uparrow}$ and $(L, \oplus, *)$-$\mathcal{CSM}od_{\downarrow}$ that consist of all continuous L-semimodules and their Scott continuous linear mappings (which therefore preserve all suprema) and their lower continuous mappings, respectively. The intersection of the categories $(L, \oplus, *)$-$\mathcal{CSM}od_{\uparrow}$ and $(L, \oplus, *)$-$\mathcal{CSM}od_{\downarrow}$ is the category $(L, \oplus, *)$-$\mathcal{CSM}od$ of all continuous L-semimodules and their Lawson continuous linear mappings.

To define more categories, we additionally require that $*\colon L \times L \to L$ distributes over the filtered infima in both variables, hence is jointly continuous w.r.t. the Lawson topology on L and w.r.t. the Scott topology on L^{op}. Then we call an L-semimodule *dually continuous* if X^{op} is a domain, and $\bar{*}\colon L \times X \to X$ is dually Scott continuous (i.e. $\bar{*}\colon L^{op} \times X^{op} \to X^{op}$ is Scott continuous). This is equivalent to the infinite distributivity of $\bar{*}$ over the filtered infima in both variables.

Similarly, a dually continuous L-semimodule $(X, \bar{\oplus}, \bar{*})$ is a dually continuous, not necessarily complete, *upper* semilattice with the least element, and $\bar{*}$ is distributive in each variable w.r.t. the finite suprema and the filtered infima. The categories $(L, \oplus, *)$-$\mathcal{DSM}od_{\downarrow}$ and $(L, \oplus, *)$-$\mathcal{DSM}od_{\uparrow}$ consist of all dually continuous L-semimodules and their linear mappings that are continuous, respectively, w.r.t. the dual Scott topologies (i.e. that preserve all filtered infima) and w.r.t. the upper topologies. Again, we denote by $(L, \oplus, *)$-$\mathcal{DSM}od$ the category with the dually continuous L-semimodules as objects and the dually Lawson continuous linear mappings as morphisms.

It a dually continuous L-semimodule X is complete, then it is a compact Hausdorff

Lawson upper semilattice in the Lawson topology on X^{op}. If $\bar{*}$ is jointly continuous w.r.t. the Lawson topologies on L and X^{op}, then we use the term "*compact Hausdorff Lawson L-semimodule*". The requirement that $(X, \bar{\oplus}, \bar{*})$ is an L-semimodule, a topology on X is given that makes it is a compact Hausdorff Lawson upper semilattice with the join $\bar{\oplus}$, and the multiplication $\bar{*}$ is continuous, is an equivalent definition in more topological terms. The full subcategories of $(L, \oplus, *)$-$\mathcal{DSM}od_{\downarrow}$, $(L, \oplus, *)$-$\mathcal{DSM}od_{\uparrow}$, and $(L, \oplus, *)$-$\mathcal{DSM}od$, with such objects, are denoted respectively by $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SM}od_{\downarrow}$, $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SM}od_{\uparrow}$, and $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SM}od$.

If we allow the *affine* mappings instead of the linear ones, then the similar categories $(L, \oplus, *)$-$\mathcal{CSA}f\!f_{\uparrow}$, $(L, \oplus, *)$-$\mathcal{CSA}f\!f_{\downarrow}$, $(L, \oplus, *)$-$\mathcal{CSA}f\!f$, $(L, \oplus, *)$-$\mathcal{DSA}f\!f_{\uparrow}$, $(L, \oplus, *)$-$\mathcal{DSA}f\!f_{\downarrow}$, $(L, \oplus, *)$-$\mathcal{DSA}f\!f$, $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SA}f\!f_{\downarrow}$, $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SA}f\!f_{\uparrow}$, and $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SA}f\!f$ are obtained. Recall that, unlike linear mappings, affine mappings does not necessarily preserve the bottom elements of L-semimodules.

Now we present some examples of (dually) continuous L-semimodules for different L. The simplest case is $L = \mathbf{2} = \{0, 1\}$, then there is a unique appropriate multiplication $\bar{*}\colon \mathbf{2} \times X \to X$, namely $1 \mathbin{\bar{*}} x \equiv x$, $0 \mathbin{\bar{*}} x \equiv \bar{0}$. Hence each linear combination is either trivial (with zero coefficients only) or affine, which in turn is a finite supremum. Thus, affine mappings are simply join-preserving ones, and linearity is preservation of the joins and the bottom elements. Clearly the continuous $\mathbf{2}$-semimodules are precisely the continuous lattices, the (complete) dually continuous $\mathbf{2}$-semimodules are the posets opposite to (complete) continuous meet-semilattices with top elements, the $\mathbf{2}$-convex compacta are complete continuous meet-semilattices (with or without top elements). All the defined above categories can be easily identified with commonly known categories for continuous (semi-) lattices.

If $(L, \oplus, \otimes)$ is a bigger completely distributive lattice, then, putting $\bar{*} = \otimes$, we obtain a quantale. E.g., let $L = \{0, \frac{1}{2}, 1\}$, then an L-semimodule X is an upper semilattice such that, for all $x \in X$, we can take a "half" of x. If L is infinite, continuity considerations also arise.

To see that there are examples that are not reduced to the above case, let $L = I$, $\oplus = \max$, $* = \cdot$. Put

$$X = \{A \underset{\mathrm{cl}}{\subset} I \times I \mid \mathrm{pr}_1(A) = I; (x, y_1) \in A, y_1 \leqslant y_2 \leqslant 1 \implies (x, y_2) \in A\},$$
$$Y = \{B \underset{\mathrm{cl}}{\subset} I \times I \mid \mathrm{pr}_1(B) = I; (x, y_1) \in B, y_1 \geqslant y_2 \geqslant 0 \implies (x, y_2) \in B\},$$
$$A_1 \leqslant A_2 \text{ in } X \iff A_1 \supset A_2, \quad B_1 \leqslant B_2 \text{ in } Y \iff B_1 \subset B_2,$$

then X is a continuous, but not dually continuous, lattice with the top element $I \times \{1\}$ and the bottom element $I \times I$, and Y is a dually continuous, but not continuous, lattice with the top element $I \times I$ and the bottom element $I \times \{0\}$. The join $\bar{\oplus}$ is the intersection in X and is the union in Y. We define multiplications as follows:

$$\alpha \mathbin{\bar{*}} A = \{(x, y') \in I \times I \mid y' \geqslant \alpha y \text{ for some } (x, y) \in A\},$$
$$\alpha \mathbin{\bar{*}} B = \{(x, y') \in I \times I \mid y' \leqslant \alpha y \text{ for some } (x, y) \in B\},$$

for all $\alpha \in I$, $A \in X$, $B \in Y$, then $(X, \bar{\oplus}, \bar{*})$ is a continuous L-semimodule and $(Y, \bar{\oplus}, \bar{*})$ is a complete dually continuous L-semimodule.

5. Lattices of Scott continuous mappings and duality. From now on $[A \to B]$ stands for the set of all Scott continuous mappings from a domain A to a domain B. It follows from [3, Theorem 4] (although called "folklore knowledge" in [6]) that, for a domain D and a completely distributive lattice L, the set $[D \to L]$ is a completely distributive lattice. If the domain D has the bottom element 0, then the subset $[D \to L]_0 = \{\varphi \in [D \to L] \mid f(0) = 0\}$ is a complete sublattice of $[D \to L]$, therefore is a completely distributive lattice as well.

We shall use the Lawson duality for domains, cf. [4]. All "triple-numbered" references in this section will be related to the latter citation, but the notation will follow [7]. For a domain D, let $D^\triangle$ be the ordered by inclusion set of all non-empty (Scott) open filters in D. The poset $D^\triangle$, which is called the *Lawson dual* of D, is a domain as well, and it is obvious that $\max D^\triangle = D$. Due to Lemma IV-2.9 the non-empty open filters in $D^\triangle$ are precisely the sets of the form $\{F \in D^\triangle \mid F \ni d\}$, for $d \in D$. Hence the correspondence $\underline{u}_D \colon D \to (D^\triangle)^\triangle$ that sends each d to $\{F \in D^\triangle \mid F \ni d\}$ is an order isomorphism.

Let $\mathcal{D}om\mathcal{F}ilt$ be the category of all domains and all mappings between them such that the preimages of all non-empty open filters are non-empty open filters as well. This implies monotonicity and Scott continuity, cf. the remark after Definition IV-2.2. The functor $(-)^\triangle \colon \mathcal{D}om\mathcal{F}ilt \to \mathcal{D}om\mathcal{F}ilt^{op}$ is obtained. It sends D to $D^\triangle$, and, for a morphism $f \colon D \to D'$ in $\mathcal{D}om\mathcal{F}ilt$, the mapping $f^\triangle \colon D'^\triangle \to D^\triangle$ takes each non-empty open filter F to its preimage $f^{-1}(F)$. By the above the identity functor is isomorphic to the composition $(-)^\triangle \circ (-)^\triangle$ via the natural transformation $\underline{u}$ that consists of all $\underline{u}_D$, thus $(-)^\triangle$ is a *duality* of the category $\mathcal{D}om\mathcal{F}ilt$ onto itself.

Observe that the category $\mathcal{CS}em_1$ of all continuous semilattices with top elements and their Scott continuous top-preserving semilattice morphisms is a subcategory of $\mathcal{D}om\mathcal{F}ilt$, and there is a restriction of $(-)^\triangle$ to the functor $\mathcal{CS}em_1 \to \mathcal{CS}em_1^{op}$, which also is a self-duality.

Let $\mathcal{D}om\mathcal{F}ilt_0$ be the category that consists of all domains with *bottom* elements, and all bottom-preserving mappings such that the preimages of open filters are (not necessarily non-empty) open filters. For each object D of $\mathcal{D}om\mathcal{F}ilt_0$, we denote by $D^\top$ the poset D with the top element $\top$ adjoined. This poset is an object of $\mathcal{D}om\mathcal{F}ilt$ such that its top element is isolated from below. For a morphism $f \colon D \to D'$ in $\mathcal{D}om\mathcal{F}ilt_0$, let $f^\top \colon D^\top \to D'^\top$ send each $d \in D$ to $f(d) \in D'$, and $\top$ to $\top$. Note that $f^\top$ is a morphism in $\mathcal{D}om\mathcal{F}ilt$, and $(-)^\top \colon \mathcal{D}om\mathcal{F}ilt_0 \to \mathcal{D}om\mathcal{F}ilt$ is a functor which is an embedding of categories. Its restriction embeds the category $\mathcal{CS}em_0$ that consists of all continuous semilattices with bottom elements, and all Scott continuous semilattice morphisms that preserve the bottom elements, into the category $\mathcal{CS}em_1$.

We also need the modified version of the Lawson duality, which was introduced in [14]. For a domain D with the bottom element, the Lawson dual $(D^\top)^\triangle$ is a domain with the top element $D^\top$ isolated from below, cf. Exercise IV.2-21, and with the bottom element $\{\top\}$. If D is also a continuous semilattice with a bottom element, then the same is valid for $(D^\top)^\triangle$. Hence a poset

$$D^\wedge = (D^\top)^\triangle \setminus \{D^\top\}.$$

is an object of respectively $\mathcal{D}om\mathcal{F}ilt_0$ or $\mathcal{CS}em_0$ as well. This assignment extends to a contravariant functor $(-)^\wedge \colon \mathcal{D}om\mathcal{F}ilt_0 \to \mathcal{D}om\mathcal{F}ilt_0$ (and its restriction $(-)^\wedge \colon \mathcal{CS}em_0 \to \mathcal{CS}em_0$) as follows: each $F \in (D'^\top)^\triangle$, $F \neq D'^\top$, does not contain the bottom element $0' \in D'$, therefore, for an arrow $f \colon D \to D'$ in $\mathcal{D}om\mathcal{F}ilt_0$ (or $\mathcal{CS}em_0$), the mapping $(f^\top)^\triangle \colon (D'^\top)^\triangle \to (D^\top)^\triangle$ takes such F to the open filter $(f^\top)^{-1}(F)$ which does not contain the bottom element $0 \in S$, hence $(f^\top)^\triangle(F) \neq D^\top$. On the other hand, $(f^\top)^\triangle(D'^\top) = D^\top$. Thus we define the mapping $f^\wedge \colon D'^\wedge \to D^\wedge$ as a restriction of $(f^\top)^\triangle$. By the above the assignment

$s \mapsto \{F \in D^\wedge \mid s \in F\}$ is an isomorphism $u_D\colon D \to D^{\wedge\wedge}$ which is a component of a natural transformation $u\colon \mathbf{1}_{\mathcal{D}om\mathcal{F}ilt_0} \to (-)^{\wedge\wedge}$ (or $u\colon \mathbf{1}_{\mathcal{CS}em_0} \to (-)^{\wedge\wedge}$). Thus $\mathcal{D}om\mathcal{F}ilt_0$ and $\mathcal{CS}em_0$ are self-dual under the contravariant functors, for which we use the same notation $(-)^\wedge$. By the above we consider these self-dualities as restrictions of the self-dualities for $\mathcal{D}om\mathcal{F}ilt$ and $\mathcal{CS}em_1$ via $(-)^\triangle$ to the subcategories $\mathcal{D}om\mathcal{F}ilt_0$ and $\mathcal{CS}em_0$. Observe that the elements of $D^\wedge$ can be identified with the not necessarily non-empty open filters in D distinct from D itself, and we shall do this in the sequel.

For domains D and D', consider the following properties of a relation $P \subset D \times D'$:

(1) for all $x_0 \in D$, $y_0 \in D'$ the sets $x_0P = \{y \in D' \mid (x_0, y) \in P\}$, $Py_0 = \{x \in D \mid (x, y_0) \in P\}$ are non-empty open filters;

(2) for all $x_1, x_2 \in D$, $x_1 \not\leq x_2$ there is $y \in D'$ such that $(x_1, y) \in P$, but $(x_2, y) \notin P$;

(3) for all $y_1, y_2 \in D'$, $y_1 \not\leq y_2$ there is $x \in D$ such that $(x, y_1) \in P$, but $(x, y_2) \notin P$.

Theorem 1. *For domains D and D' and a relation $P \subset D \times D'$ that satisfies (1)–(3), the mapping $i\colon D' \to D^\triangle$ that takes each $y \in D'$ to the open filter $Py \subset D$ is an isomorphism. Conversely, each isomorphism $i\colon D' \to D^\triangle$ is determined in the above manner by a unique relation $P \subset D \times D'$ that satisfies (1)–(3). In particular, the identity mapping $D^\triangle \to D^\triangle$ is determined by the relation $P = \{(d, F) \in D \times D^\triangle \mid d \in F\}$.*

Proof. ($\Longrightarrow$) We similarly define $i'\colon D \to D'^\triangle$ as follows: $x \mapsto xP$ for all $x \in D$. Recall that by Lemma II.2-8 ([4]) the joint Scott continuity of the characteristic mapping $P\colon D \times D' \to \mathbf{2}$ is equivalent to its Scott continuity in each variable separately, which holds due to (1). Then i and i' are Scott continuous injective mappings.

Each open filter F in $D^\triangle$ is of the form $\{F \in D^\triangle \mid F \ni x\}$ for some $x \in D$, hence its preimage under i is equal to

$$\{y \in D' \mid Py \ni x\} = xP,$$

hence is a non-empty open filter as well. Therefore i, and similarly i', are morphisms in $\mathcal{D}om\mathcal{F}ilt$. We can apply to i the contravariant functor $(-)^\triangle$. The mapping $i^\triangle\colon D'^{\triangle\triangle} \to D^\triangle$ takes each non-empty Scott open filter $\mathrm{F} \subset D'^\triangle$ to $i^{-1}(\mathrm{F})$. Then $i^\triangle \circ \underline{u}_{D'}\colon D' \to D^\triangle$ sends all y to

$$\{x \in D \mid i(x) \ni y\} = \{x \in D \mid y \in xP\} = Py = i'(y).$$

Since $\underline{u}_{D'}$ is an isomorphism and i' is injective by (2), the mapping $i^\triangle$ is injective as well, hence i is surjective. Taking into account that i is meet-preserving, we arrive at conclusion that i is an order isomorphism.

Observe that similarly $i'^\triangle \circ \underline{u}_D\colon D \to D'^\triangle$ coincides with i, hence i' is an isomorphism as well.

($\Longleftarrow$) Let $i\colon D \to D'^\triangle$ be an isomorphism. It is straightforward to verify that the relation $P = \{(x, y) \in D \times D' \mid y \notin i(x)\}$ satisfies properties (1)–(3) and is unique that determines i in the above manner. If $D' = D^\triangle$ and i is the identity mapping, then P consists of all $(d, F) \in D \times D^\triangle$ such that $d \in F$. □

Remark 1. If the domains D and D' in the latter theorem are continuous semilattices, then condition (1) can be equivalently formulated as the distributivity of the characteristic mapping $P\colon D \times D' \to \mathbf{2}$ w.r.t. meet in both variables.

Similarly, for domains D and D', with bottom elements, which are denoted by 0, consider the following properties of a relation $P \subset D \times D'$:

(1) for all $x_0 \in D$, $y_0 \in D'$ the sets $x_0 P = \{y \in D' \mid (x_0, y) \in P\}$, $Py_0 = \{x \in D \mid (x, y_0) \in P\}$ are (not necessarily non-empty) open filters, and $0P = P0 = \varnothing$;

(2) for all $x_1, x_2 \in D$, $x_1 \nleqslant x_2$ there is $y \in D'$ such that $(x_1, y) \in P$, but $(x_2, y) \notin P$;

(3) for all $y_1, y_2 \in D'$, $y_1 \nleqslant y_2$ there is $x \in D$ such that $(x, y_1) \in P$, but $(x, y_2) \notin P$.

Observe that then neither $x_0 P = D'$ nor $Py_0 = D$, hence $x_0 P \in D'^{\wedge}$, $Py_0 \in D^{\wedge}$ (we adjoin the missed top element $\top$).

Theorem 2. *For domains D and D' and a relation $P \subset D \times D'$ that satisfies the above conditions (1)–(3), the mapping $i\colon D' \to D^{\wedge}$ that takes each $y \in D'$ to the open filter $Py \cup \{\top\} \subset D^{\top}$ is an isomorphism. Conversely, each isomorphism $i\colon D' \to D^{\wedge}$ is determined in the above manner by a unique relation $P \subset D \times D'$ that satisfies (1)–(3). In particular, the identity mapping $D^{\wedge} \to D^{\wedge}$ is determined by the relation $P = \{(d, F) \in D \times D^{\wedge} \mid d \in F\}$.*

Proof is quite analogous to the proof of the previous theorem.

Remark 2. If the domains D and D' are continuous semilattices with bottom elements, then condition (1) is equivalent to the following one: the characteristic mapping $P\colon D \times D' \to \mathbf{2}$ is distributive w.r.t. meet in both variables, and $P(x, 0) = P(0, y) = 0$ for all $x \in D$, $y \in D'$.

Following [14], we call a relation P that satisfy one of the given above sets of conditions (1)–(3) (which one, will depend on a context) a *separating polarity.*

Remark 3. Observe that these conditions are symmetric w.r.t. the involved domains, hence we can always assume for simplicity that either D' is equal to $D^{\triangle}$ (or to $D^{\wedge}$) and P is the "$\in$" relation, or that D is equal to $D'^{\triangle}$ (or to $D'^{\wedge}$) and P is the "$\ni$" relation.

Definition 1 ([4])**.** If S, S' are posets and $p\colon S \to S'$, $q\colon S' \to S$ are functions such that for all $s \in S$ and $s' \in S'$

$$s \leqslant_S q(s') \text{ iff } s' \leqslant_{S'} q(s),$$

then the quadruple (S, p, q, S') is called a *contravariant Galois connection.*

Such p, q are antitone, and the latter definition is symmetric, i.e. (S', q, p, S) is a contravariant Galois connection as well.

Given domains D and D' and a separating polarity $P \subset D \times D'$, we define a relation $P^L_{\triangle} \subset [D \to L] \times [D' \to \tilde{L}]$ as follows:

$$(\varphi, \psi) \in P^L_{\triangle} \iff \varphi(x) \geqslant \psi(y) \text{ in } L \text{ for all } x \in D, y \in D' \text{ such that } (x, y) \in P.$$

By the above, we can assume that $D' = D^{\triangle}$ and $P = \in$, then the relation $\in^L_{\triangle} \subset [D \to L] \times [D^{\triangle} \to \tilde{L}]$ is the following:

$$(\varphi, \psi) \in \in^L_{\triangle} \iff \varphi(d) \geqslant \psi(F) \text{ in } L \text{ for all } d \in D, F \in D^{\triangle} \text{ such that } d \in F.$$

The characteristic mapping of $P^L_{\triangle}$ is isotone, and, for all $\varphi \in [D \to L]$, $\psi \in [D' \to \tilde{L}]$, the inclusion $(\varphi, \psi) \in P^L_{\triangle}$ is equivalent to either of the following two statements:

(*) $\varphi \geqslant q(\psi)$, where $q(\psi)\colon D \to L$ is determined by the equality

$$q(\psi)(x) = \sup\{\psi(y) \mid (x,y) \in P\}, \quad x \in D;$$

and

(**) $\psi \tilde{\geqslant} p(\varphi)$, where $p(\varphi)\colon D' \to \tilde{L}$ is determined by the equality

$$p(\varphi)(y) = \tilde{\sup}\{\varphi(x) \mid (x,y) \in P\}, \quad y \in D'.$$

Observe that $p(\varphi) \in [D' \to \tilde{L}]$, $q(\psi) \in [D \to L]$, therefore $([D \to L]^{op}, p, q, [D' \to \tilde{L}]^{op})$ is a contravariant Galois connection.

Theorem 3. *The mappings $p\colon [D \to L] \to [D' \to \tilde{L}]$ and $q\colon [D' \to \tilde{L}] \to [D \to L]$ are mutually inverse order antiisomorphisms.*

Proof. To prove that two mappings that constitute a Galois correspondence are mutually inverse, it is sufficient to show that they are injective. We prove this for p, the proof for q is dual. Assume that the considered mappings are $p\colon [D \to L] \to [D^{\triangle} \to \tilde{L}]$ and $q\colon [D^{\triangle} \to \tilde{L}] \to [D \to L]$.

Let $\varphi \nleqslant \varphi'$ in $[D \to L]$, then there is $d_0 \in D$ such that $\varphi(d_0) \nleqslant \varphi'(d_0)$. The lattice L is continuous, the element $\varphi(d_0)$ is the least upper bound of all $\alpha \in L$ way below it, hence there is $\alpha \ll \varphi(d_0)$ such that $\alpha \nleqslant \varphi'(d_0)$. Then $\varphi(d_0)$ is in the open set $\{\beta \in L \mid \alpha \ll \beta\}$.

Since D is continuous and φ is Scott continuous, and the open filters form a basis of the Scott topology in a domain (cf. Theorem II-1.14), there is an open filter $F \in D^{\triangle}$ such that $F \ni d_0$ and $\alpha \ll \varphi(d)$ for all $d \in F$. Then $p(\varphi)(F) = \inf\{\varphi(d) \mid d \in F\} \geqslant \alpha$, but $p(\varphi')(F) = \inf\{\varphi'(d) \mid d \in F\} \leqslant \varphi'(d_0) \ngeqslant \alpha$, which implies $p(\varphi')(F) \neq p(\varphi)(F)$. □

Similarly, for domains D and D' with bottom elements and a separating polarity $P \subset D \times D'$, we define a relation $P^L_\wedge \subset [D \to L]_0 \times [D' \to \tilde{L}]_0$ by the same equality:

$$(\varphi,\psi) \in P^L_\wedge \iff \varphi(x) \geqslant \psi(y) \text{ in } L \text{ for all } x \in D, y \in D' \text{ such that } (x,y) \in P.$$

Again, we can assume that $D' = D^\wedge$ and $P = \in$, then $\in^L_\wedge \subset [D \to L]_0 \times [D^{\triangle} \to \tilde{L}]_0$ is the following:

$$(\varphi,\psi) \in \in^L_\wedge \iff \varphi(d) \geqslant \psi(F) \text{ in } L \text{ for all } d \in D, F \in D^\wedge \text{ such that } d \in F,$$

which is equivalent to either of the following two statements:

(*) $\varphi \geqslant q_0(\psi)$, where $q_0(\psi)\colon D \to L$ is determined by the equality

$$q_0(\psi)(x) = \sup\{\psi(y) \mid (x,y) \in P\}, \quad x \in D;$$

and

(**) $\psi \tilde{\geqslant} p_0(\varphi)$, where $p_0(\varphi)\colon D' \to \tilde{L}$ is determined by the equality

$$p_0(\varphi)(y) = \tilde{\sup}\{\varphi(x) \mid (x,y) \in P\}, \quad y \in D'.$$

Here in (*) and (**) we assume that $\sup\varnothing = 0$, $\tilde{\sup}\varnothing = 1$. Then $([D \to L]_0^{op}, p_0, q_0, [D' \to \tilde{L}]_0^{op})$ is also a contravariant Galois connection, and the following statement holds.

Theorem 4. *The mappings $p_0\colon [D \to L]_0 \to [D' \to \tilde{L}]_0$ and $q_0\colon [D' \to \tilde{L}]_0 \to [D \to L]_0$ are mutually inverse order antiisomorphisms.*

6. Continuous and dually continuous semimodules of Scott continuous mappings and monotonic predicates. Now we describe probably the most important classes of L-semimodules, which, unlike the two previous examples, in rather general settings are simultaneously continuous and dually continuous.

For a domain D, both the finite and infinite suprema in the lattice $[D \to L]$ are calculated argumentwise:

$$(\sup_{i\in\mathcal{I}} \varphi_i)(d) = \sup_{i\in\mathcal{I}}(\varphi_i(d)),\ d \in D,$$

for each collection $\{\varphi_i \mid i \in \mathcal{I}\}$ of elements of $[D \to L]$. In particular, for $\varphi_1, \varphi_2 \in [D \to L]$ the join $\varphi_1 \bar{\oplus} \varphi_2$ is equal to $(\varphi_1 \bar{\oplus} \varphi_2)(d) = \varphi_1(d) \oplus \varphi_2(d)$, $d \in D$. Unfortunately, the argumentwise infimum $\inf_{i\in\mathcal{I}} \varphi_i(d)$, $d \in D$, need not be Scott continuous, but for each isotone function $\varphi\colon D \to L$ the greatest Scott continuous function $D \to L$ that precedes φ argumentwise is equal to

$$\varphi^l(d) = \sup\{\varphi(d') \mid d' \in D, d' \ll d\}, \quad d \in D.$$

Hence

$$(\inf_{i\in\mathcal{I}} \varphi_i)(d) = \sup\{\inf_{i\in\mathcal{I}} \varphi_i(d') \mid d' \in D, d' \ll d\}, \quad d \in D.$$

Nevertheless, the finite infima are calculated argumentwise, in particular, for $\varphi_1, \varphi_2 \in [D \to L]$ the meet $\varphi_1 \bar{\otimes} \varphi_2$ is equal to $(\varphi_1 \bar{\otimes} \varphi_2)(d) = \varphi_1(d) \otimes \varphi_2(d)$, $d \in D$.

Recall that we consider an operation $*\colon L \times L \to L$ that is associative and infinitely distributive w.r.t. supremum in both variables, and the top element 1 is a two-sided unit. Then $*$ is (separately and jointly) Scott continuous, which allows to define an operation $\bar{*}\colon L \times [D \to L] \to [D \to L]$ in a straightforward manner: for $\alpha \in L$ and $\varphi \in [D \to L]$, let $(\alpha \,\bar{*}\, \varphi)(d) = \alpha * \varphi(d)$ for all $d \in D$. It is clear that $\bar{*}$ infinitely distributes over suprema in both variables, hence is Scott continuous as well. Therefore we obtain the following assertions.

Theorem 5. *For a domain D, the triple $([D \to L], \bar{\oplus}, \bar{*})$ is a continuous L-semimodule. If, moreover, D contains a bottom element, then $([D \to L], \bar{\oplus}, \bar{*})$ is a complete sublattice and subsemimodule of $[D \to L]$, hence is a continuous L-semimodule as well.*

Theorem 6. *For a domain D, the triple $([D \to L], \bar{\oplus}, \bar{*})$ is a complete dually continuous L-semimodule if and only if the multiplication $*\colon L \times L \to L$ is distributive in both variables w.r.t. the filtered infima.*

Observe that the latter distributivity is equivalent to the continuity of $*$ w.r.t. the dual Lawson topology.

Proof. Necessity is obvious, because L can be considered as a sublattice of $[D \to L]$: each $\alpha \in L$ is identified with the constant function that maps D to α.
Sufficiency. Let $*\colon L \times L \to L$ be distributive in both variables w.r.t. the filtered infima. For each filtered set $\{\alpha_i \mid i \in \mathcal{I}\} \subset L$ and a function $\varphi \in [D \to L]$ one has

$$\inf_{i\in\mathcal{I}} \alpha_i * \varphi(d) = (\inf_{i\in\mathcal{I}} \alpha_i) * \varphi(d), \quad d \in D,$$

hence $(\inf_{i\in\mathcal{I}} \alpha_i) \,\bar{*}\, \varphi$ is the most lower bound of all $\alpha_i \,\bar{*}\, \varphi$ in $[D \to L]$, which is the required distributivity in the first variable.

Now we consider $\alpha \in L$ and a filtered collection $\{\varphi_i \mid i \in \mathcal{I}\} \subset [D \to L]$. Recall that the infimum ψ of $\{\varphi_i \mid i \in \mathcal{I}\}$ in $[D \to L]$ is equal to

$$\psi(d) = \sup\{\inf_{i\in\mathcal{I}} \varphi_i(d') \mid d' \in D, d' \ll d\}, \quad d \in D.$$

Similarly the infimum ψ' of all $\alpha \mathbin{\bar{*}} \varphi_i$ is equal to

$$\psi'(d) = \sup\{\inf_{i\in\mathcal{I}} \alpha * \varphi_i(d') \mid d' \in D, d' \ll d\} = \sup\{\alpha * \inf_{i\in\mathcal{I}} \varphi_i(d') \mid d' \in D, d' \ll d\} =$$
$$= \alpha * \sup\{\inf_{i\in\mathcal{I}} \varphi_i(d') \mid d' \in D, d' \ll d\} = \alpha * \psi(d),$$

for all $d \in D$, which completes the proof. □

Analogously we obtain the following result.

Theorem 7. *For a domain D with a bottom element but with $|D| \neq 1$, the triple $([D \to L]_0, \bar{\oplus}, \bar{*})$ is a complete dually continuous L-semimodule if and only if the multiplication $*\colon L \times L \to L$ is distributive in both variables w.r.t. the filtered infima.*

Now we consider the order dual posets to the lattices of Scott continuous functions.

Following [6], for a domain D we call elements of the set $[D \to L^{op}]^{op}$ *L-fuzzy monotonic predicates* on D. The elements of D are considered as pieces of information about the state of a certain system or process, and $a \leqslant b$ in D means that b contains more information than a (is more specific/restrictive). For $m \in [D \to L^{op}]^{op}$ and $a \in D$, we regard $m(a)$ as the truth value of a, hence it is required that $m(b) \leqslant m(a)$ for all $a \leqslant b$. The second ^{op} means that we order fuzzy predicates pointwisely, i.e. $m_1 \leqslant m_2$ iff $m_1(a) \leqslant m_2(a)$ in L (not in L^{op} !) for all $a \in D$. We denote $\underline{M}_{[L]}D = [D \to L^{op}]^{op}$, and, for D with the least element 0, consider also the subset $M_{[L]}D \subset \underline{M}_{[L]}D$ of all *normalized* predicates that take $0 \in D$ (no information) to $1 \in L$ (complete truth).

Obviously both $\underline{M}_{[L]}D$ and $M_{[L]}D$ are completely distributive lattices, hence are continuous and dually continuous lattices. It is also clear that all *infima* and *finite suprema* of functions in $\underline{M}_{[L]}D$ and $M_{[L]}D$, including the pairwise joins $m_1 \mathbin{\bar{\oplus}} m_2$, are calculated argumentwise, whereas the supremum of a collection $\{m_i \mid i \in \mathcal{I}\}$ of elements of these lattices is equal to

$$(\sup_{i\in\mathcal{I}} m_i)(d) = \inf\{\sup_{i\in\mathcal{I}} m_i(d') \mid d' \in D, d' \ll d\}, \quad d \in D.$$

Theorem 3 implies the following statement.

Corollary 1. *For domains D, D' such that there is a separating polarity $P \subset D \times D'$, the posets $\underline{M}_{[L]}D = [D \to L^{op}]^{op}$ and $\underline{M}_{[\tilde{L}]}D' = [D' \to L]^{op}$ are antiisomorphic.*

Note that the mentioned antiisomorphism $p\colon \underline{M}_{[L]}D \to \underline{M}_{[\tilde{L}]}D'$ is of the form: for $m \in \underline{M}_{[L]}D$, the monotone predicate $m' = p(m) \in \underline{M}_{[\tilde{L}]}D$ is determined by the equality

$$m'(y) = \sup\{m(x) \mid (x, y) \in P\} \text{ in } L, \quad y \in D',$$

the inverse antiisomorphism is analogous, but the supremum is taken in $\tilde{L}$. In particular, there is an isomorphism $\underline{\varkappa}_{[L]}D$ between the posets $\underline{M}_{[L]}D = [D \to L^{op}]^{op}$ and $\underline{M}_{[\tilde{L}]}D^{\triangle} = [D^{\triangle} \to L]^{op}$, namely

$$\underline{\varkappa}_{[L]}D(m)(F) = \sup\{m(d) \mid d \in F\}, \quad m \in \underline{M}_{[L]}D, F \in D^{\triangle}.$$

Analogously, one has the following consequence of Theorem 4.

Corollary 2. *For domains D, D' with bottom elements such that there is a separating polarity $P \subset D \times D'$, the posets $M_{[L]}D = [D \to L^{op}]_0^{op}$ and $M_{[\tilde{L}]}D' = [D' \to L]_0^{op}$ are antiisomorphic.*

The formulae are the same, but the convention $\sup \varnothing = 0$, $\tilde{\sup} \varnothing = 1$ is used. In particular, the posets $M_{[L]}D = [D \to L^{op}]_0^{op}$ and $M_{[\tilde{L}]}D^\wedge = [D^\wedge \to L]_0^{op}$ are antiisomorphic through the mapping $\varkappa_{[L]}D$:

$$\varkappa_{[L]}D(m)(F) = \sup\{m(d) \mid d \in F\}, \quad m \in M_{[L]}D, F \in D^\wedge.$$

If only one poset of Scott continuous mappings is "turned over", then we obtain the order *isomorphisms*

$$\underline{\varkappa}_{[L]}D\colon \underline{M}_{[L]}D = [D \to L^{op}]^{op} \to [D^\triangle \to L]$$

for each domain D and

$$\varkappa_{[L]}D\colon M_{[L]}D = [D \to L^{op}]_0^{op} \to [D^\wedge \to L]_0$$

for each domain D with a bottom element. Recall that the right-hand posets are continuous L-semimodules and (for a "sufficiently good" multiplication $*$) complete dually continuous L-semimodules. Therefore we can transfer the multiplications $\bar{*}$ via the isomorphisms to $\underline{M}_{[L]}D$ and $M_{[L]}D$, making them continuous (complete dually continuous) L-semimodules as well.

Theorem 8. *For a domain D, the unique multiplication $\bar{\circledast}\colon L \times \underline{M}_{[L]}D \to \underline{M}_{[L]}D$ that is mapped with $\underline{\varkappa}_{[L]}D\colon \underline{M}_{[L]}D \to [D^\triangle \to L]$ to the multiplication $\bar{*}\colon L \times [D^\triangle \to L] \to [D^\triangle \to L]$, is defined by the formula*

$$(\alpha \mathbin{\bar{\circledast}} m)(d) = \inf\{\alpha * m(d') \mid d' \in D, d' \ll d\},$$

$m \in \underline{M}_{[L]}D$, $d \in D$. If $*\colon L \times L \to L$ is dually Scott continuous, then a simpler equivalent formula is valid:

$$(\alpha \mathbin{\bar{\circledast}} m)(d) = \alpha * m(d), \quad m \in \underline{M}_{[L]}D, d \in D.$$

Proof is routine but straightforward, same as for the following statement:

Theorem 9. *For a domain D with the bottom element 0, the unique multiplication $\bar{\odot}\colon L \times M_{[L]}D \to M_{[L]}D$ that is mapped with $\varkappa_{[L]}D\colon M_{[L]}D \to [D^\wedge \to L]_0$ to the multiplication $\bar{*}\colon L \times [D^\wedge \to L]_0 \to [D^\wedge \to L]_0$, is defined by the formula*

$$(\alpha \mathbin{\bar{\odot}} m)(d) = \begin{cases} (\alpha \mathbin{\bar{\circledast}} m)(d), & d \neq 0, \\ 1, & d = 0, \end{cases}$$

$m \in M_{[L]}D$, $d \in D$. If $*\colon L \times L \to L$ is dually Scott continuous, then a simpler equivalent formula is valid:

$$(\alpha \mathbin{\bar{\odot}} m)(d) = \begin{cases} \alpha * m(d), & d \neq 0, \\ 1, & d = 0, \end{cases} \quad m \in M_{[L]}D, d \in D.$$

Corollary 3. *For a domain D, the triple $(\underline{M}_{[L]}D, \bar{\oplus}, \bar{\circledast})$ is a continuous L -semimodule. If $*\colon L \times L \to L$ is dually Scott continuous, then $(\underline{M}_{[L]}D, \bar{\oplus}, \bar{\circledast})$ is also a complete dually continuous L-semimodule.*

Corollary 4. *For a domain D with a bottom element, the triple $(M_{[L]}D, \bar{\oplus}, \bar{\odot})$ is a continuous L-semimodule. If $*\colon L \times L \to L$ is dually Scott continuous, then $(M_{[L]}D, \bar{\oplus}, \bar{\odot})$ is also a complete dually continuous L-semimodule.*

7. Subsemilattices and approximation. A continuous semilattice is called *stably continuous* [4] if the relation $\ll$ in it is multiplicative, i.e., $x \ll y$ and $x \ll z$ imply $x \ll y \wedge z$. This is equivalent to $x' \ll x \wedge y' \ll y \implies x' \wedge y' \ll x \wedge y$.

Lemma 1. *Let D be a stably continuous semilattice. Then the set $[D \to L]_\wedge$ of all Scott continuous meet-preserving mappings from D to L is a Lawson closed lower subsemilattice in the completely distributive lattice $[D \to L]$.*

Proof. It is obvious that the pointwise infimum of two mappings in $[D \to L]_\wedge$ is Scott continuous and meet-preserving as well, hence $[D \to L]_\wedge$ is a lower subsemilattice of $[D \to L]$.

Let $f \in [D \to L] \setminus [D \to L]_\wedge$, then there are $d_1, d_2 \in D$ such that $f(d_1 \otimes d_2) \not\geqslant f(d_1) \otimes f(d_2)$. By the Scott continuity of f there are $d_1' \ll d_1$, $d_2' \ll d_2$ such that $f(d_1 \otimes d_2) \not\geqslant f(d_1') \otimes f(d_2')$. Moreover, we can choose Scott open sets $U_1 \ni f(d_1')$, $U_2 \ni f(d_2')$ in L such that $f(d_1 \otimes d_2) \not\geqslant \inf U_1 \otimes \inf U_2$. Observe that by the stable continuity of D the set $\{d \in D \mid d_1' \otimes d_2' \ll d\}$ is an open filter, therefore the mapping $f_0\colon D \to L$,

$$f_0(d) = \begin{cases} 0, & d_1' \otimes d_2' \not\ll d, \\ \inf U_1 \otimes \inf U_2, & d_1' \otimes d_2' \ll d, \end{cases} \qquad d \in D,$$

is Scott continuous. Hence the set $V = \{g \in [D \to L] \mid g \not\geqslant f_0\}$ is lower open, the set $W = \{g \in [D \to L] \mid g(d_1') \in U_1, g(d_2') \in U_2\}$ is Scott open, and if $g \in V \cap W$, then $g(d_1' \otimes d_2') \not\geqslant \inf U_1 \otimes \inf U_2$, $g(d_1') \otimes g(d_2') \geqslant \inf U_1 \otimes \inf U_2$, hence $g(d_1' \otimes d_2') \not\geqslant g(d_1') \otimes g(d_2')$, and g does not belongs to $[D \to L]_\wedge$. This proves that the complement of $[D \to L]_\wedge$ in $[D \to L]$ is Lawson open. □

For a stably continuous semilattice D with a bottom element we denote by $[D \to L]_{\wedge 0}$ the set of all Scott continuous bottom-preserving meet-preserving mappings from D to L, and by $[D \to L]_{\wedge + 0}$ the set of all Scott continuous mappings $f\colon D \to L$ such that $f(0) = 0$ and

$$f(d_1) \otimes f(d_2) = f(d_1 \otimes d_2) \oplus \inf f(D \setminus \{0\})$$

for all $d_1, d_2 \in D \setminus \{0\}$. The latter property is equivalent to

$$f(d) = \begin{cases} g(d), & d \in D \setminus \{0\}, \\ 0, & d = 0, \end{cases}$$

for some $g \in [D \to L]_\wedge$. For convenience we call such mappings *almost meet-preserving.*

Corollary 5. *Let D be a stably continuous semilattice with a bottom element. Then the sets $[D \to L]_{\wedge 0}$ and $[D \to L]_{\wedge + 0}$ are Lawson closed lower subsemilattices in the completely distributive lattice $[D \to L]_0$ of all Scott continuous bottom-preserving mapping.*

Observe that $[D \to L]_\wedge$ and $[D \to L]_{\wedge + 0}$ have top elements, therefore:

Corollary 6. *Let D be a stably continuous semilattice. Then the set $[D \to L]_\wedge$ of all Scott continuous meet preserving mappings from D to L is a continuous lattice. If D has a bottom element, then the set $[D \to L]_{\wedge 0}$ of all Scott continuous bottom-preserving meet-preserving mappings from D to L is a complete continuous semilattice, and the set $[D \to L]_{\wedge + 0}$ of all Scott continuous bottom-preserving almost meet-preserving mappings from D to L is a continuous lattice.*

Lemma 2. *Let D be a domain, $f\colon D \to L$ be Scott continuous. For each Scott open set $U \subsetneq D$, denote by f_U the following function from D to L:*

$$f_U(d) = \begin{cases} \sup\{f(d') \mid d' \notin U\}, & d \notin U, \\ 1, & d \in U, \end{cases} \quad d \in D.$$

Then:

(a) *$f(d) = \inf\{f_U(d) \mid U \subsetneq D$ is a Scott open subset$\}$ for all $d \in D$;*

(b) *if D is a continuous semilattice and f is meet-preserving, then*

$$f(d) = \inf\{f_F(d) \mid F \subsetneq D \text{ is a Scott open filter}\}$$

for all $d \in D$.

Proof. We prove only (b), the reader can easily modify the proof to obtain (a).

Obviously $f \leqslant f_F$ for all Scott open filters F, hence the "$\leqslant$" sign is immediate.

To prove the reverse inequality, for $d \in D$ denote $\alpha = f(d)$ and choose arbitrary $\beta \in L$, $\beta \not\leqslant \alpha$. Then there is a Scott open filter $\Phi_\beta \subset L$ such that $\beta \in \Phi_\beta$, $\alpha \notin \Phi_\beta$.

Due to meet preservation, the preimage $f^{-1}(\Phi_\beta) = F_\beta$ is a Scott open filter in D, which does not contain d, and

$$f_{F_\beta}(d) = \sup\{f(d') \mid d' \notin F_\beta\} = \sup\{f(d') \mid d' \in D, f(d') \notin \Phi_\beta\} \leqslant \sup\{\gamma \in L \mid \gamma \not\geqslant \beta\},$$

therefore

$$\inf\{f_F(d) \mid F \subset D \text{ is a Scott open filter, } d \notin F\} \leqslant \inf\{f_{F_\beta}(d) \mid \beta \not\leqslant \alpha\} \leqslant \\ \leqslant \inf\bigl\{\sup\{\gamma \in L \mid \gamma \not\geqslant \beta\} \bigm| \beta \not\leqslant \alpha\bigr\}.$$

It has been proved in [10] that, for a complete lattice L, the equality

$$\inf\bigl\{\sup\{\gamma \in L \mid \gamma \not\geqslant \beta\} \bigm| \beta \not\leqslant \alpha\bigr\} = \alpha$$

for all $\alpha \in L$ is equivalent to the complete distributivity of L, which holds in our case. Thus

$$\inf\{f_F(d) \mid F \subsetneq D \text{ is a Scott open filter}\} \leqslant \alpha = f(d),$$

which completes the proof. □

Remark 4. The functions $\inf\{f_{U_1}, \dots, f_{U_n}\}$, for the finite collections $U_1, \dots, U_n$ of Scott open sets in D, form a filtered set in $[D \to L]$, hence provide an approximation from above by functions with finite ranges. In particular, for a meet-preserving function $f\colon D \to L$ and a finite collection $F_1, \dots, F_n$ of open filters in D the function $\inf\{f_{F_1}, \dots, f_{F_n}\}$ is also in $[D \to L]_\wedge$.

It is easy also to observe that, if f attains only the values in a finite sublattice $L_0 \subset L$, then f can be obtained as the *finite* infimum of functions of the form f_U:

$$f(d) = \inf\{f_U(d) \mid U = f^{-1}(\{\alpha\}\uparrow), \alpha \in L_0\}$$

for all $d \in D$. For a meet-preserving function f, all the sets $f^{-1}(\{\alpha\}\uparrow)$, $\alpha \in L_0$ are open filters, hence such f_U are meet-preserving as well.

To obtain directed approximations from below by functions with finite ranges is easier.

Lemma 3. *Let D be a domain, $f \in [D \to L]$. For each $A = \{\alpha_1, \dots, \alpha_n\} \subset L$ and $U_i = f^{-1}(\{\beta \in L \mid \alpha_i \ll \beta\})$ for $i = 1, \dots, n$, denote*

$$f_A(d) = \sup\{\alpha_i \mid 1 \leqslant i \leqslant n, d \in U_i\},\ d \in D.$$

Then $f_A \in [D \to L]$, and

$$f(d) = \sup\{f_A(d) \mid A = \{\alpha_1, \dots, \alpha_n\} \subset L, n \in \mathbb{N}\}$$

for all $d \in D$.

To keep meet preservation, the construction should be complicated:

Lemma 4. *Let D be a stably continuous semilattice, $f \in [D \to L]_\wedge$. For each $A = \{\alpha_1, \dots, \alpha_n\} \subset L$ denote*

$$U_i = f^{-1}(\{\beta \in L \mid \alpha_i \ll \beta\}),\ i = 1, \dots, n,$$
$$U_{i_1 \dots i_k} = \{d_1 \wedge \dots \wedge d_k \mid d_1 \in U_{i_1}, \dots, d_k \in U_{i_k}\},\ 1 \leqslant i_1 < \dots < i_k \leqslant n,$$

and

$$f_A(d) = \sup\{\alpha_{i_1} \otimes \ldots \otimes \alpha_{i_k} \mid d \in U_{i_1 \dots i_k}, 1 \leqslant i_1 < \dots < i_k \leqslant n\},\ d \in D.$$

Then $f_A \in [D \to L]_\wedge$, and

$$f(d) = \sup\{f_A(d) \mid A = \{\alpha_1, \dots, \alpha_n\} \subset L, n \in \mathbb{N}\}$$

for all $d \in D$.

The proofs are straightforward. Note that all $U_{i_1 \dots i_k}$ are open filters. Obviously, for a function $f \in [D \to L]_{\wedge 0}$, the approximating functions f_A provided by of the latter theorem are also meet- and bottom-preserving.

By Exercise IV-2.22 ([4]), a continuous semilattice D with a top element is complete (i.e., is a continuous lattice) if and only if the semilattice $D^\triangle$ is stably continuous. This implies that a continuous semilattice D with a bottom element is complete if and only if the semilattice $D^\wedge$ is stably continuous.

Similarly to $[D \to L]_\wedge$, for a continuous lattice D we denote

$$[D \to L]_\vee = \{f \in [D \to L] \mid f(d_1 \vee d_2) = f(d_1) \oplus f(d_2) \text{ for all } d_1, d_2 \in D\}.$$

Let D be a continuous lattice. Recall that the posets $[D^\triangle \to \tilde{L}]$ and $[D \to L]$ are antiisomorphic through the mapping q, cf. Theorem 3.

Theorem 10. *The image of* $[D^{\triangle} \to \tilde{L}]_{\wedge}$ *under the antiisomorphism* $q\colon [D^{\triangle} \to \tilde{L}] \to [D \to L]$ *is the set* $[D \to L]_{\vee}$.

Proof is trivial.

Corollary 7. *For a continuous lattice* D*, the set* $[D \to L]_{\vee}$ *is a closed w.r.t. the dual Lawson topology upper subsemilattice of* $[D \to L]$ *with a bottom element, therefore is a dually continuous lattice.*

Analogously the following assertion is true.

Theorem 11. *For a complete continuous semilattice* D *with a bottom element, the antiisomorphism* $q_0\colon [D^{\wedge} \to \tilde{L}]_0 \to [D \to L]_0$ *maps the set* $[D^{\wedge} \to \tilde{L}]_{\wedge 0}$ *onto the set*

$$[D \to L]_{\vee 01} = \left\{ f \in [D \to L]_0 \mid f(d_1) \oplus f(d_2) = \begin{cases} f(d_1 \vee d_2), & \text{if } d_1 \vee d_2 \text{ exists,} \\ 1 & \text{otherwise,} \end{cases} \; d_1, d_2 \in D \right\},$$

and the set $[D^{\wedge} \to \tilde{L}]_{\wedge + 0}$ *onto the set*

$$[D \to L]_{\vee 0} = \left\{ f \in [D \to L]_0 \mid f(d_1) \oplus f(d_2) = \begin{cases} f(d_1 \vee d_2), & \text{if } d_1 \vee d_2 \text{ exists,} \\ \sup f(D) & \text{otherwise,} \end{cases} \; d_1, d_2 \in D \right\}.$$

Corollary 8. *For a complete continuous semilattice* D *with a bottom element, the sets* $[D \to L]_{\vee 0}$ *and* $[D \to L]_{\vee 01}$ *are closed w.r.t. the dual Lawson topology upper subsemilattices of* $[D \to L]_0$*, therefore* $[D \to L]_{\vee 01}$ *is a complete dually continuous semilattice, and* $[D \to L]_{\vee 0}$ *is a dually continuous lattice.*

It is obvious how to obtain dual approximations from above and from below for the elements of $[D \to L]_{\vee}$ and $[D \to L]_{\vee 0}$.

Similarly to Lemma 2, we obtain the following statement.

Lemma 5. *For each element* x *of a domain* D *and a function* $f\colon D \to L$*, denote by* f_x *the following function from* D *to* L*:*

$$f_x(d) = \begin{cases} 0, & d \leqslant x, \\ \inf\{f(d') \mid d' \nleqslant x\}, & d \nleqslant x, \end{cases} \quad d \in D.$$

(a) If D *is a continuous lattice and* $f \in [D \to L]_{\vee}$*, then* $f(d) = \sup\{f_x(d) \mid x \in D\} \oplus f(0)$ *for all* $d \in D$*.*

(b) If D *is a complete continuous semilattice and* $f \in [D \to L]_{\vee 0}$*, then* $f(d) = \sup\{f_x(d) \mid x \in D\}$ *for all* $d \in D$*.*

Proof. (a) For each $x \in D$ we have $f_x \leqslant f$, therefore $f(d) \geqslant \sup\{f_x(d) \mid x \in D\} \oplus f(0)$ for all $d \in D$. On the other hand, for each $\alpha \in L$, $\alpha \geqslant f(0)$, the set $f^{-1}(\{\alpha\}{\downarrow}) \subset D$ is nonempty, lower, directed and Scott closed, hence is of the form $\{x_\alpha\}{\downarrow}$ for a uniquely determined $x_\alpha \in D$.

The set $L_0 = \{\alpha \in L \mid \alpha \geqslant f(0)\}$ is a complete sublattice of L, hence is a completely distributive lattice. Since $f(D \setminus \{x_\alpha\}{\downarrow}) \subset \{\beta \in L_0 \mid \beta \nleqslant \alpha\}$, we have $\inf\{f(d') \mid d' \nleqslant x_\alpha\} \geqslant \inf\{\beta \in L_0 \mid \beta \nleqslant \alpha\}$, and the inequality

$$\sup\{f_x(d) \mid x \in D\} \oplus f(0) \geqslant \sup\{f_{x_\alpha}(d) \mid \alpha \in L_0\} \oplus f(0) \geqslant$$
$$\geqslant \sup\{\inf\{\beta \in L_0 \mid \beta \nleqslant \alpha\} \mid \alpha \in L_0, \alpha \ngeqslant f(d)\} = f(d)$$

is valid for all $d \in D$.

(b) is obtained from (a) by adjoining a top element $\top$ to D and putting $f(\top) = 1$. $\square$

Taking into account Theorems 10, 11, Lemma 4, and Remark 4, we deduce the following statement.

Theorem 12. *(a) For a continuous lattice D, each join-preserving Scott continuous function $f\colon D \to L$ is an argumentwise infimum of a filtered set $\{f_i \mid i \in \mathcal{I}\}$ of join-preserving Scott continuous functions $D \to L$ with finite ranges.*

(b) For a complete continuous semilattice D, each function $f \in [D \to L]_{\vee 0}$ is an argumentwise infimum of a filtered set $\{f_i \mid i \in \mathcal{I}\}$ elements of $[D \to L]_{\vee 0}$ with finite ranges.

Following the accepted notation style, we denote $\underline{M}_{\wedge[L]}D = [D \to L^{op}]^{op}_{\wedge}$ for a stably continuous semilattice D with a top element, $\underline{M}_{\vee[L]}D = [D \to L^{op}]^{op}_{\vee}$ for a continuous lattice D, $M_{\wedge[L]}D = [D \to L^{op}]^{op}_{\wedge 0}$ for a stably continuous semilattice D, and $M_{\vee[L]}D = [D \to L^{op}]^{op}_{\vee 01}$ for a complete continuous semilattice D. Observe that under these assumptions $\underline{M}_{\wedge[L]}D$ is a dually continuous lattice, $\underline{M}_{\vee[L]}D$ is a continuous lattice, $M_{\wedge[L]}D$ is a complete dually continuous semilattice, and $M_{\vee[L]}D$ is a complete continuous semilattice.

Corollary 9. *For continuous semilattices D, D' with top elements such that D is complete and there is a separating polarity $P \subset D \times D'$, the posets $\underline{M}_{\vee[L]}D$ and $\underline{M}_{\wedge[\tilde{L}]}D'$ are antiisomorphic.*

Corollary 10. *For continuous semilattices D, D' with bottom elements such that D is complete and there is a separating polarity $P \subset D \times D'$, the posets $M_{\vee[L]}D$ and $M_{\wedge[\tilde{L}]}D'$ are antiisomorphic.*

In particular, the restrictions of $\underline{\varkappa}_{[L]}D$ and $\varkappa_{[L]}D$ provide respectively the antiisomorphisms

$$\underline{\varkappa}_{\vee[L]}D\colon \underline{M}_{\vee[L]}D \to \underline{M}_{\wedge[\tilde{L}]}D^{\triangle}, \quad \varkappa_{\vee[L]}D\colon M_{\vee[L]}D \to M_{\wedge[\tilde{L}]}D^{\wedge},$$

with the inverse mappings

$$\underline{\varkappa}_{\wedge[L]}D^{\triangle}\colon \underline{M}_{\wedge[\tilde{L}]}D^{\triangle} \to \underline{M}_{\vee[L]}D, \quad \varkappa_{\wedge[L]}D^{\triangle}\colon M_{\wedge[\tilde{L}]}D^{\wedge} \to M_{\vee[L]}D.$$

8. Semimodules of monotonic predicates as free continuous idempotent semimodules over domains. An important observation is that there is a topological and order embedding of D to $\underline{M}_{[L]}(D)$ (and $M_{[L]}(D)$, if D has a bottom element).

For an element $d_0 \in D$, we denote by $\eta_{[L]}D(d_0)$ the function $D \to L$ that sends each $d \in D$ to 1 if $d \leqslant d_0$ and to 0 otherwise. It is easy to see that $\eta_{[L]}D(d_0) \in M_{[L]}D \subset \underline{M}_{[L]}D$, and $\delta^D_L = \eta_{[L]}D(0)$ is a least element of $M_{[L]}D$.

Lemma 6 (1.1, [17])**.** *For a domain D, the mapping $\eta_{[L]}D\colon D \to \underline{M}_{[L]}D$ is Scott continuous and lower continuous.*

Moreover, if D is a continuous semilattice, then $\eta_{[L]}D$ is a semilattice morphism.

Remark. For D with a bottom element, $M_{[L]}D$ is a complete sublattice of $\underline{M}_{[L]}D$, hence we obtain that $\eta_{[L]}D$ is Scott and lower continuous also as a mapping $D \to M_{[L]}D$.

Therefore we consider D as a subspace of both $M_{[L]}D$ and $\underline{M}_{[L]}D$ w.r.t. the Scott and the lower, hence w.r.t. the Lawson topologies on the both sets. If D is a continuous semilattice, it is also a lower subsemilattice of $M_{[L]}D$ and $\underline{M}_{[L]}D$.

The following result was proved in [15].

Theorem 13. *For each Scott continuous mapping $\varphi\colon D \to K$ from a domain to a continuous L-semimodule there is a unique extension $\Phi\colon \underline{M}_{[L]}D \to K$ to a morphism in $(L,\oplus,*)$-$\mathcal{CSM}od_\uparrow$.*

Since a required extension Φ must preserve multiplication and all suprema, and each monotonic predicate $m \in \underline{M}_{[L]}D$ is the supremum of all products $m(d) \,\bar{*}\, \eta_{[L]}D(d)$, it is obvious that

$$\Phi(m) = \sup\{m(d) \,\bar{*}\, \varphi(d) \mid d \in D\}, \quad m \in \underline{M}_{[L]}D.$$

Similarly:

Theorem 14. *For each Scott continuous mapping $\varphi\colon D \to K$ from a domain with a bottom element to a continuous L-semimodule there is a unique extension $\Phi\colon M_{[L]}D \to K$ to a morphism in $(L,\oplus,*)$-$\mathcal{CSA}ff_\uparrow$. It is linear, i.e. it is a morphism in $(L,\oplus,*)$-$\mathcal{CSM}od_\uparrow$, if and only if φ preserves the bottom element.*

This extension is determined by the same formula.

As it was noted in [15], the two latter statements mean that $\underline{M}_{[L]}D$ (resp. $M_{[L]}D$) is a free object over D. The following series is their more formal equivalent reformulation.

Theorem 15 (1.4, [15])**.** *For an object D of the category $\mathcal{D}om$ the continuous L-semimodule $\underline{M}_{[L]}D$ is a free object over D in $(L,\oplus,*)$-$\mathcal{CSM}od_\uparrow$.*

Theorem 16 (1.5, [15])**.** *For an object D of the category $\mathcal{D}om_\perp$ the continuous L-semimodule $M_{[L]}D$ is a free object over D in $(L,\oplus,*)$-$\mathcal{CSA}ff_\uparrow$.*

Theorem 17 (1.6, [15])**.** *For an object D of the category $\mathcal{D}om_0$ the continuous L-semimodule $M_{[L]}D$ is a free object over D in $(L,\oplus,*)$-$\mathcal{CSM}od_\uparrow$.*

Given a Scott continuous mapping $f\colon D \to D'$ between domains, and taking into account that D' is a subspace of $\underline{M}_{[L]}D'$, by Theorem 13 we can extend the mapping f to a linear Scott continuous mapping $\underline{M}_{[L]}D \to \underline{M}_{[L]}D'$, which is unique and denoted by $\underline{M}_{[L]}f$. It is worth noting that it does not depend on the multiplication $*\colon L \times L \to L$:

$$\underline{M}_{[L]}f(m)(d') = \inf\big\{ \sup\{m(d) \mid d \in D, f(d) \geqslant d'_0\} \mid d'_0 \ll d'\big\}, \quad d' \in D'.$$

Thus the functor $\underline{M}_{[L]}\colon \mathcal{D}om \to (L,\oplus,*)$-$\mathcal{CSM}od_\uparrow$, which is a *left adjoint* ([11]) to the forgetful functor $(L,\oplus,*)$-$\mathcal{CSM}od_\uparrow \to \mathcal{D}om$, is obtained.

Similarly, a Scott continuous mapping $f\colon D \to D'$ between domains with bottom elements is uniquely extended to a Scott continuous affine mapping $M_{[L]}f\colon M_{[L]}D \to M_{[L]}D'$, namely

$$M_{[L]}f(m)(d') = \inf\big\{\sup\{m(d) \mid d \in D, f(d) \geqslant d'_0\} \mid d'_0 \ll d'\big\}, \quad d' \in D' \setminus \{0\},$$

$M_{[L]}f(m)(0) = 1$. The resulting functor $M_{[L]}\colon \mathcal{D}om_\perp \to (L,\oplus,*)$-$\mathcal{CSA}ff_\uparrow$ is left adjoint to the forgetful functor $(L,\oplus,*)$-$\mathcal{CSA}ff_\uparrow \to \mathcal{D}om_\perp$, and its restriction, for which we use the same notation, to $\mathcal{D}om_0 \to (L,\oplus,*)$-$\mathcal{CSM}od_\uparrow$, is left adjoint to the forgetful functor $(L,\oplus,*)$-$\mathcal{CSM}od_\uparrow \to \mathcal{D}om_0$.

Unfortunately, for lower topologies the straightforward analogues of the above statements fail to be valid. It seems that the requirement that D is a domain is too loose. Nevertheless, it is sufficient to add the requirement of compactness. Then, using ideas from [13], we obtain the following statement.

Theorem 18. *Let $*\colon L\times L\to L$ be Lawson continuous and let D be a complete continuous semilattice. For each lower continuous mapping $\varphi\colon D\to K$ from D to a continuous L-semimodule there is a unique extension $\Phi\colon \underline{M}_{[L]}D\to K$ to a morphism in $(L,\oplus,*)$-$\mathcal{CSM}od_\downarrow$.*

Theorem 19. *Let $*\colon L\times L\to L$ be Lawson continuous and let D be a complete continuous semilattice. For each lower continuous mapping $\varphi\colon D\to K$ from D to a continuous L-semimodule there is a unique extension $\Phi\colon M_{[L]}D\to K$ to a morphism in $(L,\oplus,*)$-$\mathcal{CSA}ff_\downarrow$. It is linear, i.e. it is a morphism in $(L,\oplus,*)$-$\mathcal{CSM}od_\downarrow$, if and only if φ preserves the bottom element.*

The formula is the same in both cases:

$$\Phi(m)=\sup\{m(d)\,\bar{*}\,\varphi(d)\mid d\in D\},$$

for $m\in \underline{M}_{[L]}D$ or $m\in M_{[L]}D$. Recall that $\mathcal{LL}aws_\downarrow$ is the category of complete continuous semilattices and their lower continuous (not necessarily meet-preserving mappings), and $\mathcal{LL}aws_{0\downarrow}$ is its subcategory obtained by taking all objects and only the bottom-preserving mappings. Then, more formally:

Theorem 20. *If $*\colon L\times L\to L$ is Lawson continuous, then for an object D of the category $\mathcal{LL}aws_\downarrow$ the continuous L-semimodule $\underline{M}_{[L]}D$ is a free object over D in the category $(L,\oplus,*)$-$\mathcal{CSM}od_\downarrow$.*

Theorem 21. *If $*\colon L\times L\to L$ is Lawson continuous, then for an object D of the category $\mathcal{LL}aws_\downarrow$ the continuous L-semimodule $M_{[L]}D$ is a free object over D in the category $(L,\oplus,*)$-$\mathcal{CSA}ff_\downarrow$.*

Theorem 22. *If $*\colon L\times L\to L$ is Lawson continuous, then for an object D of the category $\mathcal{LL}aws_{0\downarrow}$ the continuous L-semimodule $M_{[L]}D$ is a free object over D in the category $(L,\oplus,*)$-$\mathcal{CSM}od_\downarrow$.*

9. Free continuous idempotent semimodules over continuous semilatices. An important class of domains consists of continuous semilattices. Recall that $\eta_{[L]}D\colon D\hookrightarrow \underline{M}_{[L]}D$ preserves the meets. Assume that $\varphi\colon D\to K$ is a Scott continuous meet-preserving mapping from a continuous semilattice to a continuous L-semimodule. Does this guarantee that the unique Scott continuous linear extension $\Phi\colon \underline{M}_{[L]}D\to K$ of φ preserves the meets as well?

Theorem 23. *For a continuous L-semimodule $(K,\bar{\oplus},\bar{*})$ the following statements are equivalent:*

(a) K is a distributive lattice, and the multiplication $\bar{*}$ satisfies the equality

$$(\alpha\,\bar{*}\,x)\,\bar{\otimes}(\beta\,\bar{*}\,y)=(\alpha\otimes\beta)\,\bar{*}(x\,\bar{\otimes}\,y),\quad \alpha,\beta\in L,\ x,y\in K;$$

(b) *K is a distributive lattice, and there is a Scott continuous lattice morphism $p\colon L \to K$ such that the multiplication $\bar{*}$ is determined by the equality*

$$\alpha \bar{*} x = p(\alpha) \bar{\otimes} x,\ \alpha \in L,\ x \in K;$$

(c) *for each Scott continuous meet-preserving mapping φ from a continuous semilattice D to K the unique Scott continuous linear extension $\Phi\colon \underline{M}_{[L]}D \to K$ of φ preserves the meets;*

(d) *for each Scott continuous meet-preserving mapping φ from a continuous semilattice D to K the unique Scott continuous affine extension $\Phi\colon M_{[L]}D \to K$ of φ preserves the meets;*

(e) *for each Scott continuous meet-preserving bottom-preserving mapping φ from a continuous semilattice D to K the unique Scott continuous linear extension $\Phi\colon M_{[L]}D \to K$ of φ preserves the meets.*

Proof.

(a)⇒(b) The equality in (a) is equivalent to the following two equalities

$$(\alpha \bar{*} \bar{1}) \bar{\otimes} (\beta \bar{*} \bar{1}) = (\alpha \otimes \beta) \bar{*} \bar{1}, \quad \alpha \bar{*} x = (\alpha \bar{*} \bar{1}) \bar{\otimes} x,\ \alpha, \beta \in L,\ x \in K,$$

where $\bar{1}$ is the top element of K. The first of them together with the definition of an idempotent semimodule implies that the mapping $p\colon L \to K$ that sends each $\alpha \in L$ to $\alpha \bar{*} \bar{1}$ preserves all suprema, finite infima, the top and the bottom elements. Then the second equality means that $\alpha \bar{*} x = p(\alpha) \bar{\otimes} x$ for all $\alpha \in L$, $x \in K$.

(b)⇒(a) is obvious.

(a)⇒(c) Assume that $m_1, m_2 \in \underline{M}_{[L]}D$, recall that $(m_1 \bar{\otimes} m_2)(d) = m_1(d) \otimes m_2(d)$ for all $d \in D$, and compare

$$\Phi(m_1 \bar{\otimes} m_2) = \sup_{d \in D}\{(m_1(d) \otimes m_2(d)) \bar{*} \varphi(d)\}$$

and

$$\Phi(m_1) \bar{\otimes} \Phi(m_2) = \sup_{d' \in D}\{m_1(d') \bar{*} \varphi(d')\} \bar{\otimes} \sup_{d'' \in D}\{m_2(d'') \bar{*} \varphi(d'')\}.$$

Since continuous lattices are meet continuous ([4]), by the distributivity of K the latter equality is equivalent to the following one

$$\sup_{d', d'' \in D}\{(m_1(d') \bar{*} \varphi(d')) \bar{\otimes} (m_2(d'') \bar{*} \varphi(d''))\} = \sup_{d', d'' \in D}\{(m_1(d') \otimes m_2(d'')) \bar{*} \varphi(d' \otimes d''))\} \geqslant$$
$$\geqslant \sup_{\substack{d', d'' \in D,\\ d' = d''}}\{(m_1(d') \otimes m_2(d'')) \bar{*} \varphi(d' \otimes d''))\} = \sup_{d \in D}\{(m_1(d) \otimes m_2(d)) \bar{*} \varphi(d))\} = \Phi(m_1 \bar{\otimes} m_2).$$

On the other hand,

$$\sup_{d', d'' \in D}\{(m_1(d') \bar{*} \varphi(d')) \bar{\otimes} (m_2(d'') \bar{*} \varphi(d''))\} \leqslant$$
$$\leqslant \sup_{d', d'' \in D}\{(m_1(d' \bar{\otimes} d'') \bar{*} \varphi(d')) \bar{\otimes} (m_2(d' \bar{\otimes} d'') \bar{*} \varphi(d''))\} =$$
$$= \sup_{d', d'' \in D}\{(m_1(d' \bar{\otimes} d'') \otimes m_2(d' \bar{\otimes} d'')) \bar{*} \varphi(d' \otimes d'')\} =$$
$$= \sup_{d \in D}\{(m_1(d) \otimes m_2(d)) \bar{*} \varphi(d))\} = \Phi(m_1 \bar{\otimes} m_2).$$

Thus $\Phi(m_1 \mathbin{\bar{\otimes}} m_2) = \Phi(m_1) \mathbin{\bar{\otimes}} \Phi(m_2)$.

(c)⇒(a) To show that the equality $x_1 \mathbin{\bar{\otimes}}(x_2 \mathbin{\bar{\oplus}} x_3) = (x_1 \mathbin{\bar{\otimes}} x_2) \mathbin{\bar{\oplus}}(x_1 \mathbin{\bar{\otimes}} x_3)$ holds for all $x_1, x_2, x_3 \in K$, let D be the ordered reverse to inclusion set of all nonempty subsets of $\{a_1, a_2, a_3\}$.

Define $m_1, m_2 \colon D \to L$ by

$$m_1(d) = \begin{cases} 1, & d \ni a_1, \\ 0, & d \not\ni a_1, \end{cases} \quad m_2(d) = \begin{cases} 1, & d \ni a_2 \text{ or } d \ni a_3, \\ 0, & a_2, a_3 \notin d, \end{cases} \quad d \in D.$$

Observe that

$$(m_1 \mathbin{\bar{\otimes}} m_2)(d) = \begin{cases} 1, & d \supset \{a_1, a_2\} \text{ or } d \supset \{a_1, a_3\}, \\ 0 & \text{otherwise}, \end{cases} \quad d \in D.$$

Let also $\varphi \colon D \to K$ be defined as follows: $\varphi(d) = \inf\{x_i \mid a_i \in d\}$, $d \in D$. The mapping $\varphi \colon D \to K$ is meet-preserving, and

$$\Phi(m_1) = x_1,\ \Phi(m_2) = x_2 \mathbin{\bar{\oplus}} x_3,\ \Phi(m_1 \mathbin{\bar{\otimes}} m_2) = (x_1 \mathbin{\bar{\otimes}} x_2) \mathbin{\bar{\oplus}}(x_1 \mathbin{\bar{\otimes}} x_3),$$

therefore the meet preservation by Φ implies the required distributive law.

Now let $D = \{0, a, b\}$, 0 be the bottom element, a and b be incomparable, $\alpha, \beta \in L$. Define $m_1, m_2, f \colon D \to L$ as follows:

$$m_1(d) = \begin{cases} \alpha, & d \in \{0, a\}, \\ 0, & d = b, \end{cases} \quad m_2(d) = \begin{cases} \beta, & d \in \{0, b\}, \\ 0, & d = a, \end{cases} \quad f(d) = \begin{cases} x, & d = a, \\ y, & d = b, \\ x \mathbin{\bar{\otimes}} y, & d = 0, \end{cases}$$

for all $d \in D$. Then $m_1, m_2 \in \underline{M}_{[L]}D$, and $f \colon D \to K$ is continuous and meet-preserving. Observe also that

$$(m_1 \mathbin{\bar{\otimes}} m_2)(d) = \begin{cases} \alpha \otimes \beta, & d = 0, \\ 0, & d \in \{a, b\}, \end{cases} \quad d \in D.$$

Then

$$\Phi(m_1) = \alpha \mathbin{\bar{*}} x,\ \Phi(m_2) = \beta \mathbin{\bar{*}} y,\ \Phi(m_1 \mathbin{\bar{\otimes}} m_2) = (\alpha \otimes \beta) \mathbin{\bar{*}}(x \mathbin{\bar{\otimes}} y),$$

therefore the equality $\Phi(m_1 \mathbin{\bar{\otimes}} m_2) = \Phi(m_1) \mathbin{\bar{\otimes}} \Phi(m_2)$ yields the equality required by (a).

(c)⇒(d) because the mentioned affine extension of $\varphi \colon D \to K$ to $M_{[L]}D$ is the restriction of the linear extension of φ to $\underline{M}_{[L]}D$, and $M_{[L]}D$ is a complete sublattice of $\underline{M}_{[L]}D$.

(d)⇒(e) because (e) is a particular case of (d).

(e)⇒(c) For a continuous semilattice D, adjoin the bottom element $\perp$ and obtain the semilattice $D_\perp = D \cup \{\perp\}$. Each element of $M_{[L]}D_\perp$ can be obtained from some $m \in \underline{M}_{[L]}D$ as follows:

$$m_\perp(d) = \begin{cases} m(d), & d \in D, \\ 1, & d = \perp, \end{cases} \quad d \in D_\perp.$$

The correspondence $m \to m_\perp$ is a semimodule isomorphism between $\underline{M}_{[L]}D$ and $M_{[L]}D_\perp$. Similarly, the meet-preserving bottom-preserving mappings from $D_\perp$ to K are precisely those of the form

$$\varphi_\perp(d) = \begin{cases} \varphi(d), & d \in D, \\ \bar{0}, & d = \perp, \end{cases} \quad d \in D_\perp,$$

for the meet-preserving mappings $\varphi\colon D \to K$. Finally, for the Scott continuous affine extension Φ of φ and the Scott continuous linear extension $\Phi_\perp$ of $\varphi_\perp$ the equality $\Phi(m) = \Phi_\perp(m_\perp)$ is valid for all $m \in \underline{M}_{[L]}D$. Thus, if all $\Phi_\perp$ are meet-preserving, then all Φ are meet-preserving as well, i.e., (e) implies (c). □

It is easy to see that the mapping $p\colon L \to K$ in the previous statement satisfies also the equality $p(\alpha * \beta) = p(\alpha \otimes \beta)$ for all $\alpha, \beta \in L$. Given a Scott continuous lattice morphism p from a completely distributive lattice $(L, \oplus, \otimes)$ to a distributive continuous lattice K, the simplest way to make K an $(L, \oplus, *)$-semimodule that satisfies (b) above is to put $* = \otimes$.

Such a distributive continuous $(L, \oplus, \otimes)$-semimodule will be called a *continuous L-biconvex set*. We denote by $L\text{-}\mathcal{CSA}\mathit{ff}_\uparrow$ the category of all continuous L-biconvex sets and their Scott continuous affine mappings that preserve finite infima, and $L\text{-}\mathcal{CSM}\mathit{od}_\uparrow$ is its subcategory with the same objects and all bottom-preserving (i.e., linear) morphisms.

Theorem 24. *If the multiplications in continuous L-biconvex sets K and K' are determined by lattice morphisms $p\colon L \to K$ and $p'\colon L \to K'$, then $f\colon K \to K'$ is a morphism in $L\text{-}\mathcal{CSA}\mathit{ff}_\uparrow$ is and only if f is a morphism in $\mathcal{CS}\mathrm{em}$, preserves the joins and $f(p(\alpha)) = (p'(\alpha) \oplus f(\bar{0})) \bar{\otimes} f(\bar{1})$ for all $\alpha \in L$.*

Proof is straightforward.

Observe that $(\underline{M}_{[L]}D, \bar{\oplus}, \bar{\circledast})$ and $(M_{[L]}D, \bar{\oplus}, \bar{\odot})$ are continuous L-biconvex sets. The required mapping $p\colon L \to \underline{M}_{[L]}D$ takes each $\alpha \in L$ to the predicate that sends each $d \in D$ to α. For $M_{[L]}D$ the predicate $p(\alpha)$ sends 0 to 1 and all other elements of D to α.

Similarly we obtain:

Theorem 25. *For an object D of the category $\mathcal{CS}\mathrm{em}$ the continuous L-biconvex set $\underline{M}_{[L]}D$ is a free object over D in $L\text{-}\mathcal{CSM}\mathit{od}_\uparrow$.*

Theorem 26. *For an object D of the category $\mathcal{CS}\mathrm{em}_\perp$ the continuous L-biconvex set $M_{[L]}D$ is a free object over D in $L\text{-}\mathcal{CSA}\mathit{ff}_\uparrow$.*

Theorem 27. *For an object D of the category $\mathcal{CS}\mathrm{em}_0$ the continuous L-biconvex set $M_{[L]}D$ is a free object over D in $L\text{-}\mathcal{CSM}\mathit{od}_\uparrow$.*

Thus we obtain the functors, which, abusing the notation, we define $\underline{M}_{[L]}\colon \mathcal{CS}em \to L\text{-}\mathcal{CSM}od_\uparrow$, $M_{[L]}\colon \mathcal{CS}em_\perp \to L\text{-}\mathcal{CSA}\mathit{ff}_\uparrow$, $M_{[L]}\colon \mathcal{CS}em_0 \to L\text{-}\mathcal{CSM}od_\uparrow$. They are left adjoint to the forgetful functors acting in the opposite directions. In particular, for each Scott continuous meet-preserving mapping of continuous semilattices $f\colon D \to D'$ its extensions $\underline{M}_L f\colon \underline{M}_{[L]}D \to \underline{M}_{[L]}D'$ and $M_L f\colon M_{[L]}D \to M_{[L]}D'$ (if the latter one exists) preserve the meets. This can also be verified by direct calculations.

In fact, we can narrow the ranges of the obtained left adjoint functors. Let a continuous L-biconvex set $(K, \bar{\oplus}, \bar{*})$ be also dually continuous, then by Theorem VII-2.10 [4] it is completely distributive. If the mentioned correspondence $p\colon \alpha \mapsto \alpha \mathbin{\bar{*}} \bar{1}$ is also dually Scott continuous, i.e., is Lawson continuous, then $(K^{op}, \bar{\otimes}, \underline{*})$ is also a continuous L^{op}-biconvex set, where $\alpha \mathbin{\underline{*}} x = (\alpha \mathbin{\bar{*}} \bar{1}) \mathbin{\bar{\oplus}} x$ is a Lawson continuous multiplication. Obviously, we can restore the original operation by the formula $\alpha \mathbin{\bar{*}} x = (\alpha \mathbin{\underline{*}} \bar{0}) \mathbin{\bar{\otimes}} x$. Such a "two-side" L-semimodule will be called an *L-biconvex compactum*, and $\underline{M}_{[L]}D$ and $M_{[L]}D$ belong to this class. A mapping $f\colon K \to K'$ between L-biconvex compacta is affine and meet-preserving if and only if the mapping $f\colon K^{op} \to K'^{op}$ is affine and meet-preserving w.r.t. the corresponding operations on the "turned upside down" semimodules. Therefore such mappings are called "biaffine", and the category of L-biconvex compacta and their Lawson continuous biaffine (hence isotone) mappings will be denoted L-$\mathcal{B}i\mathcal{C}onv$.

10. Free dually continuous semimodules over dually continuous lattices. It has been proved in the previous section that, for a continuous semilattice D, the L-semimodules $\underline{M}_{[L]}D$ and $M_{[L]}D$ "inherit" the meets in D, and the meets are preserved by the mappings of the form $\underline{M}_{[L]}f$ and $M_{[L]}f$ for all Scott continuous semilattice morphisms f. On the other hand, meet operation is not of much importance in idempotent semimodules. Join, i.e., addition, is more important, and each join semilattice can be regarded as $\mathbf{2}$-semimodule, for $\mathbf{2} = \{0, 1\}$. In particular, a dually continuous semilattice with a bottom element is a dually continuous $\mathbf{2}$-semimodule. Thus a problem naturally arises: given such a $\mathbf{2}$-semimodule, how to "enrich" it to make it a dually continuous L-semimodule for a bigger completely distributive quantale L?

To obtain a solution for the *complete* dually continuous semimodules, i.e., the compact Hausdorff Lawson L-semimodules, in this section we assume that $*\colon L\times L \to L$ is both Scott continuous and dually Scott continuous, hence Lawson continuous.

Let S be a *dually continuous* lattice, then the meet semilattice S^{op} is a continuous lattice. It is easy to see that, for an element $s \in S$, the mapping $\eta_{\vee[L]}S(s)\colon S^{op} \to L$ that is defined as follows:

$$\eta_{\vee[L]}S(s)(d) = \begin{cases} 0, & s \leqslant d \text{ in } S, \\ 1, & s \nleqslant d \text{ in } S, \end{cases} \qquad d \in S$$

is Scott continuous, and the correspondence $\eta_{\vee[L]}S\colon S \to [S^{op} \to L]$ is join preserving and continuous w.r.t. the upper, the dual Scott, and the dual Lawson topologies. Taking into account that $\eta_{\vee[L]}S(s) \in [S^{op} \to L]_{\vee 0}$, we obtain the embedding $\eta_{\vee[L]}S\colon S \to [S_{op} \to L]_{\vee 0}$ w.r.t. the mentioned topologies.

Thus in the sequel we consider S as a subspace of $[S^{op} \to L]_{\vee 0}$ w.r.t. the dual Scott, the upper and the dual Lawson topologies. Observe also that, for $L = \mathbf{2}$, the embedding $\eta_{\vee[\mathbf{2}]}S$ is an order isomorphism and a homeomorphism.

Recall that $([S^{op} \to L]_{\vee 0}, \bar{\oplus}, \bar{*})$ is a complete dually continuous L-semimodule, and by Lemma 5 each $f \in [S^{op} \to L]_{\vee 0}$ is equal to

$$f(d) = \sup_{s\in S}\{\inf_{d'\ngeqslant s} f(d')\} * \eta_{\vee[L]}S(s)(d)$$

for all $d \in S$.

Lemma 7. *Let $f = \sup_{i\in\mathcal{I}} \alpha_i \mathbin{\bar{*}} \eta_{\vee[L]}S(s_i)$ for a subset $\{(\alpha_i, s_i) \mid i \in \mathcal{I}\} \subset L \times S$. Then, for each $s \in S$:*

$$\inf_{d\ngeqslant s} f(d) = \sup\bigl\{\inf_{i\in\mathcal{J}} \alpha_i \bigm| \mathcal{J} \subset \mathcal{I},\ \sup_{i\in\mathcal{J}} s_i \geqslant s\bigr\}.$$

Proof. It is clear that, for each $\mathcal{J} \subset \mathcal{I}$ such that $\sup_{i\in\mathcal{J}} s_i \geqslant s$, the inequality

$$\sup_{i\in\mathcal{J}} \alpha_i * \eta_{\vee[L]}S(s_i)(d) \geqslant \inf_{i\in\mathcal{J}} \alpha_i$$

holds for each $d \in S$, $d \ngeqslant s$. Thus the inequality

$$\inf_{d\ngeqslant s} f(d) \geqslant \sup\bigl\{\inf_{i\in\mathcal{J}} \alpha_i \bigm| \mathcal{J} \subset \mathcal{I},\ \sup_{i\in\mathcal{J}} s_i \geqslant s\bigr\}$$

is immediate.

Assume that there is no equality. Since L is completely distributive, each $\alpha \in L$ is the least upper bound of all $\beta \in L$ such that $\beta \lll \alpha$ (β is *way-way below* α, cf. [4]), which means that each subset $\Gamma \subset L$ such that $\sup \Gamma \geqslant \alpha$ contains an element $\gamma \geqslant \beta$. Then there is $\beta \in L$, $\beta \lll \inf_{d \ngeqslant s} f(d)$ such that β precedes no $\inf_{i\in\mathcal{J}} \alpha_i$ for $\mathcal{J} \subset \mathcal{I}$ such that $\sup_{i\in\mathcal{J}} s_i \geqslant s$.

This means that, for all $d \ngeqslant s$, there is $i \in \mathcal{I}$ such that $\alpha_i \geqslant \beta$, $d \ngeqslant s_i$. Let $\mathcal{J} = \{i \in \mathcal{I} \mid \alpha_i \geqslant \beta\}$, then

$$\bigcup_{i\in\mathcal{J}} \{d \in S \mid d \ngeqslant s_i\} \supset \{d \in S \mid d \ngeqslant s\},$$

i.e., $\bigcap\limits_{i\in\mathcal{J}} \{s_i\}\uparrow \subset \{s\}\uparrow$, hence $\sup_{i\in\mathcal{J}} s_i \geqslant s$, which contradicts the inequality $\inf_{i\in\mathcal{J}} \alpha_i \geqslant \beta$.

Thus the required equality is valid. □

Theorem 28. *Let S be a dually continuous lattice, the multiplication $*\colon L\times L \to L$ Lawson continuous, $(K, \bar{\oplus}, \bar{*})$ a dually continuous L-semimodule. Then each upper continuous, dual Scott continuous, or dual Lawson continuous join-preserving bottom-preserving mapping φ from S to K has a unique linear respectively upper continuous, dual Scott continuous, or dual Lawson continuous extension $\Phi\colon ([S^{op} \to L]_{\vee 0}, \bar{\oplus}, \bar{*}) \to (K, \bar{\oplus}, \bar{*})$.*

Proof. If $f \in [S^{op} \to L]_{\vee 0}$ has a finite range, then it is of the form

$$f = \alpha_1 \mathbin{\bar{*}} \eta_{\vee[L]}S(s_1) \mathbin{\bar{\oplus}} \ldots \mathbin{\bar{\oplus}} \alpha_n \mathbin{\bar{*}} \eta_{\vee[L]}S(s_n),$$

with $\alpha_1, \ldots, \alpha_n \in L$, $s_1, \ldots, s_n \in S$, and

$$\alpha_i = \inf\{f(d) \mid d \in S, d \ngeqslant s_i\}, \quad i = 1, \ldots, n.$$

Note that $\eta_{\vee[L]}S(\min S)$ is the constant function equal to 0, and for $s_i = 0$ we obtain $\alpha_i = 1$.

If a mapping Φ from $[S^{op} \to L]_{\vee 0}$ to a complete dually continuous L-semimodule K is linear, and $\Phi \circ \eta_{\vee[L]}S = \varphi\colon S \to K$, then φ is join-preserving, and

$$\Phi(f) = \sup_{s\in\{s_1,\ldots,s_n\}} \inf\{f(d) \mid d \in S, d \ngeqslant s\} \mathbin{\bar{*}} \varphi(s) \leqslant \sup_{s\in S} \inf\{f(d) \mid d \in S, d \ngeqslant s\} \mathbin{\bar{*}} \varphi(s).$$

On the other hand, $f \geqslant \sup\{f(d) \mid d \in S, d \ngeqslant s\} \mathbin{\bar{*}} \eta_{\vee[L]}S(s)$ for each $s \in S$, hence the reverse inequality is valid. Thus

$$\Phi(f) = \sup_{s\in S} \inf\{f(d) \mid d \in S, d \ngeqslant s\} \mathbin{\bar{*}} \varphi(s)$$

for all $f \in [S^{op} \to L]_{\vee 0}$ with finite ranges.

For each $\varphi\colon S \to K$, the latter formula defines a function $\Phi\colon [S^{op} \to L]_{\vee 0} \to K$. Let either φ preserve finite suprema and $\{(\alpha_i, s_i) \mid i \in \mathcal{I}\} \subset L \times S$ be finite, or φ preserve all suprema and $\{(\alpha_i, s_i) \mid i \in \mathcal{I}\} \subset L \times S$ be arbitrary. The previous lemma implies that the image under Φ of a function $f = \sup_{i\in\mathcal{I}} \alpha_i \mathbin{\bar{*}} \eta_{\vee[L]}S(s_i)$, for a subset $\{(\alpha_i, s_i) \mid i \in \mathcal{I}\} \subset L \times S$, is equal to

$$\Phi(f) = \sup_{s\in S}\Bigl(\sup\bigl\{\inf_{i\in\mathcal{J}} \alpha_i \bigm| \mathcal{J} \subset \mathcal{I},\ \sup_{i\in\mathcal{J}} s_i \geqslant s\bigr\} \mathbin{\bar{*}} \varphi(s)\Bigr) \leqslant$$
$$\leqslant \sup_{s\in S}\Bigl(\sup\bigl\{\inf_{i\in\mathcal{J}} \alpha_i \mathbin{\bar{*}} \varphi(\sup_{i\in\mathcal{J}} s_i) \bigm| \mathcal{J} \subset \mathcal{I},\ \sup_{i\in\mathcal{J}} s_i \geqslant s\bigr\}\Bigr) =$$
$$= \sup\bigl\{\inf_{i\in\mathcal{J}} \alpha_i \mathbin{\bar{*}} \sup_{i\in\mathcal{J}} \varphi(s_i) \bigm| \mathcal{J} \subset \mathcal{I}\bigr\} = \sup\bigl\{\sup_{i\in\mathcal{J}}\bigl(\inf_{j\in\mathcal{J}} \alpha_j \mathbin{\bar{*}} \varphi(s_i)\bigr) \bigm| \mathcal{J} \subset \mathcal{I}\bigr\} \leqslant \sup_{i\in\mathcal{I}} \alpha_i \mathbin{\bar{*}} \varphi(s_i).$$

The reverse inequality

$$\Phi(f) \geqslant \sup_{i\in\mathcal{I}} \Phi(\alpha_i \mathbin{\bar{*}} \eta_{\vee[L]}S(s_i)) = \sup_{i\in\mathcal{I}} \alpha_i \mathbin{\bar{*}} \varphi(s_i)$$

holds due to the monotonicity of Φ, which is obvious.

This and $\varphi(\min S) = \bar{0}$ implies that

$$\Phi(\alpha_1 \mathbin{\bar{*}} f_1 \mathbin{\bar{\oplus}} \ldots \mathbin{\bar{\oplus}} \alpha_n \mathbin{\bar{*}} f_n) = \alpha_1 \mathbin{\bar{*}} \Phi(f_1) \mathbin{\bar{\oplus}} \ldots \mathbin{\bar{\oplus}} \alpha_n \mathbin{\bar{*}} \Phi(f_n)$$

for all $\alpha_1, \ldots, \alpha_n \in L$, $f_1, \ldots, f_n \in [S^{op} \to L]_{\vee 0}$. If φ preserves all suprema, then the analogous equality is valid for an infinite number of functions. The equality $\Phi \circ \eta_{\vee[L]}S = \varphi$ is immediate.

Thus a linear extension Φ of φ is obtained. It is routine but straightforward to verify that, if φ is upper, dual Scott, or dual Lawson continuous, then the same holds for the extension Φ, and Lemma 5 and 12 imply that such an extension is unique. □

An equivalent formulation:

Theorem 29. *The dually continuous L-semimodule $([S^{op} \to L]_{\vee 0}, \bar{\oplus}, \bar{*})$ is a free object in the categories $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SM}od$, $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SM}od_{\uparrow}$, and $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SM}od_{\downarrow}$ over the object S of $\mathcal{DL}_0$, $\mathcal{DL}_{\uparrow 0}$, and $\mathcal{DL}_{\downarrow 0}$, respectively.*

If we are interested in *affine* extensions, a slight modification is necessary.

Theorem 30. *Let S be a dually continuous lattice, the multiplication $*\colon L \times L \to L$ Lawson continuous, $(K, \bar{\oplus}, \bar{*})$ a dually continuous L-semimodule. Then each upper continuous, dual Scott continuous, or dual Lawson continuous join-preserving mapping φ from S to K has a unique affine respectively upper continuous, dual Scott continuous, or dual Lawson continuous extension $\Phi\colon ([S^{op} \to L]_{\vee 0}, \bar{\oplus}, \bar{*}) \to (K, \bar{\oplus}, \bar{*})$.*

Or, equivalently:

Theorem 31. *The dually continuous L-semimodule $([S^{op} \to L]_{\vee 0}, \bar{\oplus}, \bar{*})$ is a free object in the categories $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SA}f\!f$, $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SA}f\!f_{\uparrow}$, and $(L, \oplus, *)$-$\mathcal{L}w\mathcal{SA}f\!f_{\downarrow}$ over the object S of $\mathcal{DL}$, $\mathcal{DL}_{\uparrow}$, and $\mathcal{DL}_{\downarrow}$, respectively.*

Proof is quite analogous, and the formula for the extensions is similar

$$\Phi(f) = \sup_{s\in S} \inf\{f(d) \mid d \in S, d \ngeqslant s\} \mathbin{\bar{*}} \varphi(s) \oplus \varphi(\min S)$$

for all $f \in [S^{op} \to L]_{\vee 0}$.

11. Concluding remarks. Free dually continuous semimodules over dually continuous *semilattices*, as well as free compact L-convex sets, will be considered in our future paper.

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Department of Mathematics and Computer Science
Vasyl' Stefanyk Precarpathian National University
oleh.nyk@gmail.com

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