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AN INVOLUTION MAP FOR THE FUNCTIONAL MONAD

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We introduce an involution map for the functional monad which generalizes the transversality map for the inclusion hyperspace monad and the conjugation map for the capacity monad.

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Определено отображение инволюции для монады функционалов, которое обобщает отображение трансверсали для монады гиперпространств включения и отображение конъюгации для монады емкостей.

1. We call a map $i\colon X \to X$ an involution iff $i\circ i=\mathrm{id}_X$. Involution maps play an important role in investigations of some monads in the category $\mathcal{C}omp$. The transversality map $\perp X\colon GX\to GX$ for the space of hyperspaces of inclusion GX is an involution (see p. 30 of [1]). Moreover, the map $\perp X$ is an antiisomorphism of the lattice GX with lattice operations intersection and union. The maps $\perp X$ are components of a natural transformation $\perp\colon \mathbb{G}\to\mathbb{G}$ which is an isomorphism of the monad $\mathbb{G}$ based on the functor G.

The conjugation map $\kappa X\colon MX\to MX$ for the space of capacities has similar properties (see [2]).

The monads $\mathbb{G}$ and $\mathbb{M}$ (based on the capacity functor M) could be represented as submonads of the functional monad $\mathbb{V}$ (see [3] and [4]). The aim of this paper is to define an involution map for the functional monad, respective restrictions of which coincide with the transversality map and the conjugation map.

The paper is arranged in the following manner. In Section 2 we give necessary definitions and facts, in Section 3 we define an involution map and investigate its properties and in Section 4 we consider how this map acts on some submonads of the monad $\mathbb{V}$.

2. By $\mathcal{C}omp$ we denote the category of compact Hausdorff spaces (compacta) and continuous maps. For each compactum X we denote by $C(X)$ the Banach space of all continuous functions $\varphi\colon X\to\mathbb{R}$ with the usual sup-norm: $\|\varphi\|=\sup\{|\varphi(x)|\mid x\in X\}$. We also consider on $C(X)$ the natural partial order.

In what follows, all spaces and maps are assumed to be in $\mathcal{C}omp$ except for $\mathbb{R}$, the spaces $C(X)$ and functionals defined on $C(X)$ with X compact Hausdorff.

We recall some categorical definitions. We define them only for the category $\mathcal{C}omp$. The central notion is the notion of monad (or triple) in the sense of S. Eilenberg and J. Moore.

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A *monad* ([5]) $\mathbb{T}=(T,\eta,\mu)$ in the category $\mathcal{C}omp$ consists of an endofunctor $T\colon \mathcal{C}omp\to\mathcal{C}omp$ and natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}omp}\to T$ (unity), $\mu\colon T^2\to T$ (multiplication) satisfying the relations $\mu\circ T\eta=\mu\circ\eta T=\mathbf{1}_T$ and $\mu\circ\mu T=\mu\circ T\mu$. (By $\mathrm{Id}_{\mathcal{C}omp}$ we denote the identity functor on the category $\mathcal{C}omp$ and T^2 is the superposition $T\circ T$.)

A natural transformation $\psi\colon T\to T'$ is called a *morphism* from a monad $\mathbb{T}=(T,\eta,\mu)$ to a monad $\mathbb{T}'=(T',\eta',\mu')$ if $\psi\circ\eta=\eta'$ and $\psi\circ\mu=\mu'\circ\eta T'\circ T\psi$. If all of the components of ψ are monomorphisms then the monad $\mathbb{T}$ is called a *submonad* of $\mathbb{T}'$ and ψ is called a *monad embedding*. If ψ has the inverse monad morphism, then ψ is called a *monad isomorphism*.

By VX we denote the product $\prod_{\varphi\in CX}[\min\varphi,\max\varphi]$, which is in fact the space of all functionals $\nu\colon C(X)\to\mathbb{R}$ (not necessarily linear) with the property $\nu(\varphi)\in[\min\varphi,\max\varphi]$ for each $\varphi\in C(X)$. For a map $\varphi\in C(X)$ we denote by π_φ or $\pi(\varphi)$ the corresponding projection $\pi_\varphi\colon VX\to\mathbb{R}$. Now, for each map $f\colon X\to Y$ we define a map $Vf\colon VX\to VY$ by the formula $\pi_\varphi\circ Vf=\pi_{\varphi\circ f}$ for $\varphi\in C(Y)$. For a compactum X we define components $hX\colon X\to VX$ and $mX\colon V^2X\to VX$ by $\pi_\varphi\circ hX=\varphi$ and $\pi_\varphi\circ mX=\pi(\pi_\varphi)$ for all $\varphi\in C(X)$). The triple $\mathbb{V}=(V,h,m)$ is a monad in the category $\mathcal{C}omp$ ([6]).

A functional $\nu\colon CX\to\mathbb{R}$ is called *weakly additive* if for each $c\in\mathbb{R}$ and $\varphi\in CX$ we have $\nu(\varphi+c_X)=\nu(\varphi)+c$; *normed* if $\nu(1_X)=1$; *order-preserving* if for each φ, $\psi\in CX$ with $\varphi\le\psi$ we have $\nu(\varphi)\le\nu(\psi)$. For a compactum X by OX we will denote the set of all order-preserving weakly additive normed functionals. The construction O defines an important submonad $\mathbb{O}$ of $\mathbb{V}$ (see [7] for more details).

3. Let us define a map $\mathcal{I}X\colon VX\to VX$ by the formula $\pi_\varphi\circ\mathcal{I}X=-\pi_{-\varphi}$. It is easy to see that $\mathcal{I}X$ is a well defined and a continuous involution.

Proposition 1. *The maps $\mathcal{I}X$ are components of a monad isomorphism $\mathcal{I}\colon\mathbb{V}\to\mathbb{V}$.*

Proof. Let $f\colon X\to Y$ be a map and $\varphi\in C(X)$. Then we have

$$\pi_\varphi\circ\mathcal{I}X\circ Vf=-\pi_{-\varphi}\circ Vf=-\pi_{-\varphi\circ f}=\pi_{\varphi\circ f}\circ\mathcal{I}X=\pi_\varphi\circ Vf\circ\mathcal{I}X.$$

Hence the maps $\mathcal{I}X$ are components of a natural transformation $\mathcal{I}\colon V\to V$.

Let us show that $\mathcal{I}$ is a monad morphism. Consider any compactum X and $\varphi\in C(X)$. Then we have

$$\begin{gathered}\pi_\varphi\circ\mathcal{I}X\circ hX=-\pi_{-\varphi}\circ hX=-(-\varphi)=\varphi=\pi_\varphi\circ hX,\\ \pi_\varphi\circ\mathcal{I}X\circ mX=-\pi_{-\varphi}\circ mX=-\pi(\pi_{-\varphi})=-\pi(-\pi_\varphi\circ\mathcal{I}X)=-\pi(-\pi_\varphi)\circ V\mathcal{I}X=\\ =\pi(\pi_\varphi)\circ\mathcal{I}VX\circ V\mathcal{I}X=\pi_\varphi\circ mX\circ\mathcal{I}VX\circ V\mathcal{I}X.\end{gathered}$$

We obtain $\mathcal{I}\circ h=h$ and $\mathcal{I}\circ m=m\circ\mathcal{I}V\circ V\mathcal{I}$, hence $\mathcal{I}$ is a monad morphism. Since $\mathcal{I}X$ is an involution, $\mathcal{I}$ is a monad isomorphism. □

There exist natural lattice operations $\vee$ and $\wedge$ on VX defined by the formulas

$$\pi_\varphi(\nu\vee\mu)=\max\{\pi_\varphi(\nu),\pi_\varphi(\mu)\},\ \ \pi_\varphi(\nu\wedge\mu)=\min\{\pi_\varphi(\nu),\pi_\varphi(\mu)\}$$

for ν, $\mu\in VX$ and $\varphi\in C(X)$. The following proposition shows that $\mathcal{I}X$ is an antiisomorphism of the lattice VX.

Proposition 2. $\mathcal{I}X(\nu\wedge\mu)=\mathcal{I}X(\nu)\vee\mathcal{I}X(\mu)$ *and* $\mathcal{I}X(\nu\vee\mu)=\mathcal{I}X(\nu)\wedge\mathcal{I}X(\mu)$.

Proof. Consider any $\varphi \in C(X)$. We have

$$\pi_\varphi(\mathcal{I}X(\nu \wedge \mu)) = -\pi_{-\varphi}(\nu \wedge \mu) = -\min\{\pi_{-\varphi}(\nu), \pi_{-\varphi}(\mu)\} =$$
$$= \max\{-\pi_{-\varphi}(\nu), -\pi_{-\varphi}(\mu)\} = \max\{\pi_\varphi \circ \mathcal{I}X(\nu), \pi_\varphi \circ \mathcal{I}X(\mu)\} = \pi_\varphi(\mathcal{I}X(\nu) \vee \mathcal{I}X(\mu)).$$

The proof of the second equality is similar. □

Let us denote

$$VPX = \{\nu \in VX \mid \pi_{-\varphi}(\nu) = -\pi_\varphi(\nu)\}.$$

The next proposition follows immediately from the definition of $\mathcal{I}X$.

Proposition 3. *We have $\mathcal{I}X(\nu) = \nu$ iff $\nu \in VPX$.*

4. Let us consider how a involution acts on some submonads of $\mathbb{V}$. We need definitions of the monads $\mathbb{G}$ and $\mathbb{M}$.

For a compactum X by $\exp X$ we denote the hyperspace of X (the set of non-void compact subsets of X provided with the Vietoris topology). An element $\mathcal{A} \in \exp^2 X$ is called an *inclusion hyperspace* if for each $A \in \mathcal{A}$ and $B \in \exp X$ with $A \subset B$ we have $B \in \mathcal{A}$. Let us denote by $GX = \{\mathcal{A} \in \exp^2 X \mid \mathcal{A} \text{ is inclusion hyperspace}\} \subset \exp^2 X$. For a map $f\colon X \to Y$ we define a map $Gf\colon GX \to GY$ by the formula $Gf(\mathcal{A}) = \{A \in \exp Y \mid f(B) \subset A \text{ for some } B \in \mathcal{A}\}$, $\mathcal{A} \in GX$. Then we define natural transformations $\eta\colon I_{\mathcal{C}omp} \to G$ and $\mu\colon G^2 \to G$ as follows: $\eta X(x) = \{A \in \exp X \mid x \in A\}$, $x \in X$ and $\mu X(\tilde{\mathcal{A}}) = \bigcup\{\bigcap \alpha \mid \alpha \in \tilde{\mathcal{A}}\}$, where $\tilde{\mathcal{A}} \in G^2 X$. It is shown in [8] that the triple $\mathbb{G} = (G, \eta, \mu)$ is a monad on the category $\mathcal{C}omp$.

The transversality map $\perp X\colon GX \to GX$ is defined by the formula $\perp X(\mathcal{A}) = \{B \in \exp X \mid B \cap A \neq \varnothing \text{ for each } A \in \mathcal{A}\}$. The map $\perp X$ is an antiisomorphism of the lattice GX and the natural transformation $\perp\colon \mathbb{G} \to \mathbb{G}$ is a monad isomorphism ([1]).

A monad embedding $l\colon \mathbb{G} \to \mathbb{V}$ could be defined as follows $\pi_\varphi \circ lX(\mathcal{A}) = \sup\{\inf \varphi(A) \mid A \in \mathcal{A}\}$, $\mathcal{A} \in GX$ and $\varphi \in CX$ ([3]).

Finally, consider the definition of a capacity monad. We follow the terminology of [2]. Let X be a compactum. A function c that sends each closed subset A of X to a real number $c(A) \in [0, 1]$ is called an *upper-semicontinuous capacity* on X if the following three properties hold for each closed subsets F and G of X:

1. $c(X) = 1$, $c(\varnothing) = 0$,
2. if $F \subset G$, then $c(F) \le c(G)$,
3. if $c(F) < a$, then there exists an open set $O \subset F$ such that $c(B) < a$ for each compactum $B \subset O$.

We extend a capacity c to all open subsets $U \subset X$ by the formula $c(U) = \sup\{c(K) \mid K$ is a closed subset of X such that $K \subset U\}$.

The space MX of all upper-semicontinuous capacities on a compactum X is a compactum as well, if a topology on MX is defined by a subbase that consists of all sets of the form $O_-(F, a) = \{c \in MX \mid c(F) < a\}$, where F is a closed subset of X, $a \in [0, 1]$, and $O_+(U, a) = \{c \in MX \mid c(U) > a\}$, where U is an open subset of X, $a \in [0, 1]$. Since all capacities we consider here are upper-semicontinuous, from now on we call elements of MX simply capacities.

Let us define the map $Mf\colon MX \to MY$ for a continuous map of compacta $f\colon X \to Y$ by the formula $Mf(c)(F) = c(f^{-1}(F))$ where $c \in MX$ and F is a closed subset of X. We obtain the functor M which is a functorial part of the monad $\mathbb{M} = (M, \eta, \mu)$ ([2]), where the components of the natural transformations are defined as follows: $\eta X(x)(F) = 1$ if $x \in F$ and $\eta X(x)(F) = 0$ if $x \notin F$;

$$\mu X(\mathcal{C})(F) = \sup\{t \in [0,1] \mid \mathcal{C}(\{c \in MX \mid c(F) \geq t\}) \geq t\},$$

where $x \in X$, F is a closed subset of X and $\mathcal{C} \in M^2(X)$.

The conjugation map $\kappa X \colon MX \to MX$ is defined by the formula $\kappa X(c)(A) = 1 - c(X \setminus A)$ for $c \in MX$ and $A \in \exp X$. The map κX is an antiisomorphism of the lattice MX and the natural transformation $\kappa \colon \mathbb{M} \to \mathbb{M}$ is a monad isomorphism [2].

Let us describe a monad embedding $\theta \colon \mathbb{M} \to \mathbb{V}$ from [4]. Let $h \colon (0;1) \to \mathbb{R}$ be an increasing homeomorphism such that $h(1-t) = -h(t)$. Then θX could be defined as follows

$$\pi_\varphi \circ \theta X(c) = \max\{t \in \mathbb{R} \mid c(\varphi^{-1}([t;+\infty))) \geq h^{-1}(t)\}$$

for a compactum X, $c \in MX$ and $\varphi \in C(X)$.

We will show that the restriction of the map $\mathcal{I}X$ to the space MX (more precisely to the image of MX under the embedding θX) coincides with the conjugation map κX. We obtain the same coincidence for GX and the transversality map.

Theorem 1. *We have $\mathcal{I}X \circ \theta X = \theta X \circ \kappa X$ for each compactum X.*

Proof. Consider any $c \in MX$ and $\varphi \in C(X)$. We have

$$\begin{gathered}
\pi_\varphi \circ \mathcal{I}X \circ \theta X(c) = -\pi_{-\varphi} \circ \theta X(c) = -\max\{t \in \mathbb{R} \mid c((-\varphi)^{-1}([t;+\infty))) \geq h^{-1}(t)\} = \\
= \min\{s \in \mathbb{R} \mid c(\varphi^{-1}((-\infty;s])) \geq h^{-1}(-s)\} = \min\{s \in \mathbb{R} \mid c(\varphi^{-1}((-\infty;s])) \geq 1 - h^{-1}(s)\} = \\
= \max\{t \in \mathbb{R} \mid c(\varphi^{-1}((-\infty;t))) \leq 1 - h^{-1}(t)\} = \\
= \max\{t \in \mathbb{R} \mid 1 - c(\varphi^{-1}((-\infty;t))) \geq h^{-1}(t)\} = \\
= \max\{t \in \mathbb{R} \mid \kappa X(c)(\varphi^{-1}([t;+\infty))) \geq h^{-1}(t)\} = \pi_\varphi \circ \theta X \circ \kappa X(c).
\end{gathered}$$

□

Theorem 2. *We have $\mathcal{I}X \circ lX = lX \circ \perp X$ for each compactum X.*

Proof. Consider any $\mathcal{A} \in GX$ and $\varphi \in C(X)$. We have

$$\begin{gathered}
\pi_\varphi \circ \mathcal{I}X \circ lX(\mathcal{A}) = -\pi_{-\varphi} \circ lX(\mathcal{A}) = -\sup\{\inf(-\varphi(A)) \mid A \in \mathcal{A}\} = \\
= \inf\{\sup(\varphi(A)) \mid A \in \mathcal{A}\} = \\
= \sup\{\inf(\varphi(B)) \mid B \cap A \neq \varnothing \text{ for each } A \in \mathcal{A}\} = \pi_\varphi \circ lX \circ \perp X(\mathcal{A}).
\end{gathered}$$

□

The following proposition shows that the restriction of $\mathcal{I}X$ on OX is an involution map for OX. Moreover, since OX is a sublattice of VX, the map $\mathcal{I}X \mid OX$ is an antiisomorphism of the lattice OX. The proof of the proposition could be done by usual checking and we omit it.

Proposition 4. *We have $\mathcal{I}X(OX) \subset OX$ for each compactum X.*

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