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UNDER A SUITABLE RENORMING EVERY NONREFLEXIVE BANACH SPACE HAS A FINITE SUBSET WITHOUT A STEINER POINT

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We present a refinement of the recent Borodin's example of a finite set without a Steiner point. Namely, we show that under a suitable renorming such an example exists in every nonreflexive Banach space.

В. Кадец. *Каждое нерефлексивное банахово пространство в подходящей перенормировке содержит конечное множество без точек Штейнера* // Мат. Студії. – 2011. – Т.36, №2. – С.197–200.

Недавно П.А. Бородин построил пример конечного множества в банаховом пространстве, не имеющего точек Штейнера. Мы уточняем этот результат, показывая, что в подходящей эквивалентной перенормировке такие примеры есть в любом нерефлексивном банаховом пространстве.

For any finite collection $A = \{x_1, \dots, x_n\}$ of (not necessarily distinct) elements of a Banach space X a *Steiner point* of A is every point $s \in X$ at which the function $x \mapsto \sum_{k=1}^{n} \|x - x_k\|$ attains its minimum. Let us say that a Banach space X has the *Steiner Point Property* ($X \in$ StPP) if every finite collection $A \subset X$ possesses a Steiner point. By weak compactness argument every reflexive space has the StPP (see [1] for the corresponding references and for a short proof). The class of spaces with the Steiner Point Property contains also some non-reflexive spaces, like dual spaces, $L_1[0,1]$, or more generally every Banach space that is 1-complemented in its bidual (see Theorem 1 below). The problem whether $C[0,1]$ in its original norm has the StPP remains open.

Recently, P. A. Borodin [1] presented the first example of a Banach space X that does not enjoy the StPP. This example is obtained by introducing an equivalent norm on $C[0,1]$ that "mixes" in a clever way the original norm of $C[0,1]$ with the L_1-norm. In this short note we use the idea of Borodin's construction in order to show that in every nonreflexive Banach space X there is an equivalent norm $\|\cdot\|_b$ such that $(X, \|\cdot\|_b) \notin$ StPP.

In the sequel, if X is a Banach space then B_X stands for its closed unit ball, X^* and X^{**} stand for the dual and bidual spaces respectively. The norm closure of a subset $D \subset X$ we denote $\mathrm{cl}(D)$. We use the word "operator" for bounded linear operators. A Banach space X is said to be *1-complemented in its bidual* if there is a linear projection $P\colon X^{**} \to X$ with $\|P\| = 1$. For standard facts about Banach spaces and properties of weak and weak* topologies we refer to [2], for more advanced Banach space theory results we refer to [3].

We start with a simple positive result.

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Theorem 1. *Let X be a Banach space that is 1-complemented in its bidual. Then $X \in StPP$.*

Proof. Let $A = \{x_1, \dots, x_n\} \subset X$ be a finite collection. Denote $r = 2\max_k \|x_k\|$, $s = \inf_{x \in X} \sum_{k=1}^n \|x - x_k\|$. Evidently, if $\|x\| > r$ then $\sum_{k=1}^n \|x - x_k\| > \sum_{k=1}^n \|x_k\| \geq s$, so the infimum in the definition of s can be searched for $x \in rB_X$. Consider $F\colon x^{**} \mapsto \sum_{k=1}^n \|x^{**} - x_k\|$ as a function on $rB_{X^{**}}$. For every $m \in \mathbb{N}$ the set $F^{-1}([0, s + 1/m]) \subset rB_{X^{**}}$ is weak* compact and not empty, so there is an $x_0^{**} \in \bigcap_m F^{-1}([0, s+1/m])$. Then $\sum_{k=1}^n \|x_0^{**} - x_k\| \leq s$. Denote $P\colon X^{**} \to X$ a norm-1 projection. Then

$$\sum_{k=1}^n \|Px_0^{**} - x_k\| = \sum_{k=1}^n \|P(x_0^{**} - x_k)\| \leq \sum_{k=1}^n \|x_0^{**} - x_k\| \leq s,$$

so $Px_0^{**} \in X$ is a Steiner point of A. □

The chain of lemmas below is a part of the main construction.

Lemma 1. *Let X, Y be infinite-dimensional Banach spaces with X non-reflexive. Then there is an operator $T\colon X \to Y$ such that $cl(T(B_X)) \setminus T(X) \neq \varnothing$.*

Proof. At first, the non-reflexivity of X implies the non-reflexivity of X^*. So, there is a sequence $(g_n) \subset B_{X^*}$ that has no weak limiting points. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. (B_{X^*}, w^*) is w^* compact, hence there is a w^* limit g of (g_n) with respect to $\mathcal{U}$. Denote $f_k = \frac{g_n - g}{\|g_n - g\|}$. $(f_k) \subset B_{X^*}$ is a w^*-convergent to zero with respect to $\mathcal{U}$ sequence of functionals that does not converge to zero weakly with respect to $\mathcal{U}$. Select a basic sequence $(e_k) \subset B_Y$ (see [3, Theorem 1.a.5]) and define T as follows:

$$Tx = \sum_{k=1}^{\infty} 2^{-k} f_k(x) e_k.$$

Let $x^{**} \in B_{X^{**}}$ be such an element that $\langle x^{**}, f_k \rangle \nrightarrow 0$ with respect to $\mathcal{U}$ as $k \to \infty$. Then $T^{**}(x^{**}) = \sum_{k=1}^{\infty} 2^{-k} \langle x^{**}, f_k \rangle e_k$ does not belong to $T(X)$. On the other hand, since B_X is w^* dense in $B_{X^{**}}$, for every $n \in \mathbb{N}$ there is $v_n \in B_X$ such that $\max\{|\langle x^{**}, f_k \rangle - f_k(v_n)| \colon 1 \leq k \leq n\} < \frac{1}{n}$. So,

$$\|T^{**}(x^{**}) - Tv_n\| \leq \sum_{k=1}^{\infty} 2^{-k} |\langle x^{**}, f_k \rangle - f_k(v_n)| < \frac{1}{n} + 2^{-n}.$$

This means that $T^{**}(x^{**}) \in \mathrm{cl}(T(B_X)) \setminus T(X)$. □

The following lemma is a well-known technical observation. We give just a sketch of proof.

Lemma 2. *Let Y be a Banach space, $\{y_k\}_{k=1}^m$, $\{z_k\}_{k=1}^m$ be two linearly independent subsets of Y. Then there is an isomorphism $G\colon Y \to Y$ such that $Gz_k = y_k$ for all $k = 1, \dots, m$.*

Proof. For every $j \in \{1, \dots, m\}$ denote $y_j^* \in Y^*$ a functional satisfying $y_j^*(y_k) = 0$ if $k \in \{1, \dots, m\} \setminus \{j\}$ and $y_j^*(y_j) = 1$ (first define it on $\mathrm{Lin}\{y_k\}_{k=1}^m$ and then extend it to the whole Y by the Hahn-Banach theorem). Recall that $\{y_k^*\}_{k=1}^m$ is called a *biorthogonal system to* $\{y_k\}_{k=1}^m$. The same way we select a biorthogonal system $\{z_k^*\}_{k=1}^m$ to $\{z_k\}_{k=1}^m$. Denote $Y_m = \bigcap_{k=1}^m \ker y_k^*$, $Z_m = \bigcap_{k=1}^m \ker z_k^*$. Being two subspaces of the same finite codimension, Y_m and Z_m are isomorphic (one can show this for subspaces of codimension 1, and then proceed by induction in codimension). Denote $W\colon Z_m \to Y_m$ the corresponding isomorphism. Now we define G as follows:

$$Gy = \sum_{j=1}^{m} z_j^*(y)y_j + W(y - \sum_{i=1}^{m} z_i^*(y)z_i).$$

The identity $Gz_k = y_k$ is evident, so G maps $\mathrm{Lin}\{z_k\}_{k=1}^m$ to $\mathrm{Lin}\{y_k\}_{k=1}^m$ bijectively. On Z_m (which is a complement to $\mathrm{Lin}\{z_k\}_{k=1}^m$) G equals W, so G maps bijectively a complement of $\mathrm{Lin}\{z_k\}_{k=1}^m$ to a complement of $\mathrm{Lin}\{y_k\}_{k=1}^m$. This implies the invertibility of G. □

Lemma 3. *Under the conditions of Lemma 1, for every linearly independent collection $y_1, y_2, \dots, y_m \in Y$ there is a bounded linear operator $V\colon X \to Y$ such that $y_1, y_2, \dots, y_{m-1} \in V(X)$, $y_m \notin V(X)$, but the closure of $V(B_X)$ contains y_m.*

Proof. Let T be the operator from Lemma 1. We select $z_m \in \mathrm{cl}(T(B_X)) \setminus T(X)$ and a linearly independent collection $z_1, \dots, z_{m-1} \in T(X)$. Denote $G\colon Y \to Y$ an isomorphism that maps each z_k to the corresponding y_k, $k = 1, \dots, m$. Then $V = G \circ T$ is the operator we need. □

The following lemma is extracted from [1].

Lemma 4. *There exists a linearly independent collection $\{y_1, y_2, y_3\} \subset L_1[0,2]$ such that y_3 is the unique Steiner point of collection $\{y_1, y_2, 0\}$.*

Proof. Take $y_1(t) = t$, $y_2(t) = t^2$, $y_3(t) = \min\{t, t^2\}$. Remark that for points $a, b, c \in \mathbb{R}$, $a \le b \le c$, the unique Steiner point of the collection $\{a, b, c\}$ is b. In particular, for every $t \in [0,2]$ the unique Steiner point of the collection $\{y_1(t), y_2(t), 0\} \subset \mathbb{R}$ is $y_3(t)$. Consequently, for every $g \in L_1[0,2]$ we have

$$\|0 - g\| + \|y_1 - g\| + \|y_2 - g\| = \int_0^2 |0 - g(t)| + |y_1(t) - g(t)| + |y_2(t) - g(t)| dt \ge$$
$$\ge \int_0^2 |0 - y_3(t)| + |y_1(t) - y_3(t)| + |y_2(t) - y_3(t)| dt = \|0 - y_3\| + \|y_1 - y_3\| + \|y_2 - y_3\|,$$

and the equality is attained only if $g = y_3$ a.e. □

Now we are ready for the main theorem.

Theorem 2. *Let X be a nonreflexive Banach space. Then there is an equivalent norm $\|\cdot\|_b$ on X and there are points $x_1, x_2 \in X$ such that in $(X, \|\cdot\|_b)$ there is no Steiner point for the collection $\{x_1, x_2, 0\}$. In particular, $(X, \|\cdot\|_b) \notin StPP$.*

Proof. Denote $Y = L_1[0,2]$ and let $\{y_1, y_2, y_3\} \subset Y$ be elements from Lemma 4. We apply Lemma 3 in order to get a bounded linear operator $V\colon X \to Y$ such that $y_1, y_2 \in V(X)$, $y_3 \notin V(X)$, but the closure of $V(B_X)$ contains y_m. We take arbitrary $x_1 \in V^{-1}y_1$, $x_2 \in V^{-1}y_2$, and select also a sequence $(z_n) \subset B_X$ such that $\|Vz_n - y_3\| \to 0$. Now we pick $M > 0$ and n_0 such that for all $n > n_0$

$$M\|Vz_n\| > \|z_n\|, M\|y_1 - Vz_n\| > \|x_1 - z_n\|, \text{ and } M\|y_2 - Vz_n\| > \|x_2 - z_n\|.$$

Finally, introduce $\|\cdot\|_b$ as follows:

$$\|x\|_b = \max\{\|x\|, M\|Vx\|\}.$$

Then, on the one hand, for every $x \in X$ we have

$$\|x\|_b + \|x_1 - x\|_b + \|x_2 - x\|_b \geq M(\|Vx\| + \|y_1 - Vx\| + \|y_2 - Vx\|) >$$
$$> M(\|y_3\| + \|y_1 - y_3\| + \|y_2 - y_3\|)$$

(the last inequality is strong because $y_3 \neq Vx$). On the other hand, thanks to the choice of M,

$$\inf_{x \in X}\{\|x\|_b + \|x_1 - x\|_b + \|x_2 - x\|_b\} \leq \inf_{n > n_0}\{\|z_n\|_b + \|x_1 - z_n\|_b + \|x_2 - z_n\|_b\} =$$
$$= M \inf_{n > n_0}\{\|Vz_n\| + \|y_1 - Vz_n\| + \|y_2 - Vz_n\|\} = M(\|y_3\| + \|y_1 - y_3\| + \|y_2 - y_3\|).$$

So there is no $x \in X$ where $\|x\|_b + \|x_1 - x\|_b + \|x_2 - x\|_b$ attains its minimum. □

Corollary 1. *A Banach space is reflexive if and only if it possesses the Steiner Point Property in all equivalent norms.*

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