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MAXIMUM MODULUS OF ENTIRE FUNCTIONS OF TWO VARIABLES AND ARGUMENTS OF COEFFICIENTS OF DOUBLE POWER SERIES

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Let $\mathcal{L}$ be the class of positive continuous functions on $(-\infty,+\infty)$ and let $\mathcal{L}^2_+$ be the class of positive continuous increasing with respect to each variable functions γ in $\mathbb{R}^2$ such that $\gamma(r_1,r_2)\to+\infty$ as $r_1+r_2\to+\infty$. We prove the following statement: for all entire functions of the form $f(z_1,z_2)=\sum_{n+m=0}^{+\infty}a_{nm}z_1^nz_2^m$ such that $|a_{nm}|\le\exp\{-(n+m)\psi(n,m)\}$ for $n+m\ge k_0(f)$ and functions $f(z_1,1), f(1,z_2)$ are transcendent, $\psi\in\mathcal{L}^2_+$, the inequality

$$\mathfrak{M}_f(r_1,r_2)=O(M_f(r_1,r_2)h(\ln M_f(r_1,r_2))),\ h\in\mathcal{L},\ r^\vee=\min\{r_1,r_2\}\to+\infty,$$

holds where $M_f(r_1,r_2)=\max\{|f(z_1,z_2)|\colon |z_1|=r_1,|z_2|=r_2\}$, $\mathfrak{M}_f(r_1,r_2)=\sum_{n+m=0}^{+\infty}|a_{nm}|\times$ $\times r_1^nr_2^m$, if and only if

$$(\forall\gamma\in\mathcal{L}^2_+)\colon\ \sqrt{r_1r_2}=O\big(h(\gamma(r_1,r_2)\psi(r_1,r_2))\big),\ r^\vee\to+\infty.$$

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Пусть $\mathcal{L}$ — класс положительных непрерывных функций на $(-\infty,+\infty)$, а $\mathcal{L}^2_+$ класс положительных непрерывных возрастающих по каждой переменной функций γ на $\mathbb{R}^2$ таких, что $\gamma(r_1,r_2)\to+\infty$ при $r_1+r_2\to+\infty$. В статье доказано следующее утверждение: для того чтобы для любой целой функции, такой что $|a_{nm}|\le\exp\{-(n+m)\psi(n,m)\}$, $n+m\ge k_0(f)$, и функции $f(z_1,1), f(1,z_2)$ — трансцендентные, $\psi\in\mathcal{L}^2_+$, имело место соотношение

$$\mathfrak{M}_f(r_1,r_2)=O(M_f(r_1,r_2)h(\ln M_f(r_1,r_2))),\ h\in\mathcal{L},\ r^\vee=\min\{r_1,r_2\}\to+\infty,$$

где $M_f(r_1,r_2)=\max\{|f(z_1,z_2)|\colon |z_1|=r_1,|z_2|=r_2\}$, $\mathfrak{M}_f(r_1,r_2)=\sum_{n+m=0}^{+\infty}|a_{nm}|r_1^nr_2^m$, необходимо и достаточно, чтобы выполнялось условие

$$(\forall\gamma\in\mathcal{L}^2_+)\colon\ \sqrt{r_1r_2}=O\big(h(\gamma(r_1,r_2)\psi(r_1,r_2))\big),\ r^\vee\to+\infty.$$

One of the classical problems of theory of entire functions is the problem of the relationships between the maximums of the modulus of an entire function and the modulus of coefficient of its power series.

As it is known, the maximum of the modulus of an entire function does not depend on the moduli only, but it also depends on the arguments of the coefficients of its power expansion.

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How essential is this dependence? One of the possible approaches for achieving the answer to this question for a function of one variable one can find in [1, 3, 4].

We denote by $M_f(r) = \max\{|f(z)|\colon |z| \le r\}$ the maximum modulus of the function

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \tag{1}$$

$\mathfrak{M}_f(r) = \sum_{n=0}^{+\infty} |a_n| r^n$. It is easy to see that $M_f(r) \le \mathfrak{M}_f(r)$ $(r \ge 0)$.

So, the question how essentially do the arguments of coefficients influence on the growth of its maximum modulus can be reformulated as follows: how quickly may $\mathfrak{M}_f(r)$ grow with respect to $M_f(r)$?

Let $\mathcal{L}$ be the class of positive continuous functions on $(-\infty, +\infty)$ and $\mathcal{L}_+$ the subclass of increasing to $+\infty$ functions of the class $\mathcal{L}$.

Also we denote by $\mathcal{E}$ the class of transcendent entire functions of the form (1) and $\mathcal{E}_\psi$, where $\psi \in \mathcal{L}_+$, the subclass of functions from $\mathcal{E}$ of form (1), for which

$$|a_n| \le \exp\{-n\psi(n)\},\ n \ge n_0(f). \tag{2}$$

In [1] H. Brinkmeier proved that for every entire function f of order ρ the relation

$$\varlimsup_{r\to+\infty} \frac{\ln \mathfrak{M}_f(r) - \ln M_f(r)}{\ln r} \le \frac{\rho}{2}$$

holds. Similar statement is also proved in [2] for Dirichlet series with arbitrary abscissa of absolute convergence.

In [3] P.V. Filevych established a necessary and sufficient condition on the maximum modulus $M_f(r)$ of the entire function f for validity of the inequality

$$\varlimsup_{r\to+\infty} \frac{\ln \mathfrak{M}_f(r) - \ln M_f(r)}{\ln M_f(r)} \le \alpha, \quad \alpha \in (0, +\infty).$$

In [4] P.V. Filevych was found a condition on the sequence $(|a_n|)$, under which for the functions $f \in \mathcal{E}_\psi$ of form (1) the relation

$$\mathfrak{M}_f(r) = O(M_f(r) h(\ln M_f(r))), \quad r \to +\infty. \tag{3}$$

holds for $h \in \mathcal{L}$. Moreover, conditions are obtained under which the relation holds for $\varphi \in \mathcal{L}_+$

$$\varphi(\ln \mathfrak{M}_f(r)) \sim \varphi(\ln M_f(r)), \quad r \to +\infty. \tag{4}$$

Theorem A ([4]). *Let $h \in \mathcal{L}$ and $\psi \in \mathcal{L}_+$. Then for all entire functions $f \in \mathcal{E}_\psi$ relation (3) holds if and only if*

$$(\forall \gamma \in \mathcal{L}_+)\ \sqrt{x} = O(h(\gamma(x)\psi(x))), \quad x \to +\infty.$$

Theorem B ([4]). *Let $\varphi, \psi \in \mathcal{L}_+$. If*

$$\varphi(t+1) \sim \varphi(t), \quad t \to +\infty,$$

then for all entire functions $f \in \mathcal{E}_\psi$ relation (4) holds if and only if

$$(\forall \gamma \in \mathcal{L}_+)\ \lim_{x\to+\infty} \frac{\varphi(\gamma(x)\psi(x) + \ln\sqrt{x})}{\varphi(\gamma(x)\psi(x))} = 1.$$

For the entire functions

$$f(z_1,z_2)=\sum_{n+m=0}^{+\infty} a_{nm}z_1^n z_2^m \tag{5}$$

we denote

$$\mathfrak{M}_f(r_1,r_2)=\sum_{n+m=0}^{+\infty}|a_{nm}|r_1^n r_2^m,\ r^{\wedge}=\max\{r_1,r_2\},$$

$$M_f(r_1,r_2)=\max\{|f(z_1,z_2)|\colon |z_1|\le r_1, |z_2|\le r_2\},\ \mu_f(r_1,r_2)=\max\{|a_{nm}|r_1^n r_2^m\colon n,m\ge 0\}$$

maximum modulus function $f(z_1,z_2)$ and maximal term of series (5) respectively.

In this paper we find condition on the sequence $(|a_{nm}|)$, under which the relations

$$\mathfrak{M}_f(r_1,r_2)=O(M_f(r_1,r_2)h(\ln M_f(r_1,r_2))),\quad r^{\vee}\to+\infty,\ h\in\mathcal{L}, \tag{6}$$

$$\varphi(\ln\mathfrak{M}_f(r_1,r_2))\sim\varphi(\ln M_f(r_1,r_2)),\quad r^{\vee}\to+\infty,\ \varphi\in\mathcal{L}_+, \tag{7}$$

where $r^{\vee}\overset{def}{=}\min\{r_1,r_2\}\to+\infty$, hold for $f(z_1,z_2)$ of form (5).

We denote by T class of entire functions of form (5) for which the functions $f(1,z_2)$ and $f(z_1,1)$ are transcendent.

It is easy to prove that for every function f of the class T one has

$$\ln r^{\wedge}=o(\ln M_f(r_1,r_2)),\quad r^{\vee}\to+\infty, \tag{8}$$

where $r^{\wedge}\overset{def}{=}\max\{r_1,r_2\}$.

Also we remark, that if only one of the functions $f(1,z_2), f(z_1,1)$ is transcendent, then relation (8) need not hold. So, we consider the function $f(z_1,z_2)=e^{z_1}+z_2$. For this function we have

$$\varlimsup_{r^{\vee}\to+\infty}\frac{\ln r^{\wedge}}{\ln\mu_f(r_1,r_2)}\ge\varlimsup_{r^{\vee}\to+\infty}\frac{\ln r^{\wedge}}{\ln M_f(r_1,r_2)}=\varlimsup_{r^{\vee}\to+\infty}\frac{\ln r^{\wedge}}{\ln(e^{r_1}+r_2)}\ge\varlimsup_{r^{\vee}\to+\infty}\frac{\ln r_2}{\ln(e^{r_1}+r_2)}\ge$$

$$\ge\varlimsup_{n\to+\infty}\frac{\ln r_2^{(n)}}{\ln(e^{r_1^{(n)}}+r_2^{(n)})}=1,$$

as $r_2^{(n)}=\exp\{r_1^{(n)}\}\to+\infty$, $n\to+\infty$.

Let $\mathcal{L}$ be the class of positive continuous functions on $\mathbb{R}^2$ and $\mathcal{L}_+^2$ the subclass of the class $\mathcal{L}$, which consists of increasing on each variable functions $\gamma\in\mathcal{L}$ such that

$$\lim_{r_1+r_2\to+\infty}\gamma(r_1,r_2)=+\infty.$$

For $\psi\in\mathcal{L}_+^2$ by T_ψ we denote the subclass of the functions from T, for which

$$|a_{nm}|\le\exp\{-(n+m)\psi(n,m)\},\quad n+m\ge k_0(f). \tag{9}$$

Theorem 1. *Let $h\in\mathcal{L},\psi\in\mathcal{L}_+^2$ and $\alpha>0$. If*

$$(\forall\gamma\in\mathcal{L}_+^2)\colon\quad\varlimsup_{r^{\vee}\to+\infty}\frac{\sqrt{r_1r_2}}{h(2\gamma(r_1,r_2)\psi(r_1,r_2))}\le\alpha, \tag{10}$$

then for any entire function $f\in T_\psi$ one has

$$\varlimsup_{r^{\vee}\to+\infty}\frac{\mathfrak{M}_f(r_1,r_2)}{M_f(r_1,r_2)h(\ln M_f(r_1,r_2))}\le\alpha.$$

Proof. Let $h \in \mathcal{L}, \psi \in \mathcal{L}_+^2$ and condition (10) hold. We suppose that there exist an entire function $f \in T_\psi$, a number $\delta > 0$ and increasing to $+\infty$ sequences $(r_1^{(p)})$, $(r_2^{(q)})$ such that

$$(\forall p, q \geq 0)\colon \quad \mathfrak{M}_f\Big(r_1^{(p)}, r_2^{(q)}\Big) \geq (\alpha+\delta) M_f\Big(r_1^{(p)}, r_2^{(q)}\Big) h\Big(\ln M_f\Big(r_1^{(p)}, r_2^{(q)}\Big)\Big). \tag{11}$$

For an entire function $f \in T$ we denote

$$S_f(r_1, r_2) = \left(\sum_{n+m=0}^{+\infty} |a_{nm}| r_1^{2n} r_2^{2m}\right)^{1/2}.$$

It is obvious that $S_f(r_1, r_2) \leq M_f(r_1, r_2)$.

By $I(r_1, r_2)$ we denote the set of pairs of numbers $(n, m) \in \mathbb{Z}_+^2$, for which $\psi(n, m) \geq 2 \ln r^\wedge$. Now we may choose $(n(r_1, r_2), m(r_1, r_2)) \in I(r_1, r_2)$ so that $(n(r_1, r_2) - 1, m(r_1, r_2) - 1) \notin I(r_1, r_2)$ and

$$\lim_{r^\vee \to +\infty} n(r_1, r_2) = +\infty, \quad \lim_{r^\vee \to +\infty} m(r_1, r_2) = +\infty.$$

It follows from (10) that

$$(\forall \gamma \in \mathcal{L}_+^2)\colon \quad \varlimsup_{r^\vee \to +\infty} \frac{\sqrt{(n(r_1, r_2) - 1)(m(r_1, r_2) - 1)}}{h(2\gamma(n(r_1, r_2) - 1, m(r_1, r_2) - 1)\psi(n(r_1, r_2) - 1, m(r_1, r_2) - 1))} \leq \alpha. \tag{12}$$

By relation $\ln r^\wedge = o(\ln M_f(r_1, r_2))$, $r^\vee \to +\infty$ we get for $f \in T_\psi$

$$\psi(n(r_1, r_2) - 1, m(r_1, r_2) - 1) < 2 \ln r^\wedge = o(\ln M_f(r_1, r_2)), \ r^\vee \to +\infty.$$

Therefore, there exist a function $\gamma \in \mathcal{L}_+^2$ and subsequences $(r_1^{(p_k)})$, $(r_2^{(q_k)})$ of the sequences $(r_1^{(p)})$, $(r_2^{(q)})$ such that:

1) $n\Big(r_1^{(p_0)}, r_2^{(q_0)}\Big) \geq n_0(f) \vee m\Big(r_1^{(p_0)}, r_2^{(q_0)}\Big) \geq m_0(f)$,

2) $r_1^{(p_0)} \geq 2, \ r_2^{(q_0)} \geq 2$,

3) $2\gamma\Big(n\Big(r_1^{(p_k)}, r_2^{(q_k)}\Big) - 1, m\Big(r_1^{(p_k)}, r_2^{(q_k)}\Big) - 1\Big) = \dfrac{\ln M_f\Big(r_1^{(p_k)}, r_2^{(q_k)}\Big)}{\psi\Big(n\Big(r_1^{(p_k)}, r_2^{(q_k)}\Big) - 1, m\Big(r_1^{(p_k)}, r_2^{(q_k)}\Big) - 1\Big)}$,

for all $k \geq 0$.

Now using the Cauchy-Bunyakovsky inequality and condition (9) we have for $r = (r_1, r_2) \in [2, +\infty)^2$ such that $n(r_1, r_2) + m(r_1, r_2) > k_0(f)$ $(K(r_1, r_2) = \{(n, m)\colon n \geq n(r_1, r_2) \vee m \geq m(r_1, r_2)\})$

$$\mathfrak{M}_f(r_1, r_2) = \left(\sum_{n < n(r_1, r_2)} \sum_{m < m(r_1, r_2)} + \sum_{(n,m) \in K(r_1, r_2)}\right) |a_{nm}| r_1^n r_2^m \leq$$

$$\leq \sqrt{n(r_1, r_2) m(r_1, r_2)} \left(\sum_{n < n(r_1, r_2)} \sum_{m < m(r_1, r_2)} |a_{nm}| r_1^{2n} r_2^{2m}\right)^{1/2} +$$

$$+ \sum_{(n,m) \in K(r_1, r_2)} \exp\{-(n+m)\psi(n, m)\} r_1^n r_2^m \leq \sqrt{n(r_1, r_2) m(r_1, r_2)} S_f(r_1, r_2) +$$

$$+ \sum_{(n,m) \in K(r_1, r_2)} \exp\{-2(n+m) \ln r^\wedge\} r_1^n r_2^m \leq \sqrt{n(r_1, r_2) m(r_1, r_2)} M_f(r_1, r_2) +$$

$$+\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\exp\{-2n\ln r_1+n\ln r_1\}\exp\{-2m\ln r_2+m\ln r_2\}\le$$

$$\le\sqrt{n(r_1,r_2)m(r_1,r_2)}M_f(r_1,r_2)+\sum_{n=0}^{+\infty}2^{-n}\sum_{m=0}^{+\infty}2^{-m}=\sqrt{n(r_1,r_2)m(r_1,r_2)}M_f(r_1,r_2)+4.$$

It follows from (12) and the third condition of choosing subsequences $(r_1^{(p_k)})$, $(r_2^{(q_k)})$ that for $n_k=n\Big(r_1^{(p_k)},r_2^{(q_k)}\Big)-1,\ m_k=m\Big(r_1^{(p_k)},r_2^{(q_k)}\Big)-1$

$$\varlimsup_{k\to+\infty}\frac{\mathfrak{M}_f\big(r_1^{(p_k)},r_2^{(q_k)}\big)}{M_f\big(r_1^{(p_k)},r_2^{(q_k)}\big)h\big(\ln M_f\big(r_1^{(p_k)},r_2^{(q_k)}\big)\big)}\le\varlimsup_{k\to+\infty}\frac{\sqrt{(n_k+1)(m_k+1)}}{h\big(2\gamma(n_k,m_k)\psi(n_k,m_k)\big)}=$$
$$=\varlimsup_{k\to+\infty}\frac{\sqrt{n_km_k}}{h\big(2\gamma(n_k,m_k)\psi(n_k,m_k)\big)}\le\alpha,$$

which contradicts inequality (11). □

Theorem 2. *Let $h\in\mathcal{L},\psi\in\mathcal{L}_+^2$. If*

$$(\exists\gamma\in\mathcal{L}_+^2):\qquad\varlimsup_{r^\vee\to+\infty}\frac{\sqrt{r_1r_2}}{h(2\gamma(r_1,r_2)\psi(r_1,r_2))}>\alpha>0,\tag{13}$$

then there exists an entire function $f\in T_\psi$ such that

$$\varlimsup_{r^\vee\to+\infty}\frac{\mathfrak{M}_f(r_1,r_2)}{M_f(r_1,r_2)h(\ln M_f(r_1,r_2))}>\frac{3-2\sqrt{2}}{4}\alpha.\tag{14}$$

We need the following lemma from [5] (see also [4]).

Lemma 1 ([5])**.** *For all $n\in\mathbb{N}$ there exist numbers $e_0(n),\dots,e_{n-1}(n)$ of a set $\{-1;1\}$ such that*

$$\max_{t\in[0,2\pi]}\Big|e_0(n)+e_1(n)e^{it}+\ldots+e_{n-1}(n)e^{i(n-1)t}\Big|\le\frac{2\sqrt{n}}{\sqrt{2}-1}.$$

Proof of Theorem 2. Without loss of generality we may and do assume that $\alpha=1$. Let $h\in\mathcal{L},\psi\in\mathcal{L}_+^2$ and condition (13) hold. Then there exist a function $\gamma\in\mathcal{L}_+^2$ and a number $\varepsilon>0$, for which

$$\varlimsup_{r^\vee\to+\infty}\frac{\sqrt{[r_1][r_2]}}{h(\gamma(r_1,r_2)\psi(r_1,r_2))}>1+\varepsilon,\tag{15}$$

where $[x]=\max\{n\in\mathbb{Z}\colon\ n\le x\}$ is the integer part of x.

Now we construct a function $f\in T_\psi$, for which inequality (14) holds.

Firstly we remark that for every function $\psi\in\mathcal{L}_+$ the series

$$\sum_{n+m=0}^{+\infty}\exp\{-(n+m)\psi(n,m)\}z_1^nz_2^m$$

converges in $\mathbb{C}^2$. Then for all $\delta > 0,\ x > 0,\ y > 0$ there exist positive integers $P(\delta, x, y)$, $Q(\delta, x, y)$ such that

$$\sum_{(n,m)\in V(\delta,x,y)} \exp\{-(n+m)\psi(n,m)\}x^n y^m < \delta,$$

where $V(\delta, x, y) = \{(n,m)\colon n \ge P(\delta,x,y) \vee m \ge Q(\delta,x,y)\}$. Since $h \in \mathcal{L}$, for all $x > 0, y > 0$

$$h(2\gamma(x,y)\psi(x,y)) > h(\ln\exp\{2\gamma(x,y)\psi(x,y)\}) \cdot \frac{x}{x+1} \cdot \frac{y}{y+1}.$$

By $\delta(x,y) \in (0,1)$ we denote the number such that $(\forall z \in [-\delta(x,y), \delta(x,y)])$:

$$h(2\gamma(x,y)\psi(x,y)) > h(2\ln(\exp\{\gamma(x,y)\psi(x,y)\} + z)) \cdot \frac{x}{x+1} \cdot \frac{y}{y+1}.$$

We consider the set

$$E = \{x \ge 4,\ y \ge 4\colon \sqrt{[xy]} > (1+\varepsilon)h(\gamma(x,y)\psi(x,y))\}.$$

Since inequality (15) holds, the set E is unbounded, such that for all $(x_0, y_0) \in \mathbb{R}^2_+$:

$$E \cap ((x_0, +\infty) \times (y_0, +\infty)) \ne \varnothing.$$

Since $\gamma \in \mathcal{L}^2_+$, then there exist increasing to $+\infty$ sequences (x_k) and (y_k) such that for all $k \in \mathbb{N}$: $(x_k, y_k) \in E$ and

$$\gamma(x_{k+1}, y_{k+1}) \ge 2\max\{[x_k], [y_k]\}.$$

Thus for $k \in \mathbb{N}$

$$\Big(\frac{\gamma(x_{k+1}, y_{k+1})}{[x_k]} - 2\Big)\psi(x_{k+1}, y_{k+1}) + l_k^{(1)}\psi(x_k, y_k) \ge 0,$$
$$\Big(\frac{\gamma(x_{k+1}, y_{k+1})}{[y_k]} - 2\Big)\psi(x_{k+1}, y_{k+1}) + l_k^{(2)}\psi(x_k, y_k) \ge 0,$$

i.e.

$$\frac{\gamma(x_{k+1}, y_{k+1})\psi(x_{k+1}, y_{k+1})}{n_k} + l_k^{(1)}\psi(x_k, y_k) \ge 2\psi(x_{k+1}, y_{k+1}), \tag{16}$$
$$\frac{\gamma(x_{k+1}, y_{k+1})\psi(x_{k+1}, y_{k+1})}{m_k} + l_k^{(2)}\psi(x_k, y_k) \ge 2\psi(x_{k+1}, y_{k+1}), \tag{17}$$

where

$$l_0^{(1)} = l_0^{(2)} = 1,\ n_k = [x_k],\ m_k = [y_k],\ p_k = [\sqrt{n_k}],\ q_k = [\sqrt{m_k}],$$
$$l_{k+1}^{(1)} = \frac{\gamma(x_{k+1}, y_{k+1})(n_{k+1} - n_k)}{n_{k+1}n_k} + l_k^{(1)}\frac{\psi(x_k, y_k)}{\psi(x_{k+1}, y_{k+1})}, \tag{18}$$
$$l_{k+1}^{(2)} = \frac{\gamma(x_{k+1}, y_{k+1})(m_{k+1} - m_k)}{m_{k+1}m_k} + l_k^{(2)}\frac{\psi(x_k, y_k)}{\psi(x_{k+1}, y_{k+1})}. \tag{19}$$

For $k\in\mathbb{N}$ we define

$$c_k^{(1)}=\exp\Big\{\frac{\gamma(x_{k+1},y_{k+1})\psi(x_{k+1},y_{k+1})}{n_k}+l_k^{(1)}\psi(x_k,y_k)\Big\}, \tag{20}$$

$$c_k^{(2)}=\exp\Big\{\frac{\gamma(x_{k+1},y_{k+1})\psi(x_{k+1},y_{k+1})}{m_k}+l_k^{(2)}\psi(x_k,y_k)\Big\}. \tag{21}$$

By the definitions of $c_k^{(1)}$ and $c_k^{(2)}$ the inequalities (16), (17) imply

$$\ln c_k^{(1)}\geq 2\psi(x_{k+1},y_{k+1}),\ \ln c_k^{(2)}\geq 2\psi(x_{k+1},y_{k+1}). \tag{22}$$

Moreover, sequences $\{x_k\}_{k=0}^{+\infty}$ and $\{y_k\}_{k=0}^{+\infty}$ have to increase so rapidly that the following inequalities are true

$$p_{k+1}\geq 2n_k,\ q_{k+1}\geq 2m_k,\ \big(\big[\sqrt{[x_{k+1}]}\big]\geq[x_k],\ \big[\sqrt{[y_{k+1}]}\big]\geq[y_k]\big), \tag{23}$$

$$c_{k+1}^{(1)}\geq 2(k+1)c_k^{(1)},\ c_{k+1}^{(2)}\geq 2(k+1)c_k^{(2)}, \tag{24}$$

$$p_{k+2}\geq P(\delta,c_k^{(1)},c_k^{(2)}),\ q_{k+2}\geq Q(\delta,c_k^{(1)},c_k^{(2)}). \tag{25}$$

For $k\geq 0$ we define

$$b_k=\exp\Big\{-l_k^{(1)}n_k\psi(x_k,y_k)\Big\},\ N_k=n_k-p_k, \tag{26}$$

$$d_k=\exp\Big\{-l_k^{(2)}m_k\psi(x_k,y_k)\Big\},\ M_k=m_k-p_k. \tag{27}$$

Also for $(p,q)\in[p_{k+1},n_{k+1}-1]\times[q_{k+1},m_{k+1}-1]$ we suppose that

$$a_{pq}=e_{p-p_{k+1}}(N_{k+1})b_k\Big(c_k^{(1)}\Big)^{n_k-p}e_{q-q_{k+1}}(M_{k+1})d_k\Big(c_k^{(2)}\Big)^{m_k-q}, \tag{28}$$

where $e_j(N_{k+1})$ and $e_\nu(M_{k+1})$, $j\in\{1,\ldots,N_{k+1}-1\},\nu\in\{1,\ldots,M_{k+1}-1\}$ are numbers from the set $\{-1;1\}$ from Lemma 1 when $n=N_{k+1}$ and $m=M_{k+1}$, respectively. For all other pairs (p,q) for which a_{pq} is not defined, we suppose that $a_{pq}=0$.

Let us consider double power series

$$g(z_1,z_2)=\sum_{p+q=0}^{+\infty}a_{pq}z_1^pz_2^q=\sum_{k=0}^{+\infty}\Bigg(\sum_{p=p_{k+1}}^{n_{k+1}-1}\sum_{q=q_{k+1}}^{m_{k+1}-1}a_{pq}z_1^pz_2^q\Bigg)=\sum_{k=0}^{+\infty}g_k(z_1,z_2). \tag{29}$$

Now we prove that $g\in T_\psi$. Using one after another (28), (23) and (22), we obtain for all $(p,q)\in[p_{k+1},n_{k+1}-1]\times[q_{k+1},m_{k+1}-1]$

$$-\ln|a_{pq}|=-\ln b_k+(p-n_k)\ln c_k^{(1)}-\ln d_k+(q-m_k)\ln c_k^{(2)}>$$
$$>(p-n_k)\ln c_k^{(1)}+(q-m_k)\ln c_k^{(2)}\geq\Big(\frac{p}{2}+\frac{p_{k+1}}{2}-n_k\Big)\ln c_k^{(1)}+\Big(\frac{q}{2}+\frac{q_{k+1}}{2}-m_k\Big)\ln c_k^{(2)}\geq$$
$$\geq\frac{p}{2}\cdot 2\psi(x_{k+1},y_{k+1})+\frac{q}{2}\cdot 2\psi(x_{k+1},y_{k+1})\geq(p+q)\psi(n_{k+1},m_{k+1})\geq(p+q)\psi(p,q).$$

Therefore, $|a_{pq}|\leq\exp\{-(p+q)\psi(p,q)\}$. By the definition of T_ψ, $g(z_1,z_2)$ is an entire function in $\mathbb{C}^2$ and $g\in T_\psi$.

Now we construct the required function $f(z_1, z_2)$. Let

$$\mu_s = b_s\left(c_s^{(1)}\right)^{n_s} d_s\left(c_s^{(2)}\right)^{m_s}.$$

So, from (28) it follows that

$$|a_{pq}|\left(c_s^{(1)}\right)^p\left(c_s^{(2)}\right)^q = b_s\left(c_s^{(1)}\right)^{n_s-p} d_s\left(c_s^{(2)}\right)^{m_s-q}\left(c_s^{(1)}\right)^p\left(c_s^{(2)}\right)^q = b_s\left(c_s^{(1)}\right)^{n_s} d_s\left(c_s^{(2)}\right)^{m_s} = \mu_s$$

for $(p,q) \in [p_{s+1}, n_{s+1}-1] \times [q_{s+1}, m_{s+1}-1],\ s > 0$.
Then

$$S_{g_s}\left(c_s^{(1)}, c_s^{(2)}\right) = \left(\sum_{p=p_{s+1}}^{n_{s+1}-1}\sum_{q=q_{s+1}}^{m_{s+1}-1} |a_{pq}|^2\left(c_s^{(1)}\right)^{2p}\left(c_s^{(2)}\right)^{2q}\right)^{1/2} = \left(\sum_{p=p_{s+1}}^{n_{s+1}-1}\sum_{q=q_{s+1}}^{m_{s+1}-1} \mu_s^2\right)^{1/2} = $$
$$= \mu_s\sqrt{N_{s+1}M_{s+1}}. \tag{30}$$

By Lemma 1 we can estimate the maximum modulus of $g_k(z_1, z_2)$ on the sequence $\left(c_s^{(1)}, c_s^{(2)}\right)$. That is, for $\theta = (\theta_1, \theta_2)$ we get

$$M_{g_s}\left(c_s^{(1)}, c_s^{(2)}\right) = \max_{\theta\in[0,2\pi]^2}\left|\sum_{p=p_{s+1}}^{n_{s+1}-1}\sum_{q=q_{s+1}}^{m_{s+1}-1} e_{p-p_{s+1}}(N_{s+1}) b_k\left(c_s^{(1)}\right)^{n_s-p}\times\right.$$
$$\left.\times e_{q-q_{s+1}}(M_{s+1}) d_k\left(c_s^{(2)}\right)^{m_s-q}\left(c_s^{(1)}\right)^p\left(c_s^{(2)}\right)^q \exp\{i(p\theta_1+q\theta_2)\}\right| =$$
$$= \mu_s \max_{\theta\in[0,2\pi]^2}\left|\sum_{p=p_{s+1}}^{n_{s+1}-1}\sum_{q=q_{s+1}}^{m_{s+1}-1} e_{p-p_{s+1}}(N_{s+1}) e_{q-q_{s+1}}(M_{s+1}) \exp\{i(p\theta_1+q\theta_2)\}\right| =$$
$$= \mu_s \max_{\theta_1\in[0,2\pi]}\left|\sum_{p=p_{s+1}}^{n_{s+1}-1} e_{p-p_{s+1}}(N_{s+1})\exp\{ip\theta_1\}\right| \cdot \max_{\theta_2\in[0,2\pi]}\left|\sum_{q=q_{s+1}}^{m_{s+1}-1} e_{q-q_{s+1}}(M_{s+1})\exp\{iq\theta_2\}\right| \le$$
$$\le \mu_s\frac{2\sqrt{N_{s+1}}}{\sqrt{2}-1}\cdot\frac{2\sqrt{M_{s+1}}}{\sqrt{2}-1} = \mu_s\cdot\frac{4\sqrt{N_{s+1}M_{s+1}}}{3-2\sqrt{2}}. \tag{31}$$

From (18), (20) and (26) it follows that

$$-\ln b_{k+1} \overset{(26)}{=} l_{k+1}^{(1)} n_{k+1}\psi(x_{k+1}, y_{k+1}) \overset{(18)}{=}$$
$$= n_{k+1}\psi(x_{k+1}, y_{k+1})\left\{\frac{\gamma(x_{k+1}, y_{k+1})(n_{k+1}-n_k)}{n_{k+1}n_k} + l_k^{(1)}\frac{\psi(x_k, y_k)}{\psi(x_{k+1}, y_{k+1})}\right\} =$$
$$= \frac{\psi(x_{k+1}, y_{k+1})\gamma(x_{k+1}, y_{k+1})}{n_k}(n_{k+1}-n_k) + l_k^{(1)} n_{k+1}\psi(x_k, y_k) \overset{(20)}{=}$$
$$= (\ln c_k^{(1)} - l_k^{(1)}\psi(x_k, y_k))(n_{k+1}-n_k) + l_k^{(1)} n_{k+1}\psi(x_k, y_k) =$$
$$= (n_{k+1}-n_k)\ln c_k^{(1)} + l_k^{(1)} n_k\psi(x_k, y_k) \overset{(26)}{=} (n_{k+1}-n_k)\ln c_k^{(1)} - \ln b_k.$$

Similarly we may prove that $-\ln d_{k+1} = (m_{k+1}-m_k)\ln c_k^{(2)} - \ln d_k$. Thus,

$$\frac{b_k}{b_{k+1}} = \left(c_k^{(1)}\right)^{n_{k+1}-n_k},\quad \frac{d_k}{d_{k+1}} = \left(c_k^{(2)}\right)^{m_{k+1}-m_k},\ k \ge 0. \tag{32}$$

Let $s\ge 1,\ k\le s-1,\ (p,q)\in[p_{k+1},n_{k+1}-1]\times[q_{k+1},m_{k+1}-1]$. By (28), (32) and (24) we obtain

$$|a_{pq}|\Big(c_s^{(1)}\Big)^p\Big(c_s^{(2)}\Big)^q\overset{(28)}{=}b_k\Big(c_k^{(1)}\Big)^{n_k-p}d_k\Big(c_k^{(2)}\Big)^{m_k-q}\Big(c_s^{(1)}\Big)^p\Big(c_s^{(2)}\Big)^q\overset{(32)}{=}$$
$$=\Big(c_k^{(1)}\Big)^{n_k-p}\Big(c_s^{(1)}\Big)^p b_s\prod_{j=k}^{s-1}\Big(c_j^{(1)}\Big)^{n_{j-1}-n_j}\cdot\Big(c_k^{(2)}\Big)^{m_k-q}\Big(c_s^{(2)}\Big)^q d_s\prod_{j=k}^{s-1}\Big(c_j^{(2)}\Big)^{m_{j-1}-m_j}\le$$
$$\le\Big(c_k^{(1)}\Big)^{n_k-p}\Big(c_s^{(1)}\Big)^p b_s\Big(c_k^{(1)}\Big)^{n_{k-1}-n_k}\Big(c_{s-1}^{(1)}\Big)^{n_s-n_{k-1}}\times$$
$$\times\Big(c_k^{(2)}\Big)^{m_k-q}\Big(c_s^{(2)}\Big)^q d_s\Big(c_k^{(2)}\Big)^{m_{k-1}-m_k}\Big(c_{s-1}^{(2)}\Big)^{m_s-m_{k-1}}\le$$
$$\le\Big(c_{s-1}^{(1)}\Big)^{n_k-p}\Big(c_s^{(1)}\Big)^p b_s\Big(c_{s-1}^{(1)}\Big)^{n_{k-1}-n_k}\Big(c_{s-1}^{(1)}\Big)^{n_s-n_{k-1}}\times$$
$$\times\Big(c_{s-1}^{(2)}\Big)^{m_k-q}\Big(c_s^{(2)}\Big)^q d_s\Big(c_{s-1}^{(2)}\Big)^{m_{k-1}-m_k}\Big(c_{s-1}^{(2)}\Big)^{m_s-m_{k-1}}=$$
$$=\Big(c_{s-1}^{(1)}\Big)^{n_s-p}\Big(c_s^{(1)}\Big)^p b_s\cdot\Big(c_{s-1}^{(2)}\Big)^{m_s-q}\Big(c_s^{(2)}\Big)^q d_s=$$
$$=b_s\Big(c_s^{(1)}\Big)^{n_s}d_s\Big(c_s^{(2)}\Big)^{m_s}\left(\frac{c_{s-1}^{(1)}}{c_s^{(1)}}\right)^{n_s-p}\left(\frac{c_{s-1}^{(2)}}{c_s^{(2)}}\right)^{m_s-q}\overset{(24)}{\le}\mu_s\left(\frac{1}{2s}\right)^{n_s-p+m_s-q}.$$

Therefore,

$$\sum_{p\le n_s-1}\sum_{q\le m_s-1}|a_{pq}|\Big(c_s^{(1)}\Big)^p\Big(c_s^{(2)}\Big)^q\le\mu_s\sum_{p=1}^{n_s-1}\left(\frac{1}{2s}\right)^{n_s-p}\cdot\sum_{q=1}^{m_s-1}\left(\frac{1}{2s}\right)^{m_s-q}\le$$
$$\le\mu_s\left(\sum_{j=1}^{+\infty}\left(\frac{1}{2s}\right)^j\right)^2=\mu_s\left(\frac{1}{2s-1}\right)^2\le\frac{\mu_s}{s^2},\ \ s\ge 1. \tag{33}$$

We define by induction a sequence of numbers $(t_k)_{k=0}^{+\infty}$ from $[0,1]$ in the following way. Let $t_0=1$. Also we suppose that for all $s\ge 1$ $t_0,\dots,t_{s-1}$ are already defined. Now we consider the function

$$\alpha_s(t,\theta)=\left|\sum_{k=0}^{s-1}t_k g_k\Big(c_s^{(1)}e^{i\theta_1},c_s^{(2)}e^{i\theta_2}\Big)+tg_s\Big(c_s^{(1)}e^{i\theta_1},c_s^{(2)}e^{i\theta_2}\Big)\right|.$$

It is obvious, that this function is continuous on $[0,1]\times[0,2\pi]^2$ and then the function

$$\beta_s(t)=\max_{\theta\in[0,2\pi]^2}|\alpha_s(t,\theta)|$$

is also continuous on $[0,1]$. Then (33) and (30) yield

$$\beta_s(0)\le\sum_{p\le n_s-1}\sum_{q\le m_s-1}|a_{pq}|\Big(c_s^{(1)}\Big)^p\Big(c_s^{(2)}\Big)^q\le\frac{\mu_s}{s^2}.$$
$$\beta_s(1)\ge M_{g_s}\Big(c_s^{(1)},c_s^{(2)}\Big)-\sum_{p\le n_s-1}\sum_{q\le m_s-1}|a_{pq}|\Big(c_s^{(1)}\Big)^p\Big(c_s^{(2)}\Big)^q\ge$$
$$\ge S_{g_s}\Big(c_s^{(1)},c_s^{(2)}\Big)-\frac{\mu_s}{s^2}=\mu_s\Big(\sqrt{N_{s+1}M_{s+1}}-\frac{1}{s^2}\Big).$$

Then there exists $t_s \in [0,1]$ such that $\beta_s(t_s) = \mu_s,\ s \geq 1$.

We may consider the power series

$$f(z_1, z_2) = \sum_{k=0}^{+\infty} t_k g_k(z_1, z_2).$$

Since $g \in T_\psi$, one has that $f \in T_\psi$.

It remains to prove that the function f satisfies inequality (14). Let $\Delta_s = M_f\left(c_s^{(1)}, c_s^{(2)}\right) - \mu_s$. By inequalities (25) we get

$$|\Delta_s| = \left|M_f\left(c_s^{(1)}, c_s^{(2)}\right) - \beta_s(t_s)\right| \leq \sum_{(p,q)\in W(p,q,s)} |a_{pq}|\left(c_s^{(1)}\right)^p\left(c_s^{(2)}\right)^q \leq$$

$$\leq \sum_{(p,q)\in V\left(\delta, c_s^{(1)}, c_s^{(2)}\right)} \exp\{-(n+m)\psi(n,m)\}\left(c_s^{(1)}\right)^p\left(c_s^{(2)}\right)^q < \delta\left(c_s^{(1)}, c_s^{(2)}\right) < 1,$$

where $W(p,q,s) = \{(p,q)\colon p \geq p_{s+2} \ \vee q \geq q_{s+2}\}$. Now from (20), (21), (26) and (27) it follows that

$$\mu_s = b_s\left(c_s^{(1)}\right)^{n_s} d_s\left(c_s^{(2)}\right)^{m_s} = \exp\Big\{ - l_s^{(1)} n_s\psi(x_s, y_s) + n_s \ln c_s^{(1)} -$$

$$- l_s^{(2)} m_s\psi(x_s, y_s) + m_s \ln c_s^{(2)}\Big\} = \exp\{2\gamma(x_{s+1}, y_{s+1})\psi(x_{s+1}, y_{s+1})\}.$$

Therefore,

$$\ln M_f\left(c_s^{(1)}, c_s^{(2)}\right) = \ln(\mu_s + \Delta_s) = \ln(\exp\{2\gamma(x_{s+1}, y_{s+1})\psi(x_{s+1}, y_{s+1})\} + \Delta_s).$$

By the definition of $\delta(x,y)$ we get

$$h(2\gamma(x_{s+1}, y_{s+1})\psi(x_{s+1}, y_{s+1})) > h\Big(\ln M_f\left(c_s^{(1)}, c_s^{(2)}\right)\Big) \cdot \frac{x_{s+1}}{x_{s+1}+1} \cdot \frac{y_{s+1}}{y_{s+1}+1}. \tag{34}$$

Moreover, by the definition β_s and inequality (33) we have

$$\mu_s = \beta_s(t_s) \leq \sum_{p\geq n_s-1}\sum_{q\geq m_s-1} |a_{pq}|\left(c_s^{(1)}\right)^p\left(c_s^{(2)}\right)^q + t_s M_{g_s}\left(c_s^{(1)}, c_s^{(2)}\right) \leq \frac{\mu_s}{s^2} + t_s M_{g_s}\left(c_s^{(1)}, c_s^{(2)}\right).$$

It follows from inequality (31) that

$$t_s \geq \left(1 - \frac{1}{s^2}\right)\cdot\frac{\mu_s}{M_{g_s}\left(c_s^{(1)}, c_s^{(2)}\right)} \geq \left(1 - \frac{1}{s^2}\right)\cdot\frac{3-2\sqrt{2}}{4}\cdot\frac{1}{\sqrt{N_{s+1}M_{s+1}}}. \tag{35}$$

So, by inequalities (34) and (35) we obtain as $s \to +\infty$

$$\mathfrak{M}_f\left(c_s^{(1)}, c_s^{(2)}\right) \geq t_s \sum_{p=p_{s+1}}^{n_{s+1}-1}\sum_{q=q_{s+1}}^{m_{s+1}-1} |a_{pq}|\left(c_s^{(1)}\right)^p\left(c_s^{(2)}\right)^q =$$

$$= t_s\mu_s N_{s+1}M_{s+1} \overset{(35)}{\geq} (1+o(1))\frac{3-2\sqrt{2}}{4}\cdot\frac{1}{\sqrt{N_{s+1}M_{s+1}}}\Big(M_f\left(c_s^{(1)}, c_s^{(2)}\right) + \Delta_s\Big)N_{s+1}M_{s+1} =$$

$$= (1+o(1))\frac{3-2\sqrt{2}}{4}\cdot\Big(M_f\Big(c_s^{(1)},c_s^{(2)}\Big)+\Delta_s\Big)\sqrt{N_{s+1}M_{s+1}} \overset{(26),(27)}{=}$$
$$= (1+o(1))\frac{3-2\sqrt{2}}{4}\cdot M_f\Big(c_s^{(1)},c_s^{(2)}\Big)\sqrt{[x_{s+1}][y_{s+1}]} \geq$$
$$\geq (1+\varepsilon+o(1))\frac{3-2\sqrt{2}}{4}\cdot M_f\Big(c_s^{(1)},c_s^{(2)}\Big)h(2\gamma(x_{s+1},y_{s+1})\psi(x_{s+1},y_{s+1})) \overset{(34)}{\geq}$$
$$\geq (1+\varepsilon+o(1))\frac{3-2\sqrt{2}}{4}\cdot M_f\Big(c_s^{(1)},c_s^{(2)}\Big)h\Big(\ln M_f\Big(c_s^{(1)},c_s^{(2)}\Big)\Big).$$

This completes the proof of Theorem 2. □

The following theorem is a consequence of theorems 1 and 2.

Theorem 3. *Let $h\in\mathcal{L},\psi\in\mathcal{L}_+^2$. Then for all entire function $f\in T_\psi$ relation (6) holds if and only if*

$$(\forall\gamma\in\mathcal{L}_+^2)\ \sqrt{r_1r_2}=O(h(\gamma(r_1,r_2)\psi(r_1,r_2))),\ r^\vee\to+\infty. \tag{36}$$

If we choose $h(r)\equiv 1$, then we have the following corollary.

Corollary 1. *Let $\psi\in\mathcal{L}_+^2$. Then exists a function $f\in T_\psi$ such that*

$$\varlimsup_{r^\vee\to+\infty}\frac{\mathfrak{M}_f(r_1,r_2)}{M_f(r_1,r_2)}=+\infty.$$

In the case when $h(r)$ is an increasing function we obtain the following statement.

Corollary 2. *Let $\psi\in\mathcal{L}_+^2$. Then for all entire functions $f\in T_\psi$ relation (6) hold if and only if*

$$(\exists\beta>0)\ \sqrt{r_1r_2}=O(h(\beta\psi(r_1,r_2))),\ r^\vee\to+\infty. \tag{37}$$

Now we consider the function

$$f(z_1,z_2)=\sum_{n+m=0}^{+\infty}\exp\{-(n+m)(nm)^{1/2\alpha}\}z_1^nz_2^m,\ \alpha>0.$$

Then for $\psi(n,m)=(nm)^{1/2\alpha}$ and $h(r)=r^\alpha$ condition (37) holds.

Corollary 3. *Let $\alpha>0$. For every entire function $f\in T$ the inequality*

$$|a_{nm}|\leq\exp\{-(n+m)(nm)^{1/2\alpha}\},\ n+m\geq k_0(f),$$

holds if and only if

$$\mathfrak{M}_f(r_1,r_2)=O(M_f(r_1,r_2)\ln^\alpha M_f(r_1,r_2))),\ r^\vee\to+\infty.$$

The following theorems concern relation (7).

Theorem 4. *Let function $\varphi\in\mathcal{L}_+$ be such that the condition*

$$\varphi(t+1)\sim\varphi(t),\ t\to+\infty, \tag{38}$$

does not hold. For any function $\psi\in\mathcal{L}_+^2$ there exists an entire function $f\in T_\psi$ such that relation (7) does not hold.

Proof. If condition (38) does not hold, then there exist a number $\delta > 0$ and an increasing to $+\infty$ sequence (σ_n) such that

$$\varphi(\sigma_n + 1) > (1+\delta)\varphi(\sigma_n),\ n \geq 0. \tag{39}$$

Also we consider the function $h_\delta \in \mathcal{L}$ such that

$$h_\delta(x) = \exp\{\varphi^{-1}((1+\delta)\varphi(x)) - x\},\ x \in \mathbb{R}. \tag{40}$$

Now from the definition of the function h_δ from (39) it follows that $h_\delta(\sigma_n) < e,\ n \geq 0$. Then the function $h(x) = e^4 h_\delta(x)$ does not satisfy condition (10) for all functions $\psi \in \mathcal{L}^2_+$. Indeed, let the sequence $r^{(n)} = (r_1^{(n)}, r_2^{(n)})$ be such that

$$\min\{r_1^{(n)}, r_2^{(n)}\} \to +\infty,\ n \to +\infty,$$

and for all $n \geq n_1$

$$2\gamma(r_1^{(n)}, r_2^{(n)})\psi(r_1^{(n)}, r_2^{(n)}) = \sigma_n,$$

where $\gamma, \psi \in \mathcal{L}^2_+$ are arbitrary functions from condition (10). But

$$\frac{\sqrt{r_1^{(n)} r_2^{(n)}}}{h\big(2\gamma(r_1^{(n)}, r_2^{(n)})\psi(r_1^{(n)}, r_2^{(n)})\big)} = \frac{\sqrt{r_1^{(n)} r_2^{(n)}}}{h(\sigma_n)} \geq e^{-5}\sqrt{r_1^{(n)} r_2^{(n)}} \to +\infty$$

as $n \to +\infty$.

Therefore, by Theorem 2 there exist a function $f \in T_\psi$, and increasing to $+\infty$ sequences $(x_n), (y_n)$ such that for all $n \geq 0$

$$\mathfrak{M}_f(x_n, y_n) > e^{-4} M_f(x_n, y_n) h(\ln M_f(x_n, y_n)) = M_f(x_n, y_n) h_\delta(\ln M_f(x_n, y_n)),$$

i.e.

$$\varphi(\ln \mathfrak{M}_f(x_n, y_n)) > (1+\delta)\varphi(\ln M_f(x_n, y_n)),\ n \geq 0.$$

So, inequality (7) is not valid. □

In the following theorem we find a condition on the sequence $(|a_{nm}|)_{n+m=0}^{+\infty}$, under which relation (7) holds.

Theorem 5. *Let $\varphi \in \mathcal{L}_+, \psi \in \mathcal{L}^2_+$. If $\varphi(t+1) \sim \varphi(t),\ t \to +\infty$, then for all entire functions $f \in T_\psi$ inequality (7) holds if and only if*

$$(\forall \gamma \in \mathcal{L}^2_+):\quad \lim_{r^\vee \to +\infty} \frac{\varphi(\gamma(r_1, r_2)\psi(r_1, r_2) + \ln\sqrt{r_1 r_2})}{\varphi(\gamma(r_1, r_2)\psi(r_1, r_2))} = 1. \tag{41}$$

Proof. Sufficiency. Let $\varphi \in \mathcal{L}_+, \psi \in \mathcal{L}^2_+$ and conditions (38), (41) hold. Then we will prove, that for all entire functions $f \in T_\psi$ inequality (7) is satisfied.

We suppose that $\delta > 0$. Then condition (41) implies that

$$(\forall \gamma \in \mathcal{L}^2_+):\quad \varphi(\gamma(x_1, x_2)\psi(x_1, x_2) + \ln\sqrt{x_1 x_2}) < (1+\delta)\varphi(\gamma(x_1, x_2)\psi(x_1, x_2)).$$

as $x_1 > x_1(\gamma),\ x_2 > x_2(\gamma)$. Now by the definition of the function h_δ we have

$$(\forall \gamma \in \mathcal{L}^2_+):\quad \sqrt{x_1 x_2} < h_\delta(\gamma(x_1, x_2)\psi(x_1, x_2)),\ x_1 > x_1(\gamma),\ x_2 > x_2(\gamma).$$

For the function $h(x) = h_\delta(x)$ condition (10) holds. Then by Theorem 1 for all entire functions $f \in T_\psi$ we obtain

$$\mathfrak{M}_f(r_1, r_2) < eM_f(r_1, r_2)h_\delta(\ln M_f(r_1, r_2)),\ r_1 > r_1(\gamma),\ r_2 > r_2(\gamma).$$

From (40) it follows that

$$\varphi(\ln \mathfrak{M}_f(r_1, r_2) - 1) < (1+\delta)\varphi(\ln M_f(r_1, r_2)),\ r_1 > r_1(\gamma),\ r_2 > r_2(\gamma). \tag{42}$$

By (38) and (42) we get

$$1 \leq \varlimsup_{r^\vee \to +\infty} \frac{\varphi(\ln \mathfrak{M}_f(r_1, r_2))}{\varphi(\ln M_f(r_1, r_2))} = \varlimsup_{r^\vee \to +\infty} \frac{\varphi(\ln \mathfrak{M}_f(r_1, r_2) - 1)}{\varphi(\ln M_f(r_1, r_2))} \leq 1 + \delta.$$

Since δ is an arbitrary positive number, then relation (7) holds.
Necessity. Firstly we suppose that condition (41) is not valid, i.e. there exist a function $\forall \gamma \in \mathcal{L}_+^2$, a number $\delta > 0$ and increasing to $+\infty$ sequences $\{x_n\}_{n=0}^{+\infty}$ and $\{y_n\}_{n=0}^{+\infty}$, for which we have

$$\varphi(\gamma(x_k, y_k)\psi(x_k, y_k) + \ln\sqrt{x_k y_k}) > (1+\delta)\varphi(\gamma(x_k, y_k)\psi(x_k, y_k)),\ k \geq 0. \tag{43}$$

We prove, that there exists a function $f \in T_\psi$, for which inequality (7) does not hold. So, by the definition of the function $h_\delta(x)$ one can rewrite inequality (43) in the following form

$$\sqrt{x_k y_k} > h_\delta(\gamma(x_k, y_k)\psi(x_k, y_k)),\ k \geq 0.$$

Now the function $h(x) = e^{-1}h_\delta(x)$ satisfies condition (13). Therefore, by Theorem 2 there exist an entire function $f \in T_\psi$ and increasing to $+\infty$ sequences $\big(r_n^{(1)}\big)$, $\big(r_n^{(2)}\big)$ such that for all $n \geq 0$

$$\mathfrak{M}_f(r_n^{(1)}, r_n^{(2)}) \geq e^{-4}M_f(r_n^{(1)}, r_n^{(2)})h(\ln M_f(r_n^{(1)}, r_n^{(2)})) = e^{-5}M_f(r_n^{(1)}, r_n^{(2)})h_\delta(\ln M_f(r_n^{(1)}, r_n^{(2)})),$$
$$\varphi(\ln \mathfrak{M}_f(r_n^{(1)}, r_n^{(2)}) - 5) > \varphi(\ln M_f(r_n^{(1)}, r_n^{(2)})).$$

It remains to use (38).

$$\varlimsup_{r^\vee \to +\infty} \frac{\varphi(\ln \mathfrak{M}_f(r_1, r_2))}{\varphi(\ln M_f(r_1, r_2))} = \varlimsup_{r^\vee \to +\infty} \frac{\varphi(\ln \mathfrak{M}_f(r_1, r_2) - 5)}{\varphi(\ln M_f(r_1, r_2))} \geq$$
$$\geq \varlimsup_{n \to +\infty} \frac{\varphi\Big(\ln \mathfrak{M}_f\Big(r_n^{(1)}, r_n^{(2)}\Big) - 5\Big)}{\varphi\Big(\ln M_f\Big(r_n^{(1)}, r_n^{(2)}\Big)\Big)} \geq 1 + \delta.$$

Therefore, relation (7) does not hold. □

REFERENCES

1. Brinkmeier H. *Über das Mass der Bestimmtheit des Wachstum einer ganzen transzendenten Funktion durch die absoluten Beträge der Koeffizienten ihrer Potenzreihe*// Math. Ann. – 1926. – B.96, №1. – S. 108–118.

2. Zelisko M.M., Sheremeta M.M. *On influence of coefficients arguments for Dirichlet series on its growth*// Mat. Stud. – 2006. – V.26, №1. – P. 81–85. (in Ukrainian)

3. Filevych P.V. *Wiman-Valiron type inequalities for entire and random entire functions of finite logarithmic order*// Sib. Mat. Zh. – 2001. – V.42, №3. – P. 683–692. (in Russian) English transl. in Sib. Math. J. 2001, V.42, №3, 579–586.

4. Filevych P.V. *On influence of the arguments of coefficients of a power series expansion of an entire function on the growth of the maximum of its modulus*// Sib. Mat. Zh. – 2003. – V.44, №3. – P. 674–685. (in Russian) English transl. in Sib. Math. J. 2003, V.44, №3, 529–538.

5. Kahane J.-P. Séries de Fourier absolument convergentes. – Ergebnisse der Mathematik und ihrer Grenzgebiete. Bd. 50. – Berlin-Heidelberg-New York: Springer Verlag, 1970.

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