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BOUNDEDNESS OF l-INDEX OF ANALYTIC CURVES

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We introduce a concept of an analytic curve of bounded l-index and investigate possible growth of such curves. Moreover, l-index boundedness of analytic curves satisfying linear differential equations is investigated.

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Рассматривается понятие аналитической кривой ограниченного l-индекса, изучается возможный рост таких кривых. Также исследуется ограниченность l-индекса аналитических кривых, удовлетворяющих линейным дифференциальным уравнениям.

Let $0 < R \le +\infty$, $D_R = \{z\colon |z| < R\}$, $m \in \mathbb{N}$ and $F = (f_1, f_2, \ldots, f_m)$ be an analytic curve in D_R, i.e. $F\colon D_R \to \mathbb{C}^m$ is a vector-valued function, where each function f_j is analytic in D_R. We put $F^{(n)} = (f_1^{(n)}, f_2^{(n)}, \ldots, f_m^{(n)})$, and let $\|F(z)\|_S = \max\{|f_j(z)|\colon 1 \le j \le m\}$ be the sup-norm and $\|F(z)\|_E = \sqrt{|f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_m(z)|^2}$ the Euclidean norm.

Let l be a positive continuous function on $[0, R)$ such that

$$l(r) > \frac{\beta}{R-r} \quad (0 \le r < R), \quad \beta > 1. \tag{1}$$

For entire curves ($R = +\infty$) condition (1) is unnecessary.

An analytic curve F is said to be of bounded l-index by sup-norm if there exists $N \in \mathbb{Z}_+$ such that

$$\frac{\|F^{(n)}(z)\|_S}{n!l^n(|z|)} \le \max\left\{\frac{\|F^{(k)}(z)\|_S}{k!l^k(|z|)}\colon 0 \le k \le N\right\} \tag{2}$$

for all $n \in \mathbb{Z}_+$ and $z \in D_R$. The least such integer N is called the l-index by sup-norm and is denoted by $N_S(l; F)$. If we replace $\|F^{(k)}(z)\|_S$ with $\|F^{(k)}(z)\|_E$ then we obtain the concept of an analytic curve in D_R of bounded l-index by the Euclidean norm with l-index $N_E(l; F)$. The entry $\|F(z)\|$ means either $\|F(z)\| = \|F(z)\|_S$ or $\|F(z)\| = \|F(z)\|_E$ and in accordance with this meaning in the definition of the boundedness of l-index we omit the words "by the sup-norm" and "by the Euclidean norm" and denote l-index by $N(l; F)$.

We remark that if $m = 1$ then $\|F(z)\|_S = \|F(z)\|_E = |F(z)| = |f_1(z)|$ and from both definitions we obtain the definition of the boundedness of l-index of an analytic function F in D_R (see, for example [1], pp. 7, 71). We remark also that if $R = +\infty$ and $l(r) \equiv 1$ then

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from the definitions of the boundedness of l-index by the sup-norm and by the Euclidean norm we obtain the definitions of the index boundedness of an analytic curve accordingly to the sup-norm and the Euclidean norm [2–3].

As in [1, p. 71] for $t \in [0, \beta]$ we put

$$\lambda_1(t) = \inf\{l(r)/l(r_0)\colon |r - r_0| \le t/l(r_0), 0 \le r_0 < R\},$$
$$\lambda_2(t) = \sup\{l(r)/l(r_0)\colon |r - r_0| \le t/l(r_0), 0 \le r_0 < R\}$$

and by $Q_\beta(D_R)$ we denote the class of positive continuous functions on $[0, R)$ satisfying conditions (1) and $0 < \lambda_1(t) \le \lambda_2(t) < +\infty$ for all $t \in [0, \beta]$. We remark that if $l \in Q_\beta(D_R)$ and $z_0 \in D_R$ then for all $t \in [0, \beta]$ the inequality $|z - z_0| \le t/l(|z_0|)$ implies the inequalities

$$\lambda_1(t)l(|z_0|) \le l(|z|) \le \lambda_2(t)l(|z_0|). \tag{3}$$

We begin the investigation of the l-index boundedness of analytic curves with a proof of an analog of Hayman's theorem (see [1, p. 20] and [5]) for entire functions of bounded index. Note that in the case $R = \infty$ and $l(r) = r^\nu$, $r > r_0$, R. Roj and S. M. Shah ([4]) consider condition (4) as a definition of a vector entire function of ν-bounded index.

Theorem 1. *Let $\beta > 1$ and $l \in Q_\beta(D_R)$, $0 < R \le +\infty$. An analytic curve F in D_R is of bounded l-index if and only if there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that for all $z \in D_R$*

$$\frac{\|F^{(p+1)}(z)\|}{l^{p+1}(|z|)} \le C \max\left\{\frac{\|F^{(k)}(z)\|}{l^k(|z|)}\colon 0 \le k \le p\right\}. \tag{4}$$

Proof. If $N(l; F) = N < +\infty$ then from (2) we obtain (4) with $p = N$ and $C = (N + 1)!$ that is, the necessity of (4) is obvious.

Now, let (4) hold, $z_0 \in D_R$, $1 \le j \le m$ and $s \in \mathbb{N}$. Then by the Cauchy formula

$$|f_j^{(p+1+s)}(z_0)| = \frac{s!}{2\pi}\left|\int\limits_{|z-z_0|=\beta/l(|z_0|)} \frac{f_j^{(p+1)}(z)}{(z - z_0)^{s+1}}dz\right| \le \frac{s!l^s(|z_0|)}{\beta^s}|f_j^{(p+1)}(z_j^*)|,$$

where z_j^* is a point from $\{z\colon |z-z_0| = \beta/l(|z_0|)\}$ such that $|f_j^{(p+1)}(z_j^*)| = \max\{|f_j^{(p+1)}(z)|\colon |z-z_0| = \beta/l(|z_0|)\}$.

Hence, in view of (3) and (4) we have

$$\frac{|f_j^{(p+1+s)}(z_0)|}{(p+1+s)!l^{p+1+s}(|z_0|)} \le \frac{s!\lambda_2^{p+1}(\beta)}{(p+1+s)!\beta^s}\frac{|f_j^{(p+1)}(z_j^*)|}{l^{p+1}(|z_j^*|)} \le \frac{s!\lambda_2^{p+1}(\beta)}{(p+1+s)!\beta^s}\frac{\|F^{(p+1)}(z_j^*)\|}{l^{p+1}(|z_j^*|)} \le$$
$$\le \frac{s!\lambda_2^{p+1}(\beta)}{(p+1+s)!\beta^s}C\max\left\{\frac{\|F^{(k)}(z_j^*)\|}{l^k(|z_j^*|)}\colon 0 \le k \le p\right\} \le$$
$$\le \frac{Cs!\lambda_2^{p+1}(\beta)}{(p+1+s)!\beta^s\lambda_1^p(\beta)}\max\left\{\frac{\|F^{(k)}(z_j^*)\|}{l^k(|z_0|)}\colon 0 \le k \le p\right\}. \tag{5}$$

On $[0, \beta/l(|z_0|)]$ we consider the function

$$G(x) = \max\left\{\frac{\|F^{(k)}(z_0 + x\alpha)\|}{l^k(|z_0|)}\colon\ 0 \le k \le p\right\},$$

where $\alpha \in \mathbb{C}$ and $|\alpha| = 1$. Since $|f_j^{(k)}(z_0 + x\alpha)|$ is an analytic function of real x, the function G is continuously differentiable on $[0, \beta/l(|z_0|)]$, with exception, perhaps, of countable set of points, and

$$G'(x) \le \begin{cases} \max\limits_{0\le k\le p} \dfrac{\frac{d}{dx}\|F^{(k)}(z_0 + x\alpha)\|_E}{l^k(|z_0|)} \\ \max\limits_{0\le k\le p} \dfrac{\frac{d}{dx}\|F^{(k)}(z_0 + x\alpha)\|_S}{l^k(|z_0|)} \end{cases} \le \begin{cases} \max\limits_{0\le k\le p} \dfrac{\sum\limits_{j=1}^{m} |f_i^{(k)}(z_0 + x\alpha)||f_j^{(k+1)}(z_0 + x\alpha)|}{l^k(|z_0|)\|F^{(k)}(z_0 + x\alpha)\|_E} \\ \max\limits_{0\le k\le p} \dfrac{\max\{|f_j^{(k+1)}(z_0 + x\alpha)|:\ 1 \le j \le m\}}{l^k(|z_0|)} \end{cases} \le$$

$$\le \begin{cases} ml(|z_0|) \max\limits_{0\le k\le p} \dfrac{\|F^{(k+1)}(z_0 + x\alpha)\|_E}{l^{k+1}(|z_0|)} \\ l(|z_0|) \max\limits_{0\le k\le p} \dfrac{\|F^{(k+1)}(z_0 + x\alpha)\|_S}{l^{k+1}(|z_0|)} \end{cases} \le ml(|z_0|) \max\left\{ \frac{\|F^{(p+1)}(z_0 + x\alpha)\|}{l^{p+1}(|z_0|)}, G(x) \right\} \le$$

$$\le ml(|z_0|) \max\left\{ \lambda_2^{p+1}(\beta) \frac{\|F^{(p+1)}(z_0 + x\alpha)\|}{l^{p+1}(|z_0 + x\alpha|)}, G(x) \right\} \le$$

$$\le ml(|z_0|) \max\left\{ G(x), C\lambda_2^{p+1}(\beta) \max\left\{ \frac{\|F^{(k)}(z_0 + x\alpha)\|}{l^k(|z_0 + x\alpha|)} : 0 \le k \le p \right\} \right\} \le$$

$$\le ml(|z_0|) \max\left\{ G(x), \frac{C\lambda_2^{p+1}(\beta)}{\lambda_1^p(\beta)} \max\left\{ \frac{\|F^{(k)}(z_0 + x\alpha)\|}{l^k(|z_0|)} : 0 \le k \le p \right\} \right\} =$$

$$= ml(|z_0|) \max\left\{ 1, \frac{C\lambda_2^{p+1}(\beta)}{\lambda_1^p(\beta)} \right\} G(x),$$

whence for all $x \in [0, \beta/l(|z_0|)]$

$$G(x) \le G(0) \exp\{K_1 l(|z_0|)x\}, \quad K_1 = m \max\{1, C\lambda_2^{p+1}(\beta)\lambda_1^{-p}(\beta)\}$$

and, thus,

$$\max\left\{ \frac{\|F^{(k)}(z_j^*)\|}{l^k(|z_0|)} : 0 \le k \le p \right\} \le e^{K_1\beta} \max\left\{ \frac{\|F^{(k)}(z_0)\|}{l^k(|z_0|)} : 0 \le k \le p \right\}.$$

Therefore, from (5) for all $1 \le j \le m$ and $s \in \mathbb{N}$ we obtain

$$\frac{|f_j^{(p+1+s)}(z_0)|}{(p+1+s)!l^{p+1+s}(|z_0|)} \le \frac{K_2 s!}{(p+1+s)!\beta^s} \max\left\{ \frac{\|F^{(k)}(z_0)\|}{l^k(|z_0|)} : 0 \le k \le p \right\},$$

where $K_2 = \frac{C\lambda_2^{p+1}(\beta)e^{K_1\beta}}{\lambda_1^p(\beta)}$. Since $\|F^{(k)}(z_0)\|_E \le \sqrt{m}\|F^{(k)}(z_0)\|_S = \sqrt{m} \max\limits_{1\le j\le m} |f_j^{(k)}(z_0)|$ and $K_2 s!p!\sqrt{m} \le (p+1+s)!\beta^s$ for $s \ge s_0$, hence it follows that for all $s \ge s_0$

$$\frac{\|F^{(p+1+s)}(z_0)\|}{(p+1+s)!l^{p+1+s}(|z_0|)} \le \frac{K_2 s!p!\sqrt{m}}{(p+1+s)!\beta^s} \max\left\{ \frac{\|F^{(k)}(z_0)\|}{k!l^k(|z_0|)} : 0 \le k \le p \right\} \le$$

$$\le \max\left\{ \frac{\|F^{(k)}(z_0)\|}{k!l^k(|z_0|)} : 0 \le k \le p \right\},$$

that is, $N(l; F) \le p + 1 + s_0$. Theorem 1 is proved. □

Corollary 1. *Let $\beta > 1$ and $l \in Q_\beta(D_R)$, $0 < R \le +\infty$. An analytic curve F in D_R is of bounded l-index by the sup-norm if and only if F is of bounded l-index by the Euclidean norm.*

Proof. The assertion follows from Theorem 1 (4) for both norms and inequalities

$$\|F^{(k)}(z)\|_S \le \|F^{(k)}(z)\|_E \le \sqrt{m}\|F^{(k)}(z)\|_S. \qquad \square$$

Between the boundedness of l-index of an analytic curve and the boundedness of l-index of its components the following connection exists.

Proposition 1. *Let l be a positive continuous function on $[0,R)$ satisfying condition (1) and each component f_j of an analytic curve F in D_R is of bounded l-index $N(l;f_j)$. Then F is of bounded l-index by the sup-norm with $N_S(l;F) \le \max\{N(l;f_j)\colon 1 \le j \le m\}$ and F is of bounded l_*-index by the Euclidean norm with $l_*(r) = \sqrt{m}l(r)$ and $N_S(l_*;F) \le \max\{N(l;f_j)\colon\ 1 \le j \le m\}$.*

Proof. For all $n \ge N = \max\{N(l;f_j)\colon 1 \le j \le m\}$ we have

$$\frac{\|F^{(n)}(z)\|_S}{n!l^n(|z|)} = \frac{\max\{|f_j^{(n)}(z)|\colon 1 \le j \le m\}}{n!l^n(|z|)} \le$$
$$\le \max\left\{\frac{|f_j^{(k)}(z)|}{k!l^k(|z|)}\colon 0 \le k \le N, 1 \le j \le m\right\} = \max\left\{\frac{\|F^{(k)}(z)\|_S}{k!l^k(|z|)}\colon 0 \le k \le N\right\},$$

that is $N_S(l;F) \le N = \max\{N(l;f_j)\colon 1 \le j \le m\}$. Also

$$\frac{\|F^{(n)}(z)\|_E}{n!l^n(|z|)} = \sqrt{\sum_{j=1}^m \left(\frac{|f_j^{(n)}(z)|}{n!l^n(|z|)}\right)^2} \le \sqrt{\sum_{j=1}^m \left(\max\left\{\frac{|f_j^{(k)}(z)|}{k!l^k(|z|)}\colon 0 \le k \le N\right\}\right)^2} \le$$
$$\le \sqrt{m}\max\left\{\frac{|f_j^{(k)}(z)|}{k!l^k(|z|)}\colon 0 \le k \le N, 1 \le j \le m\right\} \le \sqrt{m}\max\left\{\frac{\|F^{(k)}(z)\|_E}{k!l^k(|z|)}\colon 0 \le k \le N\right\}$$

and, thus, for $n \ge N+1$

$$\frac{\|F^{(n)}(z)\|_E}{n!l_*^n(|z|)} \le \frac{\|F^{(n)}(z)\|_E}{n!l^n(|z|)}\frac{1}{(\sqrt{m})^{N+1}} \le \frac{1}{(\sqrt{m})^N}\max\left\{\frac{\|F^{(k)}(z)\|_E}{k!l^k(|z|)}\colon 0 \le k \le N\right\} \le$$
$$\le \max\left\{\frac{\|F^{(k)}(z)\|_E}{k!l_*^k(|z|)}\colon 0 \le k \le N\right\},$$

that is $N_S(l_*;F) \le N = \max\{N(l;f_j)\colon 1 \le j \le m\}$. Proposition 1 is proved. $\square$

Remark 1. If F is of bounded l-index then its components need not be of bounded l-index. For $l(r) \equiv 1$ and entire curves the proposition is proved in [2–3].

The following theorem describes a possible growth of analytic curves in D_R of bounded l-index.

Theorem 2. *Let $0 < R \le +\infty$ and l be a positive analytic function of real $t \in [0, R)$ satisfying (1). Suppose that $(-l'(t))^+ = o(l^2(t)), t \uparrow R$, where $a^+ = \max\{a, 0\}$. If an analytic curve F in D_R is of bounded l-index then*

$$\varlimsup_{r\uparrow R} \frac{\ln M(r, \|F\|_S)}{L(r)} \le N_S(l; F) + 1, \tag{6}$$

where $M(r, \|F\|_S) = \max\{\|F(z)\|_S \colon |z| = r\}$ and $L(r) = \int\limits_0^r l(t)dt$.

Proof. We put $N = N_S(l; F)$ and for fixed $\theta \in [0, 2\pi]$ we consider the function

$$G(t) = \max\left\{\frac{\|F^{(k)}(te^{i\theta})\|_S}{l^k(t)} \colon 0 \le k \le N\right\}, \quad t \in [0, R).$$

Since

$$G(t) = \max\left\{\frac{|f_j^{(k)}(te^{i\theta})|}{l^k(t)} \colon 0 \le k \le N, 1 \le j \le m\right\},$$

as in the proof of Theorem 3.4 from [1, p. 73], we obtain the inequality

$$G(r) \le G(0)\exp\{(N + 1)(1 + o(1))L(r)\}, \quad r \uparrow R.$$

Therefore, for every $\theta \in [0, 2\pi]$

$$\|F(re^{i\theta})\|_S \le G(r) \le G(0)\exp\{(N + 1)(1 + o(1))L(r)\}, \quad r \uparrow R,$$

whence we obtain (6). Theorem 2 is proved. □

Note that for an entire curve of ν-bounded index Roy and Shah [4] obtained the estimate

$$\varlimsup_{r\to\infty} \frac{\ln M(r, \|F\|_S)}{r^{\nu+1}} \le \max\{1, C\},$$

where C is the number used in (4) for sup-norm.

Finally, we use Theorem 1 to the study of l-index boundedness of analytic solutions of differential equations, but at first we remark that Theorem 5.1 from [1, p.88] implies the following proposition.

Proposition 2. *Let $\beta > 1$, $l \in Q_\beta(D_R)$, $0 < R \le +\infty$ and h be an analytic in D_R function of bounded l-index. Suppose that an analytic in D_R function f satisfies the differential equation $w^{(n)} + q_1 w^{(n-1)} + \dots + q_n w = h(z)$ with constant coefficients q_s. If $|q_s| \le Tl^s(|z|)$, $T \equiv const > 0$, for all $1 \le s \le n$ and $z \in D_R$ then f is of bounded l-index.*

Remark 2. If $R < +\infty$ then (1) implies $|q_s| \le Tl^s(|z|)$ for all $1 \le s \le n$ and $z \in D_R$. If $R = +\infty$ then $|q_s| \le Tl^s(|z|)$ for all $1 \le s \le n$ and $z \in \mathbb{C}$ provided $l(r) \ge \beta > 0$ for all $r \in [0, +\infty)$. Finally, if $\varliminf\limits_{r\to+\infty} l(r) = 0$ then the condition $|q_s| \le Tl^s(|z|)$ implies $q_s = 0$ for all $1 \le s \le n$ and the differential equation has the form $w^{(n)} = h(z)$. For an entire solution f of this equation $N(l; f) \le N(l; h) + n$.

Now we consider the vector equation

$$W^{(n)} + Q_1 W^{(n-1)} + \dots + Q_n W = H(z), \tag{7}$$

where

$$H(z) = \begin{pmatrix} h_1(z) \\ \dots \\ h_m(z) \end{pmatrix}, \quad W = \begin{pmatrix} w_1 \\ \dots \\ w_m \end{pmatrix}, \quad Q_s = \begin{pmatrix} q_{11}^{(s)} \dots q_{1m}^{(s)} \\ \dots\dots\dots \\ q_{m1}^{(s)} \dots q_{mm}^{(s)} \end{pmatrix}$$

and all $q_{ik}^{(s)}$ are constant numbers.

Theorem 3. *Let $\beta > 1$, $l \in Q_\beta(D_R)$ and $l(r) \geq \tilde{\beta} > 0$ for all $r \in [0, +\infty)$ in the case of $R = +\infty$. Suppose that an analytic in D_R curve H is of bounded l-index. Then an analytic in D_R solution $F = \begin{pmatrix} f_1 \\ \dots \\ f_m \end{pmatrix}$ of equation (7) is a curve of bounded l-index.*

Proof. By Theorem 1 there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that for all $z \in D_R$

$$\frac{\|H^{(p+1)}(z)\|_S}{l^{p+1}(|z|)} \leq C \max\left\{ \frac{\|H^{(k)}(z)\|_S}{l^k(|z|)} : 0 \leq k \leq p \right\}. \tag{8}$$

If F satisfies (7) then $H^{(k)}(z) \equiv F^{(n+k)}(z) + Q_1 F^{(n+k-1)}(z) + \dots + Q_n F^{(k)}(z)$ and, therefore, from (8) we obtain

$$\frac{\|F^{(n+p+1)}(z)\|_S - \|Q_1 F^{(n+p)}(z) + \dots + Q_n F^{(p+1)}(z)\|_S}{l^{p+1}(|z|)} \leq$$
$$\leq \frac{\|F^{(n+p+1)}(z) + Q_1 F^{(n+p)}(z) + \dots + Q_n F^{(p+1)}(z)\|_S}{l^{p+1}(|z|)} \leq$$
$$\leq C \max\left\{ \frac{\|F^{(n+k)}(z) + Q_1 F^{(n+k-1)}(z) + \dots + Q_n F^{(k)}(z)\|_S}{l^k(|z|)} : 0 \leq k \leq p \right\}.$$

If we denote $q = \max\{|q_{ik}^{(s)}| : 1 \leq s \leq n, 1 \leq k \leq m, 1 \leq i \leq m\}$ and $B = \sum_{j=1}^{n} (\frac{R}{\beta})^j$ if $R \neq +\infty$, $B = \sum_{j=1}^{n} (\tilde{\beta})^{-j}$ if $R = +\infty$, then hence we have

$$\frac{\|F^{(n+p+1)}(z)\|_S}{l^{n+p+1}(|z|)} \leq \frac{mq(\|F^{(n+p)}(z)\|_S + \dots + \|F^{(p+1)}(z)\|_S)}{l^{n+p+1}(|z|)} +$$
$$+ C \max\left\{ \frac{\|F^{(n+k)}(z)\|_S}{l^{n+k}(|z|)} + \frac{mq(\|F^{(n+k-1)}(z)\|_S + \dots + \|F^{(k)}(z)\|_S)}{l^{n+k}(|z|)} : 0 \leq k \leq p \right\} \leq$$
$$\leq mqB \max\left\{ \frac{\|F^{(k)}(z)\|_S}{l^k(|z|)} : p+1 \leq k \leq p+n \right\} + C \max\left\{ \frac{\|F^{(n+k)}(z)\|_S}{l^{n+k}(|z|)} : 0 \leq k \leq p \right\} +$$
$$+ C \max\left\{ mqB \max\left\{ \frac{\|F^{(j)}(z)\|_S}{l^j(|z|)} : k \leq j \leq n+k-1 \right\} : 0 \leq k \leq p \right\} \leq$$
$$\leq ((C+1)mqB + C) \max\left\{ \frac{\|F^{(k)}(z)\|_S}{l^k(|z|)} : 0 \leq k \leq p+n \right\},$$

whence by Theorem 1 it follows that F is of bounded l-index. Theorem 3 is proved. □

The boundedness of index and ν-index by the sup-norm of entire curves satisfying differential equation (7), where Q_s are matrices whose entries are rational functions, is considered in [3–4]. We propose some results concerning analytic curves in D_R satisfying differential equation with meromorphic coefficients.

Let $\beta > 1$, $l \in Q_\beta(D_R)$, $S = \{c_k\}$ be a set of points from D_R, $n(r, z_0, S) = \sum_{|c_k - z_0| \le r} 1$, $r > 0$, and $G_r(S) = \bigcup_k \left\{z\colon\ |z - c_k| \le \frac{r}{l(|c_k|)}\right\}$, $r \in (0, \beta]$. A set S will be called an l-set if for every $r \in (0, \beta]$ there exists $\tilde{n}(r) \in \mathbb{Z}_+$ such that $n(\frac{r}{l(|z_0|)}, z_0, S) \le \tilde{n}(r)$ for each $z_0 \in D_R$.

For an analytic in D_R function f let S_f be the set of its zeros.

We need the following lemmas.

Lemma 1 ([1, p.27]). *Let $\beta > 1$, $l \in Q_\beta(D_R)$. If an analytic in D_R function $f(z)$ is of bounded l-index then S_f is an l-set.*

Lemma 2 ([1, p.87]). *Let $\beta > 1$, $l \in Q_\beta(D_R)$. If an analytic in D_R function $f(z)$ is of bounded l-index then for every $r \in (0, \beta]$ and every $m \in \mathbb{N}$ there exists $P = P(r) > 0$ such that for each $z \in D_R \setminus G_r(S_f)$*

$$|f^{(m)}(z)| \le P l^m(|z|)|f(z)|. \tag{9}$$

Lemma 3 ([1, p.88–92]). *Let $\beta > 1$, $l \in Q_\beta(D_R)$, let S be an l-set and $p \in \mathbb{Z}_+$. If for an analytic in D_R function f for every $r \in (0, \beta]$ there exists $P = P(r) > 0$ such that for all $z \in D_R \setminus G_r(S)$*

$$\frac{|f^{(p+1)}(z)|}{l^{p+1}(|z|)} \le P \max\left\{\frac{|f^{(k)}(z)|}{l^k(|z|)} \colon 0 \le k \le p\right\} \tag{10}$$

then f is of bounded l-index.

The last lemma is proved, in fact, in the second part of the proof of Theorem 5.1 ([1]).

Theorem 4. *Let $\beta > 1$, $l \in Q_\beta(D_R)$. Assume that an analytic in D_R curve $F(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}$ satisfies the differential equation*

$$W' + Q(z)W = 0, \tag{11}$$

where $Q(z) = \begin{pmatrix} a_1(z) & a_2(z) \\ a_3(z) & a_4(z) \end{pmatrix}$, and a_j, $j = \overline{1,4}$, are meromorphic functions of the form $a_j = \frac{A_j}{B_j}$, where A_j, B_j are analytic in D_R functions of bounded l-index and for every $r \in (0, \beta]$ there exists $M = M(r) > 0$ such that for each $z \in D_R \setminus \bigcup_{j=1}^{4} G_r(S_{B_j})$

$$|a_j(z)| \le M l(|z|). \tag{12}$$

Then f_1, f_2, and, therefore, F are of bounded l-index in D_R.

Proof. We write (11) as

$$f_1' + a_1 f_1 + a_2 f_2 = 0, \tag{13}$$

$$f_2' + a_3 f_1 + a_4 f_2 = 0 \tag{14}$$

We differentiate (13) and then use f_2' from (14) and f_2 from (13) and get

$$f_1'' + \left(a_1 + a_4 - \frac{a_2'}{a_2}\right) f_1' + \left(a_1' + a_1a_4 - a_2a_3 - a_1\frac{a_2'}{a_2}\right) f_1 = 0 \tag{15}$$

Let $S = \bigcup\limits_{j=1}^{4}(S_{A_j} \bigcup S_{B_j})$, $r \in (0,\beta]$. We can assume that $D_R \setminus G_r(S) \neq \varnothing$. We estimate coefficients in (15) for $z \in D_R \setminus G_r(S)$. In view of (12) and Lemma 1 for A_j and B_j we have

$$|a_1'(z)| = \left|\frac{A_1'(z)}{B_1(z)} - \frac{A_1(z)B_1'(z)}{B_1^2(z)}\right| \leq \left|\frac{A_1'(z)}{A_1(z)}\frac{A_1(z)}{B_1(z)}\right| + \left|\frac{A_1(z)}{B_1(z)}\frac{B_1'(z)}{B_1(z)}\right| \leq 2PMl^2(|z|),$$

$$\left|\frac{a_2'(z)}{a_2(z)}\right| = \left|\frac{A_2'(z)}{A_2(z)} - \frac{B_2'(z)}{B_2(z)}\right| \leq 2Pl(|z|).$$

Thus, for every $r \in (0,\beta]$ there exists $M^* = M^*(r) > 0$ such that for each $z \in D_R \setminus G_r(S)$

$$\frac{|f_1''(z)|}{l^2(|z|)} \leq (2M+2P)\frac{|f_1'(z)|}{l(|z|)} + (4MP + 2M^2)|f_1(z)| \leq M^* \max\left\{\frac{|f_1^{(k)}(z)|}{l^k(|z|)} : 0 \leq k \leq 1\right\}.$$

Since S is an l-set as a union of l-sets by Lemma 1, therefore by Lemma 3 we obtain the boundedness of l-index of f_1 (and, by analogy, f_2) in D_R. The proof is complete. □

Theorem 5. *Let $\beta > 1$, $l \in Q_\beta(D_R)$ and $l(r) \geq \tilde{\beta} > 0$ for all $r \in [0,+\infty)$ in the case of $R = +\infty$. Assume that an analytic in D_R curve $F(z) = \begin{pmatrix} f_1(z) \\ f_2(z)\end{pmatrix}$ satisfies the differential equation*

$$W'' + Q_1(z)W' + Q_2(z)W = 0, \tag{16}$$

where $Q_1(z) = \begin{pmatrix} a_1(z) & a_2(z) \\ a_3(z) & a_4(z)\end{pmatrix}$, $Q_2(z) = \begin{pmatrix} a_5(z) & 0 \\ 0 & a_6(z)\end{pmatrix}$, and a_j, $j = \overline{1,6}$, are meromorphic functions of the form $a_j = \frac{A_j}{B_j}$, where A_j, B_j are analytic in D_R functions of bounded l-index and for every $r \in (0,\beta]$ there exists $M = M(r) > 0$ such that for each $z \in D_R \setminus \bigcup\limits_{j=1}^{6} G_r(S_{B_j})$

$$|a_j(z)| \leq Ml(|z|). \tag{17}$$

Then f_1, f_2, and, therefore, F are of bounded l-index in D_R.

Proof. We write (16) as

$$f_1'' + a_1f_1' + a_2f_2' + a_5f_1 = 0, \tag{18}$$
$$f_2'' + a_3f_1' + a_4f_2' + a_6f_2 = 0 \tag{19}$$

Let

$$D_2(f_1) = f_1'' + a_1f_1' + a_5f_1 = -a_2f_2'.$$

We differentiate $D_2(f_1)$ and then use f_2'' from (19) and get

$$f_1''' + a_1f_1'' + (a_1' + a_5 - a_2a_3)\, f_1' + a_5'f_1 = (a_2a_4 - a_2')f_2' + a_2a_6f_2 \tag{20}$$

Using f_2' from $D_2(f_1)$ we write (20) as

$$D_3(f_1) = f_1''' + g_2 f_1'' + g_1 f_1' + g_0 f_1 = a_2 a_6 f_2,$$

where

$$g_2 = a_1 + a_4 - \frac{a_2'}{a_2},\ g_1 = a_1' + a_5 - a_2 a_3 + a_1 a_4 - \frac{a_2'}{a_2} a_1,\ g_0 = a_5' + a_4 a_5 - \frac{a_2'}{a_2} a_5.$$

We differentiate $D_3(f_1)$

$$f_1^{IV} + g_2 f_1''' + (g_2' + g_1) f_1'' + (g_1' + g_0) f_1' + g_0' f_1 = a_2 a_6 f_2' + (a_2 a_6)' f_2,$$

and remark that

$$a_2 f_2' = -D_2(f_1), \quad (a_2 a_6)' f_2 = \left(\frac{a_2'}{a_2} + \frac{a_6'}{a_6}\right) a_2 a_6 f_2 = \left(\frac{a_2'}{a_2} + \frac{a_6'}{a_6}\right) D_3(f_1),$$

we get, finally,

$$D_4(f_1) = f_1^{IV} + h_3 f_1''' + h_2 f_1'' + h_1 f_1' + h_0 f_1 = 0, \tag{21}$$

where

$$h_3 = a_1 + a_4 - 2\frac{a_2'}{a_2} - \frac{a_6'}{a_6},$$
$$h_2 = 2a_1' + a_4' - \frac{a_2''}{a_2} + \frac{a_2'^2}{a_2^2} + a_5 + a_6 - a_2 a_3 + a_1 a_4 - \frac{a_2'}{a_2} a_1 - \left(a_1 + a_4 - \frac{a_2'}{a_2}\right)\left(\frac{a_2'}{a_2} + \frac{a_6'}{a_6}\right),$$
$$h_1 = a_1'' + a_4 a_5 + 2a_5' - a_2' a_3 - a_2 a_3' + a_1' a_4 + a_1 a_4' - \left(\frac{a_2''}{a_2} - \frac{a_2'^2}{a_2^2}\right) a_1 - \frac{a_2'}{a_2} a_1' - \frac{a_2'}{a_2} a_5 + a_1 a_6 +$$
$$+ \left(a_1' + a_5 - a_2 a_3 + a_1 a_4 - \frac{a_2'}{a_2}\right)\left(\frac{a_2'}{a_2} + \frac{a_6'}{a_6}\right),$$
$$h_0 = a_5'' + a_4' a_5 + a_4 a_5' + a_5 a_6 - \frac{a_2''}{a_2} a_5 + \frac{a_2'^2}{a_2^2} a_5 - \frac{a_2'}{a_2} a_5' - \left(a_5' + a_4 a_5 - \frac{a_2'}{a_2} a_5\right)\left(\frac{a_2'}{a_2} + \frac{a_6'}{a_6}\right).$$

Let $S = \cup_{j=1}^{6}(S_{A_j} \cup S_{B_j})$, $r \in (0, \beta]$. We may assume that $D_R \setminus G_r(S) \neq \varnothing$. As in the proof of Theorem 4 for every $r \in (0, \beta]$ there exists $M = M(r) > 0$ such that for each $z \in D_R \setminus G_r(S)$ and $j = \overline{1, 6}$

$$|a_j(z)| \le M l(|z|), \quad |a_j'(z)| \le M l^2(|z|), \quad \left|\frac{a_j'(z)}{a_j(z)}\right| \le M l(|z|)$$

and, similarly,

$$|a_j''(z)| = \left|\frac{A_j''(z) A_j(z)}{A_j(z) B_j(z)} - 2\frac{A_j'(z) B_j'(z)}{B_j^2(z)} - \frac{A_j(z) B_j''(z)}{B_j^2(z)} + 2\frac{A_j(z) B_j'^2(z)}{B_j^3(z)}\right| \le (2M^3 + 4M^2) l^3(|z|),$$
$$\left|\frac{a_j''(z)}{a_j(z)}\right| = \left|\frac{B_j(z)}{A_j(z)}\left(\frac{A_j(z)}{B_j(z)}\right)''\right| \le \left|\frac{A_j''(z)}{A_j(z)}\right| + 2\left|\frac{A_j'(z) B_j'(z)}{A_j(z) B_j(z)}\right| + \left|\frac{B_j''(z)}{B_j(z)}\right| + 2\left|\frac{B_j'^2(z)}{B_j^2(z)}\right| \le$$
$$\le (4M^2 + 2M) l^2(|z|).$$

So long as $\dfrac{Rl(|z|)}{\beta}\geq 1$ if $R\neq +\infty$ and $\dfrac{l(|z|)}{\tilde\beta}\geq 1$ if $R=+\infty$, so

$$|h_3(z)|\leq M_1l(|z|),\quad |h_2(z)|\leq M_2l^2(|z|),\quad |h_1(z)|\leq M_3l^3(|z|),\quad |h_0(z)|\leq M_4l^4(|z|).$$

Thus (21) implies (10) with $p=3$ and we complete the proof by the same way as in Theorem 4. □

Remark 3. It is well known that the sum of functions of bounded l-index need not be of bounded l-index. Using Lemma 3, Theorem 5.1 from [1] in homogeneous case can be reformulated in broader terms (linear combination of functions instead of one of the coefficients).

Proposition 3. *Let $\beta>1$, $l\in Q_\beta(D_R)$ and let $g_0, g_{11},\dots,g_{1s_1},\dots,g_{n1},\dots,g_{ns_n}$ be analytic functions of bounded l-index in D_R. Suppose that for every $r\in(0,\beta]$ there exists $T=T(r)>0$ such that for each $z\in D_R\setminus G_r(S_{g_0})$ and $m=1,2,\dots,n$*

$$|g_{mk}(z)|\leq Tl^m(|z|)|g_o(z)|,\quad k=1,2,\dots,s_m.$$

Then an analytic function f satisfying the differential equation

$$g_0w^{(n)}+\sum_{k=1}^{s_1}\lambda_{1k}g_{1k}w^{(n-1)}+\dots+\sum_{k=1}^{s_n}\lambda_{nk}g_{nk}w=0,$$

where $\lambda_{mk}\in\mathbb{C}$, is of bounded l-index in D_R.

The proof is similar to that of Theorem 4.

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