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ON ENTIRE SOLUTIONS WITH A TWO-MEMBER RECURRENT FORMULA FOR TAYLOR'S COEFFICIENTS OF LINEAR DIFFERENTIAL EQUATIONS

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It is proved that the differential equation

$$z^n w^{(n)} + (a_1^{(n-1)} z + a_2^{(n-1)}) z^{n-1} w^{(n-1)} + \sum_{k=0}^{n-2} (a_{n-1-k}^{(k)} z^2 + a_{n-k}^{(k)} z + a_{n+1-k}^{(k)}) z^k w^{(k)} = 0$$

has an entire solution f with a two-member recurrent formula for its Taylor's coefficients. The growth of such function f is studied. The conditions for coefficients $a_k^{(j)}$ are obtained, under which the solution f is convex or close-to-convex in $\mathbb{D} = \{z : |z| < 1\}$.

Я. С. Магола. *О целых решениях с двухчленной рекуррентной формулой для тейлоровских коэффициентов линейных дифференциальных уравнений* // Мат. Студії. – 2011. – Т.36, №2. – C.133–141.

Доказано, что дифференциальное уравнение

$$z^n w^{(n)} + (a_1^{(n-1)} z + a_2^{(n-1)}) z^{n-1} w^{(n-1)} + \sum_{k=0}^{n-2} (a_{n-1-k}^{(k)} z^2 + a_{n-k}^{(k)} z + a_{n+1-k}^{(k)}) z^k w^{(k)} = 0$$

имеет целое решение f с двухчленной рекуррентной формулой для тейлоровских коэффициентов. Изучен рост такой функции f. Указаны условия на параметры $a_k^{(j)}$, при выполнении которых такое решение f является выпуклой или близкой к выпуклой в $\mathbb{D} = \{z : |z| < 1\}$ функцией.

1. Introduction. A function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \qquad (1)$$

analytic and univalent in $\mathbb{D} = \{z\colon |z| < 1\}$ is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [1, p. 38], that the condition $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for convexity of f in $\mathbb{D}$. A function f is said to be close-to-convex in $\mathbb{D}$ [1, p. 64] if there exists a function Φ convex in $\mathbb{D}$ such that $\mathrm{Re}\{f'(z)/\Phi'(z)\} > 0$ $(z \in \mathbb{D})$. Every function close-to-convex in $\mathbb{D}$ is univalent in $\mathbb{D}$ [1, p. 64] and $f_1 \neq 0$. A function f close-to-convex in $\mathbb{D}$ has the characteristic property that the complement G of $f(\mathbb{D})$ can be filled with rays L which go from $\partial f(\mathbb{D})$ and lie in G [1, p. 71]. Since $f_1 \neq 0$, it follows that the function (1) is

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close-to-convex in $\mathbb{D}$ if and only if the function $\tilde{f}(z) = z + \sum_{n=2}^{\infty} (f_n/f_1)z^n$ is close-to-convex in $\mathbb{D}$.

S. M. Shah [2, 3] studied properties of entire solutions of the differential equation

$$z^2 w'' + (a_1^{(1)} z^2 + a_2^{(1)} z)w' + (a_1^{(0)} z^2 + a_2^{(0)} z + a_3^{(0)})w = 0. \tag{2}$$

In particular, he obtained [3] the conditions under which entire solutions with a one-member recurrent formula for Taylor's coefficients of differential equation (2) is a function close-to-convex in $\mathbb{D}$.

The general case of a two-member recurrent formula in a number of papers is investigated by Z. M. Sheremeta [4–7] and M. M. Sheremeta with Z. M. Sheremeta [8]. Particularly, in the case when the parameters $a_k^{(j)}$ are complex, they have obtained the following result.

Theorem A. *Let $a_2^{(1)} + a_3^{(0)} = 0$, $|a_2^{(1)}| < 2$ and*

$$2\frac{|a_1^{(1)}| + |a_2^{(0)}|}{2 - |a_2^{(1)}|} + \frac{3|a_1^{(0)}|}{2(3 - |a_2^{(1)}|)} + \frac{6|a_1^{(1)}| + 3|a_2^{(0)}|}{4(3 - |a_2^{(1)}|)} + \frac{2|a_1^{(0)}|}{3(4 - |a_2^{(1)}|)} < 1.$$

Then differential equation (2) has an entire solution $f(z) = z + \sum_{s=2}^{\infty} f_s z^s$ which is a function close-to-convex in $\mathbb{D}$ and $\ln M_f(r) = (1 + o(1))\sigma r$, $r \to \infty$, where either

$$\sigma = \sigma_1 := \frac{1}{2}\left| -a_1^{(1)} + \sqrt{(a_1^{(1)})^2 - 4a_1^{(0)}} \right|,$$

or

$$\sigma = \sigma_2 := \frac{1}{2}\left| -a_1^{(1)} - \sqrt{(a_1^{(1)})^2 - 4a_1^{(0)}} \right|.$$

The straightforward generalization of Shah's equation is the differential equation

$$z^n w^{(n)} + \sum_{j=1}^{n} \left(\sum_{k=1}^{j+1} a_k^{(n-j)} z^{n-k+1} \right) w^{(n-j)} = 0. \tag{3}$$

The following theorem is proved in [9].

Theorem B. *A function (1) analytic at the origin is a solution of differential equation (3) if and only if for each $s \in \mathbb{Z}_+$*

$$\sum_{m=0}^{\min\{s,n\}} \sum_{k=0}^{\min\{s,n\}-m} a_{n+1-k-m}^{(k)} \frac{(s-m)!}{(s-k-m)!} f_{s-m} = 0, \tag{4}$$

where $a_1^{(n)} = 1$.

In the case when formula (4) reduces to a one-member recurrent formula for two neighboring coefficients f_s in [9, 10] it is investigated convexity, close-to-convexity in $\mathbb{D}$ and possible growth of a function f. In [11] it is studied the case of two non-neighboring coefficients f_s. Here we consider the conditions under which the function f has the same properties in the case when formula (4) reduces to a two-member recurrent formula for neighboring coefficients. Further we assume that $n \geq 3$.

We may rewrite differential equation (3) in the form

$$\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{n+1-k-m}^{(k)} z^{k+m} w^{(k)} = 0. \tag{5}$$

Let $a_{n+1-k-m}^{(k)} = 0$ for $m = \overline{3,n}$ and $k = \overline{0,n-m}$. Then differential equation (5) takes the following form

$$\sum_{k=0}^{n} a_{n+1-k}^{(k)} z^k w^{(k)} + \sum_{k=0}^{n-1} a_{n-k}^{(k)} z^{k+1} w^{(k)} + \sum_{k=0}^{n-2} a_{n-1-k}^{(k)} z^{k+2} w^{(k)} = 0,$$

that is

$$z^n w^{(n)} + (a_1^{(n-1)} z + a_2^{(n-1)}) z^{n-1} w^{(n-1)} + \sum_{k=0}^{n-2} (a_{n-1-k}^{(k)} z^2 + a_{n-k}^{(k)} z + a_{n+1-k}^{(k)}) z^k w^{(k)} = 0. \tag{6}$$

Proposition 1. *Let $n \geq 3$. A function (1) analytic at the origin is a solution of differential equation (6) if and only if*

$$a_{n+1}^{(0)} f_0 = 0, \quad (a_{n+1}^{(0)} + a_n^{(1)}) f_1 + a_n^{(0)} f_0 = 0 \tag{7}$$

and for all $s \geq 2$

$$\sum_{k=0}^{\min\{s,n\}} a_{n+1-k}^{(k)} \frac{s!}{(s-k)!} f_s + \sum_{k=0}^{\min\{s,n\}-1} a_{n-k}^{(k)} \frac{(s-1)!}{(s-k-1)!} f_{s-1} + \\ + \sum_{k=0}^{\min\{s,n\}-2} a_{n-1-k}^{(k)} \frac{(s-2)!}{(s-k-2)!} f_{s-2} = 0, \tag{8}$$

where $a_1^{(n)} = 1$.

Indeed, if $s = 0$ and $s = 1$ then from (4) we obtain (7). On the other hand, if $s \geq 2$ then from (4) in view of $a_{n+1-k-m}^{(k)} = 0$ for $m = \overline{3,n}$ and $k = \overline{0,n-m}$ we obtain (8).

Assuming that for all $s \geq 2$

$$\sum_{k=0}^{\min\{s,n\}} a_{n+1-k}^{(k)} \frac{s!}{(s-k)!} \neq 0$$

we may rewrite recurrent formula (8) in the form

$$f_s = -\frac{\displaystyle\sum_{k=0}^{\min\{s,n\}-1} \frac{a_{n-k}^{(k)}}{(s-k-1)!}}{\displaystyle s \sum_{k=0}^{\min\{s,n\}} \frac{a_{n+1-k}^{(k)}}{(s-k)!}} f_{s-1} - \frac{\displaystyle\sum_{k=0}^{\min\{s,n\}-2} \frac{a_{n-1-k}^{(k)}}{(s-k-2)!}}{\displaystyle s(s-1) \sum_{k=0}^{\min\{s,n\}} \frac{a_{n+1-k}^{(k)}}{(s-k)!}} f_{s-2},$$

that is

$$f_s = \frac{1}{s} \xi_s f_{s-1} + \frac{1}{s(s-1)} \eta_s f_{s-2}, \quad s \geq 2, \tag{9}$$

where

$$\xi_s = -\frac{\sum\limits_{k=0}^{\min\{s,n\}-1} \dfrac{a_{n-k}^{(k)}}{(s-k-1)!}}{\sum\limits_{k=0}^{\min\{s,n\}} \dfrac{a_{n+1-k}^{(k)}}{(s-k)!}}, \quad s \geq 2, \tag{10}$$

$$\eta_s = -\frac{\sum\limits_{k=0}^{\min\{s,n\}-2} \dfrac{a_{n-1-k}^{(k)}}{(s-k-2)!}}{\sum\limits_{k=0}^{\min\{s,n\}} \dfrac{a_{n+1-k}^{(k)}}{(s-k)!}}, \quad s \geq 2. \tag{11}$$

2. Close-to-convexity of a solution. For the investigation of the close-to-convexity for a solution of differential equation (6) we use following lemma [8, 12].

Lemma 1. *If* $\sum\limits_{s=2}^{\infty} s|f_s| \leq 1$, *then the function*

$$f(z) = z + \sum_{s=2}^{\infty} f_s z^s \tag{12}$$

is close-to-convex in $\mathbb{D}$.

In view of this lemma, we search for a solution of differential equation (6) in the form of (12). Suppose that $a_{n+1}^{(0)} + a_n^{(1)} = 0$. Choosing $f_0 = 0$ and $f_1 = 1$, condition (7) holds. For $s \geq 2$ from recurrent formula (9) we obtain

$$\sum_{s=2}^{\infty} s\,|f_s| = \sum_{s=2}^{\infty} s \left| \frac{\xi_s}{s} f_{s-1} + \frac{\eta_s}{s(s-1)} f_{s-2} \right| \leq \sum_{s=2}^{\infty} |\xi_s f_{s-1}| + \sum_{s=2}^{\infty} \left| \frac{\eta_s}{s-1} f_{s-2} \right| =$$

$$= |\xi_2 f_1| + \sum_{s=3}^{\infty} |\xi_s f_{s-1}| + |\eta_2 f_0| + \left| \frac{\eta_3}{2} f_1 \right| + \sum_{s=4}^{\infty} \left| \frac{\eta_s}{s-1} f_{s-2} \right| =$$

$$= |\xi_2| + \sum_{s=2}^{\infty} |\xi_{s+1} f_s| + \left| \frac{\eta_3}{2} \right| + \sum_{s=2}^{\infty} \left| \frac{\eta_{s+2}}{s+1} f_s \right| = \sum_{s=2}^{\infty} s \left| \frac{\xi_{s+1}}{s} f_s \right| + \sum_{s=2}^{\infty} s \left| \frac{\eta_{s+2}}{s(s+1)} f_s \right| + |\xi_2| + \left| \frac{\eta_3}{2} \right|,$$

whence

$$\sum_{s=2}^{\infty} \left(1 - \frac{|\xi_{s+1}|}{s} - \frac{|\eta_{s+2}|}{s(s+1)} \right) s\,|f_s| \leq |\xi_2| + \left| \frac{\eta_3}{2} \right|. \tag{13}$$

Now we put $\xi^* = \max\left\{ \frac{|\xi_{s+1}|}{s}; s \geq 2 \right\}$, $\eta^* = \max\left\{ \frac{|\eta_{s+2}|}{s(s+1)}; s \geq 2 \right\}$. The following theorem is true.

Theorem 1. *Let* $n \geq 3$ *and* $a_{n+1}^{(0)} + a_n^{(1)} = 0$. *If*

$$\xi^* + |\xi_2| + \eta^* + \left| \frac{\eta_3}{2} \right| < 1, \tag{14}$$

then entire solution (12) of differential equation (6) is close-to-convex in $\mathbb{D}$.

Proof. Indeed, (14) implies $1-\xi^*-\eta^*>0$ and so from (13) we obtain the inequality

$$(1-\xi^*-\eta^*)\sum_{s=2}^{\infty} s\,|f_s| \le |\xi_2| + \left|\frac{\eta_3}{2}\right|,$$

whence in view of (14), $\sum_{s=2}^{\infty} s\,|f_s| \le 1$, i.e. by Lemma 1 the function f is close-to-convex in $\mathbb{D}$. □

3. Convexity of a solution. For investigation of the convexity of a solution of differential equation (6), as in [13], we use following lemma [12].

Lemma 2. *If* $\sum\limits_{s=2}^{\infty} s^2|f_s| \le 1$, *then the function (12) is convex in* $\mathbb{D}$.

In view of this lemma we again search for a solution of differential equation (6) in the form of (12). Suppose that $a_{n+1}^{(0)}+a_n^{(1)}=0$. Choosing $f_0=0$ and $f_1=1$, condition (7) holds. For $s\ge 2$ from recurrent formula (9) we obtain

$$\sum_{s=2}^{\infty} s^2\,|f_s| \le \sum_{s=2}^{\infty} s\,|\xi_s f_{s-1}| + \sum_{s=2}^{\infty} s\left|\frac{\eta_s}{s-1} f_{s-2}\right| =$$
$$= 2\,|\xi_2 f_1| + \sum_{s=3}^{\infty} s\,|\xi_s f_{s-1}| + 2\,|\eta_2 f_0| + 3\left|\frac{\eta_3}{2} f_1\right| + \sum_{s=4}^{\infty} s\left|\frac{\eta_s}{s-1} f_{s-2}\right| =$$
$$= 2\,|\xi_2| + \sum_{s=2}^{\infty} (s+1)\,|\xi_{s+1} f_s| + \frac{3}{2}\,|\eta_3| + \sum_{s=2}^{\infty} (s+2)\left|\frac{\eta_{s+2}}{s+1} f_s\right| =$$
$$= 2\,|\xi_2| + \sum_{s=2}^{\infty} \frac{s+1}{s}\left|\frac{\xi_{s+1}}{s}\right| s^2\,|f_s| + \frac{3}{2}\,|\eta_3| + \sum_{s=2}^{\infty} \frac{s+2}{s}\left|\frac{\eta_{s+2}}{s(s+1)}\right| s^2\,|f_s| \le$$
$$\le \sum_{s=2}^{\infty} \frac{3}{2}\left|\frac{\xi_{s+1}}{s}\right| s^2\,|f_s| + \sum_{s=2}^{\infty} 2\left|\frac{\eta_{s+2}}{s(s+1)}\right| s^2\,|f_s| + 2\,|\xi_2| + \frac{3}{2}\,|\eta_3|,$$

whence

$$\sum_{s=2}^{\infty}\left(1-\frac{3}{2}\frac{|\xi_{s+1}|}{s} - 2\frac{|\eta_{s+2}|}{s(s+1)}\right) s^2\,|f_s| \le 2\,|\xi_2| + \frac{3}{2}\,|\eta_3|. \tag{15}$$

Define ξ^* and η^* as above. The following theorem is true.

Theorem 2. *Let* $n\ge 3$ *and* $a_{n+1}^{(0)}+a_n^{(1)}=0$. *If*

$$\frac{3}{2}\xi^* + 2\,|\xi_2| + 2\eta^* + \frac{3}{2}\,|\eta_3| < 1, \tag{16}$$

then entire solution (12) of differential equation (6) is convex in $\mathbb{D}$.

Proof. Indeed, (16) implies $1-3\xi^*/2-2\eta^*>0$ and so from (15) we obtain the inequality

$$\left(1-\frac{3}{2}\xi^* - 2\eta^*\right)\sum_{s=2}^{\infty} s^2\,|f_s| \le 2\,|\xi_2| + \frac{3}{2}\,|\eta_3|,$$

whence, in view of (16), $\sum\limits_{s=2}^{\infty} s^2\,|f_s| \le 1$, i.e. by Lemma 2, the function f is convex in $\mathbb{D}$. □

4. Growth of a solution. For investigation of the growth for a solution of the differential equation (6) we use Wiman-Valiron's method. Let $\mu_f(r)$ be the maximal term of series (12) and let $\nu_f(r)$ be its central index. Let ζ be a point on the circle $\{z\colon |z| = r\}$ such that $|f(\zeta)| = M_f(r) = \max\{|f(z)|\colon |z| = r\}$. Then [14, Ch. 1] the equality

$$f^{(j)}(\zeta) = \left(\frac{\nu_f(r)}{\zeta}\right)^j f(\zeta)(1+\delta_j(\zeta)), \quad j = 1, 2, ..., \tag{17}$$

where $\delta_j(\zeta) = O\left(\nu_f(r)^{-1/5}\right)$, holds as $r \to +\infty$ outside a set $E \subset [1, +\infty)$ of finite logarithmic measure, moreover this set E is contained in a union of intervals $[\sigma'_{n-1}, \sigma_n)$ such that $\sigma_n/\sigma'_{n-1} \to 1$ as $n \to \infty$ (see [14, Ch. 1]).

If f is a solution of differential equation (6) then in view of (17) we have the equality

$$\zeta^n \left(\frac{\nu_f(r)}{\zeta}\right)^n (1+\delta_n(\zeta)) + \left(a_1^{(n-1)}\zeta + a_2^{(n-1)}\right)\zeta^{n-1}\left(\frac{\nu_f(r)}{\zeta}\right)^{n-1}(1+\delta_{n-1}(\zeta)) +$$
$$+ \sum_{k=0}^{n-2}\left(a_{n-1-k}^{(k)}\zeta^2 + a_{n-k}^{(k)}\zeta + a_{n+1-k}^{(k)}\right)\zeta^k\left(\frac{\nu_f(r)}{\zeta}\right)^k(1+\delta_k(\zeta)) = 0, \tag{18}$$

which in general is an equation of n-th order for finding $\nu_f(r)$. It is clear that asymptotic behavior of $\nu_f(r)$ depends on vanishing of parameters $a_k^{(j)}$.

The following theorem is true.

Theorem 3. *Let $n \geq 3$ and $|a_1^{(n-1)}| + |a_1^{(n-2)}| > 0$. Then a transcendental solution (12) of differential equation (6) has regular growth and*

$$\lim_{r\to+\infty}\frac{\ln M_f(r)}{r} = \gamma, \tag{19}$$

where either $\gamma = |\gamma_1|$ or $\gamma = |\gamma_2|$, and

$$\gamma_1 = \frac{-a_1^{(n-1)} + \sqrt{\left(a_1^{(n-1)}\right)^2 - 4a_1^{(n-2)}}}{2}, \quad \gamma_2 = \frac{-a_1^{(n-1)} - \sqrt{\left(a_1^{(n-1)}\right)^2 - 4a_1^{(n-2)}}}{2}.$$

Proof. Firstly, suppose that $a_1^{(n-1)} \neq 0$ and $a_1^{(n-2)} \neq 0$. Then formula (18) yields

$$\zeta^n\left(\frac{\nu_f(r)}{\zeta}\right)^n(1+o(1)) + a_1^{(n-1)}\zeta^n\left(\frac{\nu_f(r)}{\zeta}\right)^{n-1}(1+o(1)) +$$
$$+ a_1^{(n-2)}\zeta^n\left(\frac{\nu_f(r)}{\zeta}\right)^{n-2}(1+o(1)) + O\left(\zeta^2\nu_f^{n-3}(r)\right) = 0, \quad r\to+\infty, r\notin E,$$

that is

$$\left(\frac{\nu_f(r)}{\zeta}\right)^2 + (1+o(1))a_1^{(n-1)}\frac{\nu_f(r)}{\zeta} + (1+o(1))a_1^{(n-2)} = 0, \quad r\to+\infty, r\notin E,$$

whence it follows that either $\nu_f(r) = -(1+o(1))\gamma_1\zeta$ or $\nu_f(r) = -(1+o(1))\gamma_2\zeta$ as $r \to +\infty$ and $r \notin E$.

If $a_1^{(n-2)} = 0$, then from (18) likewise we obtain that $\nu_f(r)/\zeta + (1+o(1))a_1^{(n-1)} = 0$ i.e. $\nu_f(r) = -(1+o(1))a_1^{(n-1)}\zeta$ as $r \to +\infty, r \notin E$. Finally, if $a_1^{(n-1)} = 0$, then (18) implies that $(\nu_f(r)/\zeta)^2 + (1+o(1))a_1^{(n-2)} = 0$, i.e. either $\nu_f(r) = (1+o(1))\sqrt{-a_1^{(n-2)}}\zeta$ or $\nu_f(r) = -(1+o(1))\sqrt{-a_1^{(n-2)}}\zeta$ as $r \to +\infty, r \notin E$.

So, in all three cases $\nu_f(r) = (1+o(1))\gamma r$ as $r \to +\infty, r \notin E$, where either $\gamma = |\gamma_1|$ or $\gamma = |\gamma_2|$.

If $r \in E$ i.e. $r \in [\sigma'_{n-1}, \sigma_n)$ for some $n \in \mathbb{N}$ then

$$\nu_f(r) \ge \nu_f(\sigma'_{n-1}) = (1+o(1))\gamma\sigma'_{n-1} = (1+o(1))\gamma\frac{\sigma'_{n-1}}{\sigma_n}\sigma_n = \\ = (1+o(1))\gamma\sigma_n \ge (1+o(1))\gamma r, \quad r \to +\infty,$$

and

$$\nu_f(r) \le \nu_f(\sigma_n) = (1+o(1))\gamma\sigma_n = (1+o(1))\gamma\frac{\sigma_n}{\sigma'_{n-1}}\sigma'_{n-1} = \\ = (1+o(1))\gamma\sigma'_{n-1} \le (1+o(1))\gamma r, \quad r \to +\infty,$$

that is $\nu_f(r) = (1+o(1))\gamma r$ as $r \to +\infty$. Therefore,

$$\ln\, \mu_f(r) = \ln\, \mu_f(1) + \int_1^r \frac{\nu_f(t)}{t}dt = (1+o(1))\gamma r, \quad r \to +\infty,$$

and by Borel' theorem $\ln\, M_f(r) = (1+o(1))\ln\, \mu_f(r) = (1+o(1))\gamma r$ as $r \to +\infty$. □

5. The main theorem. Using Theorems 1–3 we prove the following theorem.

Theorem 4. *Let $a_{n+1}^{(0)} = a_n^{(1)} = 0$, $a_{n-1}^{(2)} > 0$ and $a_{n+1-k}^{(k)} \ge 0$ for $k = \overline{3,n}$, $|a_{n-k}^{(k)}| \le \varkappa a_{n-k}^{(k+1)}$ for $k = \overline{0, n-1}$ and $|a_{n-1-k}^{(k)}| \le \varkappa a_{n-1-k}^{(k+2)}$ for $k = \overline{0, n-2}$, where $\varkappa \equiv const > 0$. Then differential equation (6) has an entire solution (12) such that:*

1) *if $\varkappa < 6/13$ then f is close-to-convex in $\mathbb{D}$;*
2) *if $\varkappa < 12/55$ then f is convex in $\mathbb{D}$;*
3) *if $|a_1^{(n-1)}| + |a_1^{(n-2)}| > 0$ then equality (19) holds.*

Proof. Since $a_{n+1}^{(0)} = a_n^{(1)} = 0$, $a_{n-1}^{(2)} > 0$ and $a_{n+1-k}^{(k)} \ge 0$ for $k = \overline{3,n}$ we obtain for all $s \ge 2$

$$B_s = \sum_{k=0}^{\min\{s,n\}} \frac{a_{n+1-k}^{(k)}}{(s-k)!} > 0.$$

In view of (10) and (11) for $s \ge 2$ we have

$$|\xi_s| \le \frac{\displaystyle\sum_{k=0}^{\min\{s,n\}-1} \frac{|a_{n-k}^{(k)}|}{(s-k-1)!}}{\displaystyle\frac{a_{n+1}^{(0)}}{s!} + \sum_{k=1}^{\min\{s,n\}} \frac{a_{n+1-k}^{(k)}}{(s-k)!}} = \frac{\displaystyle\sum_{k=0}^{\min\{s,n\}-1} \frac{|a_{n-k}^{(k)}|}{(s-k-1)!}}{\displaystyle\sum_{k=0}^{\min\{s,n\}-1} \frac{a_{n-k}^{(k+1)}}{(s-k-1)!}} \le \varkappa$$

and

$$|\eta_s| \le \frac{\sum\limits_{k=0}^{\min\{s,n\}-2} \frac{|a_{n-1-k}^{(k)}|}{(s-k-2)!}}{\frac{a_{n+1}^{(0)}}{s!} + \frac{a_n^{(1)}}{(s-1)!} + \sum\limits_{k=2}^{\min\{s,n\}} \frac{a_{n+1-k}^{(k)}}{(s-k)!}} = \frac{\sum\limits_{k=0}^{\min\{s,n\}-2} \frac{|a_{n-1-k}^{(k)}|}{(s-k-2)!}}{\sum\limits_{k=0}^{\min\{s,n\}-2} \frac{a_{n-1-k}^{(k+2)}}{(s-k-2)!}} \le \varkappa,$$

that is $\xi^* \le \varkappa/2$, $\eta^* \le \varkappa/6$ and since $\xi_2 \le \varkappa$ and $\eta_3/2 \le \varkappa/2$ we have $\xi^* + |\xi_2| + \eta^* + |\eta_3/2| \le 13\varkappa/6 < 1$ provided $\varkappa < 6/13$. Therefore, by Theorem 1, claim 1) of Theorem 4 is true. Likewise from Theorem 2 we obtain claim 2), and Theorem 3 implies claim 3). □

Since equality $a_{n+1}^{(0)} + a_n^{(1)} = 0$ is one of the conditions of Theorems 1–2 we get

$$B_s = \frac{a_n^{(1)}}{(s-1)!}\frac{s-1}{s} + \sum_{k=2}^{\min\{s,n\}} \frac{a_{n+1-k}^{(k)}}{(s-k)!},$$

and if $a_n^{(1)} > 0$ and $a_{n+1-k}^{(k)} \ge 0$ for $k = \overline{2,n}$ then $B_s \ge \frac{1}{2}\frac{a_n^{(1)}}{(s-1)!} + \sum\limits_{k=2}^{\min\{s,n\}} \frac{a_{n+1-k}^{(k)}}{(s-k)!}$. Therefore, the conclusion of Theorem 4 remains true if $a_{n+1}^{(0)} + a_n^{(1)} = 0$, $a_n^{(1)} > 0$, $a_{n-1}^{(2)} > 0$ and $a_{n+1-k}^{(k)} \ge 0$ for $k = \overline{3,n}$, $|a_n^{(0)}| \le \varkappa a_n^{(1)}/2$, $|a_{n-k}^{(k)}| \le \varkappa a_{n-k}^{(k+1)}$ for $k = \overline{1,n-1}$ and $|a_{n-1-k}^{(k)}| \le \varkappa a_{n-1-k}^{(k+2)}$ for $k = \overline{0,n-2}$.

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