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ON SOME MODULES OVER GROUP RINGS OF LOCALLY SOLUBLE GROUPS WITH RANK RESTRICTIONS ON SUBGROUPS

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The author studies an $\mathbf{R}G$-module A such that $\mathbf{R}$ is an integral domain, G is a locally soluble group of infinite section p-rank (or infinite 0-rank), $C_G(A) = 1$, $A/C_A(G)$ is not a noetherian $\mathbf{R}$-module, and for every proper subgroup H of infinite section p-rank (or infinite 0-rank respectively), the quotient module $A/C_A(H)$ is a noetherian $\mathbf{R}$-module. It is proved that under the above conditions, G is a soluble group. Some properties of soluble groups of this type are obtained.

О. Ю. Дашкова. *О некоторых модулях над групповыми кольцами локально разрешимых групп с ранговыми ограничениями на подгруппы* // Мат. Студiї. – 2011. – Т.36, №2. – С.119–127.

Автор изучает $\mathbf{R}G$-модуль A такой, что $\mathbf{R}$ — область целостности, G — локально разрешимая группа бесконечного секционного p-ранга (или бесконечного 0-ранга), $C_G(A) = 1$, $A/C_A(G)$ не является нетеровым $\mathbf{R}$-модулем, и для каждой собственной подгруппы H бесконечного секционного p-ранга (или бесконечного 0-ранга соответственно), фактор-модуль $A/C_A(H)$ — нетеров $\mathbf{R}$-модуль. Доказано, что при выполнении указанных условий, G — разрешимая группа. Получены некоторые свойства разрешимых групп данного типа.

1. Introduction. Let A be a vector space over a field F. The subgroups of the group $GL(F, A)$ of all automorphisms of A are called linear groups. If A has finite dimension over F then $GL(F, A)$ can be considered as the group of non-singular $n \times n$-matrices, where $n = \dim_F A$. Finite dimensional linear groups have played an important role in various fields of mathematics, physics and natural sciences, and have been well-studied. When A is infinite dimensional over F, the situation is totally different. Infinite dimensional linear groups have been investigated little. The study of this class of groups requires additional restrictions. In [1] the definition of the central dimension of an infinite dimensional linear group was introduced. Let H be a subgroup of $GL(F, A)$. H acts on the quotient space $A/C_A(H)$ in a natural way. The authors define $\operatorname{centdim}_F H = \dim_F(A/C_A(H))$. The subgroup H is said to have a finite central dimension if $\operatorname{centdim}_F H$ is finite and H has infinite central dimension otherwise. Let $G \leq GL(F, A)$, G be a locally soluble group of infinite central dimension and infinite rank. Suppose that every proper subgroup $H \leq G$ of infinite rank has a finite central dimension. It is proved that under the above conditions, G is a soluble group [2].

If $G \leq GL(F, A)$ then A can be considered as an FG-module. The natural generalization of this case is the study of an $\mathbf{R}G$-module A, where $\mathbf{R}$ is a ring whose structure is close to

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the structure of a field. Here the role of central dimension of a subgroup of a linear group is played by the cocentralizer of the subgroup, which is introduced in [3].

Let A be an $\mathbf{R}G$-module, $\mathbf{R}$ a ring and G a group. If $H \le G$, then the quotient module $A/C_A(H)$ considered as an $\mathbf{R}$-module is called the cocentralizer of a subgroup H in the module A.

Investigation of algebraic systems satisfying maximal and minimal conditions still remains very actual. Examples of such systems are the classes of noetherian modules and artinian modules. We recall that a module is called an artinian module if the ordered set of all submodules of this module satisfies the minimal condition. A module is called a noetherian module if the ordered set of all submodules of this module satisfies the maximal condition.

The author has studied an $\mathbf{R}G$-module A such that $\mathbf{R}$ is a dedekind domain, G is a locally soluble group, $C_G(A) = 1$, $A/C_A(G)$ is not an artinian $\mathbf{R}$-module and the cocentralizer of H in A is an artinian $\mathbf{R}$-module for every proper subgroup H of infinite rank. It was proved that under the above conditions, G is a soluble group [4]. In [5] it was investigated an $\mathbf{R}G$-module A such that $\mathbf{R} = \mathbb{Z}$ is the ring of integers, G is a locally soluble group, $C_G(A) = 1$, $A/C_A(G)$ is not a noetherian $\mathbf{R}$-module and the cocentralizer of H in A is a noetherian $\mathbf{R}$-module for every proper subgroup H of infinite rank. Also it was proved that, under the above conditions, G is a soluble group.

In this paper we consider different ranks of a group. A group G is said to have finite 0-rank (or finite torsion-free rank) $r_0(G) = r$, if G has a finite subnormal series with exactly r infinite cyclic factors, all other factors are periodic. It is well known that the 0-rank is independent of the chosen series, and if G is a group of finite 0-rank, $H \le G$ and L is a normal subgroup of G, then H and G/L also have finite 0-rank. Furthermore, $r_0(H) \le r_0(G)$ and $r_0(G) = r_0(L) + r_0(G/L)$.

Now let p be a prime. A group G has finite section p-rank $r_p(G) = r$, if every elementary abelian p-section of G is finite of order, at most p^r, and there is an elementary abelian p-section U/V such that $|U/V| = p^r$. It is known that if G is a group of finite section p-rank, $H \le G$ and L is a normal subgroup of G, then H and G/L have finite section p-rank, too. Furthermore, $r_p(H) \le r_p(G)$ and $r_p(G) \le r_p(L) + r_p(G/L)$.

In this paper we study an $\mathbf{R}G$-module A such that $\mathbf{R}$ is an integral domain, G is a locally soluble group of infinite rank, $C_G(A) = 1$, $A/C_A(G)$ is not a noetherian $\mathbf{R}$-module and the cocentralizer of H in A is a noetherian $\mathbf{R}$-module for every proper subgroup H of infinite rank.

In Section 2 basic results concerning $\mathbf{R}G$-modules of considered type are obtained. Section 3 covers the investigation of an $\mathbf{R}G$-module A such that $r_p(G)$ is infinite, $p \ge 0$, G is a soluble group, $C_G(A) = 1$, $A/C_A(G)$ is not a noetherian $\mathbf{R}$-module and the cocentralizer of H in A is a noetherian $\mathbf{R}$-module for every proper subgroup H such that $r_p(H)$ is infinite.

Recall that a group G has finite abelian section rank if $r_p(G)$ is finite for all prime p. Furthermore, a group G has finite special rank $r(G) = r$, if every finitely generated subgroup of G is generated by r elements, where r is the least positive integer with this property. This notion is due to Mal'cev [6].

In section 4 we concentrate on the study of an $\mathbf{R}G$-module A of considered type such that G is locally soluble. It is proved that under such conditions, G is soluble (Theorem 5). This result generalizes the abelian section rank and the special rank (Theorems 6, 7).

2. Preliminary results. In Lemmas 1, 2, 6, 7, 9, 11, Theorems 1–3, 5–7 we consider an $\mathbf{R}G$-module A such that $C_G(A) = 1$ and $A/C_A(G)$ is not a noetherian $\mathbf{R}$-module.

Lemma 1. *Let A be an* $\mathbf{R}G$*-module, and let $r_p(G)$ be infinite for some $p \geq 0$. Suppose that the cocentralizer of M in A is a noetherian* $\mathbf{R}$*-module for every proper subgroup M of G such that $r_p(M)$ is infinite. Then:*

(i) If U, V are proper subgroups of G and $G = \langle U, V \rangle$, then the cocentralizer of at least one of U, V in A is a noetherian $\mathbf{R}$*-module.*

(ii) If a proper subgroup H of G has infinite $r_p(H)$, then the cocentralizer of every subgroup of H in A and the cocentralizer of every proper subgroup of G containing H in A are noetherian $\mathbf{R}$*-modules.*

(iii) If K, L are proper subgroups of G containing H, then $\langle K, L \rangle$ is a proper subgroup of G.

Lemma 2. *Let A be an* $\mathbf{R}G$*-module, $r_p(G)$ be infinite for some $p \geq 0$. Suppose that the cocentralizer of M in A is a noetherian* $\mathbf{R}$*-module for every proper subgroup M such that $r_p(M)$ is infinite. If K is a proper normal subgroup of G such that $r_p(K)$ is infinite and G/K is finitely generated, then G/K is a cyclic q-group for some prime q.*

Proof. Suppose that $G = \langle K, S \rangle$ for some finite set S with the following property: if T is a proper subset of S then $G \neq \langle K, T \rangle$. Let $S = \langle x_1, x_2, ..., x_n \rangle$. If $n > 1$, then $\langle K, x_1, x_2, ..., x_{n-1} \rangle$ and $\langle K, x_n \rangle$ are proper subgroups. Lemma 1 provides a contradiction. It follows that G/K is cyclic. If G/K is either infinite or finite but $|\pi(G/K)| > 1$, then G is a product of two proper subgroups G_1 and G_2 such that $r_p(G_1)$ and $r_p(G_2)$ are infinite. Lemma 1 again gives a contradiction. Hence, G/K is a finite q-group for some prime q. □

Lemma 3 ([2])**.** *Let G be a group and let q be a prime. Suppose that A is an infinite normal elementary abelian q-subgroup of G such that G/A is finite. Then G is generated by two proper subgroups having infinite section q-rank.*

Lemma 4 ([2])**.** *Let G be a group and q be a prime. Let A be a normal divisible abelian q-subgroup of G such that G/A is finite. If A has infinite section q-rank, then G is generated by two proper subgroups having infinite section q-rank.*

We shall also use the following result.

Lemma 5 ([2])**.** *Let G be a group. Suppose that A is a normal subgroup of G such that G/A is infinite, periodic and abelian-by-finite. If $|\pi(G/A)| > 1$, then G is a product of two proper subgroups containing A.*

We apply these results to prove the following lemma.

Lemma 6. *Let A be an* $\mathbf{R}G$*-module, $r_p(G)$ be infinite for some $p \geq 0$. Suppose that the cocentralizer of M in A is a noetherian* $\mathbf{R}$*-module for every proper subgroup M such that $r_p(M)$ is infinite. If K is a normal subgroup of H, $H \leq G$, H/K is abelian-by-finite and $r_p(H/K)$ is infinite, then the cocentralizer of H in A is a noetherian* $\mathbf{R}$*-module.*

Proof. First suppose that $p = 0$. Let L be a normal subgroup of H such that H/L is finite and L/K is abelian. Since $r_0(L/K)$ is infinite, L/K contains a free abelian subgroup B/K such that $r_0(B/K)$ is infinite and L/B is periodic. Since H/L is finite, B has only finitely many conjugates in H, $B_1, ..., B_m$ say, and if $C = \mathrm{core}_H\, B$, then L/C is isomorphically embedded into $L/B_1 \times L/B_2 \times \cdots \times L/B_m$. It follows that L/C is periodic and hence $r_0(C/K)$ is infinite as $r_0(C)$. Note also that C/K is free abelian. If H/C is finite or $|\pi(H/C)| = 1$, then choose a prime $q \notin \pi(H/C)$ and set $D/K = (C/K)^q$. If H/C is infinite and $|\pi(H/C)| > 1$, then let $D = C$. Then, in each case, H/D is infinite, periodic and $|\pi(H/D)| > 1$. Also $r_0(D)$ is infinite. Applying lemmas 5 and 1 we see that the cocentralizer of H in A is a noetherian **R**-module.

Now suppose that $p > 0$ and let L be defined as before. Choose a free abelian subgroup B/K of L/K such that L/B is periodic. If $r_0(B/K)$ is infinite, then we can proceed as in the case $p = 0$, so we suppose that $r_0(B/K)$ is finite. As above, if $C = \operatorname{core}_H B$, then L/C is periodic and $r_p(L/C)$ is infinite. Factoring by the Sylow p'-subgroup of L/C if necessary, we may assume that L/C is a p-group. If L/L^pC is infinite, then H/L^pC satisfies the hypotheses of Lemma 3, so H is a product of two proper subgroups, each of infinite section p-rank, and so their cocentralizers in A are noetherian **R**-modules. Thus, the cocentralizer of H in A is a noetherian **R**-module in this case. If L/L^pC is finite, then using properties of basic subgroups we have that $L/C = E/C \times D/C$ for some finite subgroup E/C and divisible subgroup D/C. Since H/L is finite and L/C is abelian, $F/C = (E/C)^{H/C}$ is also finite. Furthermore, L/F is a divisible abelian p-group of infinite section p-rank. Now we can apply Lemma 4 for H/F to deduce that H is a product of two proper subgroups, each of infinite section p-rank, and therefore their cocentralizers in A are noetherian **R**-modules. Thus, the cocentralizer of H in A is a noetherian **R**-module. □

Lemma 7. *Let A be an* **R***G-module, and let $r_p(G)$ be infinite for some $p \geq 0$. Suppose that the cocentralizer of M in A is a noetherian* **R***-module for every proper subgroup M such that $r_p(M)$ is infinite. If H is a normal subgroup of G and G/H is abelian-by-finite, then G/H is isomorphic to a subgroup of C_{q^∞} for some prime q.*

Proof. We may assume that $G \neq H$. If $r_p(G/H)$ is infinite then the cocentralizer of G in A is a noetherian **R**-module by Lemma 6. Thus, $r_p(G/H)$ is finite and therefore $r_p(H)$ is infinite. Moreover, if G/H is finite, then the result follows from Lemma 2. Thus, we may suppose that G/H is infinite.

First suppose that G/H is an abelian group. By Lemma 1(iii) G/H cannot be free abelian. Let B/H be a free abelian subgroup of G/H such that G/B is periodic. Since $r_p(H)$ is infinite, $r_p(B)$ is also infinite. If $|\pi(G/B)| > 1$, then clearly G is a product of proper subgroups G_1 and G_2 such that $r_p(G_1)$ and $r_p(G_2)$ are infinite. We have the contradiction with Lemma 1(iii). Thus, G/B is a q-group for some prime q. Suppose that B/H is nontrivial. Let r be a prime different from q, and let $C/H = (B/H)^r \neq B/H$. Then G/C is periodic, $\pi(G/C) = \{q, r\}$ and we have a contradiction. It follows that G/H is a periodic q-group. If G/H is divisible, then it is a direct product of copies of Prüfer q-groups and Lemma 1(iii) implies that $G/H \simeq C_{q^\infty}$. Otherwise $(G/H)/(G/H)^q$ is a nontrivial elementary abelian q-group and Lemma 1(iii) shows that $|(G/H)/(G/H)^q| = q$. It is easy to see that $G/H = (E/H) \times (D/H)$ where D/H is divisible, $|E/H| = q$ and Lemma 1(iii) again gives a contradiction.

In the general case let L/H be a normal abelian subgroup of G/H such that G/L is finite. Let U/H be an arbitrary subgroup of finite index in G/H. If $V/H = \operatorname{core}_{G/H} U/H$, then G/V is also finite and $r_p(V)$ is infinite. By Lemma 2, G/V is a cyclic q-group for some prime q, so $G' \leq V \leq U$. Thus, if W/H is the finite residual of L/H, then G/W is abelian, $r_p(W)$ is infinite and, by the first part of the argument, G/W is finite, since it is residually finite. Thus, $G = WK$ for some subgroup K containing H such that K/H is finitely generated. Since G/H is infinite, Lemma 2 implies that $G \neq K$. Then from Lemma 1(iii) we deduce that $G = W$, and therefore G/H is abelian, and the result follows by the first part of the proof. □

3. Modules over group rings of soluble groups. In this section we apply the results of Section 2 to soluble groups.

Theorem 1. *Let A be an $\mathbf{R}G$-module, G a soluble group and $r_p(G)$ be infinite for some $p \geq 0$. Suppose that the cocentralizer of M in A is a noetherian $\mathbf{R}$-module for every proper subgroup M such that $r_p(M)$ is infinite. Then G has a series of normal subgroups $H \leq N \leq G$ such that the subgroup H is abelian, the quotient group N/H is nilpotent and the quotient group G/N is isomorphic to C_{q^∞} for some prime q.*

Proof. If

$$G = D_0 \;\geq\; D_1 \;\geq\; \cdots \;\geq\; D_{n-1} \;\geq\; D_n = E$$

is the derived series of G, then there is a positive integer m such that G/D_m is finite, but D_m/D_{m+1} is infinite. Let $K = D_m$. By Lemma 7 $G/K' \simeq C_{q^\infty}$ for some prime q. Since $r_p(K')$ is infinite, the cocentralizer of K' in A is a noetherian $\mathbf{R}$-module. Let $C = C_A(K')$. Hence, A/C is a noetherian $\mathbf{R}$-module. Now $K' \leq C_G(C)$, and since the cocentralizer of G in A is not a noetherian $\mathbf{R}$-module, $G/C_G(C) \simeq C_{q^\infty}$. Since K' is a normal subgroup of G, C is an $\mathbf{R}G$-submodule of A. The quotient module A/C is a noetherian $\mathbf{R}$-module. It follows that A has a finite series of $\mathbf{R}G$-submodules

$$\langle 0\rangle = C_0 \;\leq\; C_1 = C \;\leq\; C_2 = A,$$

where C_2/C_1 is a finitely generated $\mathbf{R}$-module. Since $\mathbf{R}$ is an integral domain, $\mathbf{R}$ can be embedded in the field F. Then $G/C_G(C_2/C_1)$ is isomorphic to some subgroup of $GL(r, F)$. By Theorem 3.6 [7] $G/C_G(C_2/C_1)$ is nilpotent-by-(abelian-by-finite).

Let

$$H \;=\; C_G(C_1) \;\cap\; C_G(C_2/C_1).$$

Every element of H acts trivially on every factor $C_{j+1}/C_j, j = 0, 1$. It follows that H is abelian. By Remak's Theorem

$$G/H \;\leq\; G/C_G(C_1) \;\times\; G/C_G(C_2/C_1).$$

Therefore, G/H is nilpotent-by-(abelian-by-finite). Since $H \leq C_G(C_1)$ and $G/C_G(C_1) \simeq C_{q^\infty}$ it follows that G/H is infinite. Then G has a series of normal subgroups $H \leq N \leq G$ such that the subgroup H is abelian, the quotient group N/H is nilpotent, and G/N is an infinite abelian-by-finite quotient group. It follows from Lemma 7 that G/N is isomorphic to C_{q^∞} for some prime q. □

Theorem 2. *Let A be an $\mathbf{R}G$-module, and let G be a soluble group of infinite abelian section rank. Suppose that the cocentralizer of every proper subgroup of infinite abelian section rank in A is a noetherian $\mathbf{R}$-module. Then G has a series of normal subgroups $H \leq N \leq G$ such that the subgroup H is abelian, the quotient group N/H is nilpotent and the quotient group G/N is isomorphic to C_{q^∞} for some prime q.*

Proof. Since G is soluble and has infinite abelian section rank, there is a prime p such that $r_p(G)$ is infinite. If H is a proper subgroup and $r_p(H)$ is infinite, then H has infinite abelian section rank and the cocentralizer of H in A is a noetherian $\mathbf{R}$-module. We may apply Theorem 1. □

Theorem 3. *Let A be an $\mathbf{R}G$-module, and let G be a soluble group of infinite special rank. Suppose that the cocentralizer of every proper subgroup of infinite special rank in A is a noetherian $\mathbf{R}$-module. Then G has a series of normal subgroups $H \le N \le G$ such that the subgroup H is abelian, the quotient group N/H is nilpotent and the quotient group G/N is isomorphic to C_{q^∞} for some prime q.*

Proof. If G has infinite abelian section rank and X is a proper subgroup of infinite abelian section rank, then X has infinite special rank. Hence, the cocentralizer of X in A is a noetherian $\mathbf{R}$-module. The result follows from Theorem 2. Therefore, we suppose that G has finite abelian section rank.

Let U be a normal subgroup of G such that G/U is infinite abelian-by-finite and let V/U be a normal abelian subgroup of G/U such that G/V is finite. Since $r_0(G)$ is finite, V/U contains a finitely generated subgroup B/U such that V/B is periodic. If $C/U = (B/U)^{G/U}$, then C/U is finitely generated as well. Suppose that G/U has infinite special rank. Since G has finite abelian section rank, it follows that p-subgroups of V/C are Chernikov groups for all primes. Thus, $\pi(V/C)$ is infinite. If D/C is a Sylow $\pi(G/V)$-subgroup of V/C, then V/D has infinite special rank. By Lemma 1.D.4 [8], $G/D = (V/D)(W/D)$ where V/D is a normal subgroup of G/D, $(V/D) \cap (W/D) = E$, W/D is finite, and $\pi(V/D) \cap \pi(W/D)$ is empty. Then V/D is a product of two G-invariant proper subgroups of infinite special rank. Therefore, the cocentralizer of G in A is a noetherian $\mathbf{R}$-module. Thus, G/U has finite special rank, and so U has infinite special rank. As in Section 2 we deduce that $G/U \simeq C_{q^\infty}$ for some prime q. As in Theorem 1, G has the required properties. □

4. Modules over group rings of locally soluble groups. In this section we show that locally soluble groups of the type under discussion are soluble.

Lemma 8. *Let A be an $\mathbf{R}G$-module, and let G be a locally soluble group. Suppose that the cocentralizer of G in A is a noetherian $\mathbf{R}$-module. Then G is soluble.*

Proof. Let $C = C_A(G)$. Then A has a finite series of $\mathbf{R}G$-submodules

$$\langle 0 \rangle = C_0 \le C_1 = C \le C_2 = A,$$

where C_2/C_1 is a finitely generated $\mathbf{R}$-module. Since $\mathbf{R}$ is an integral domain, $\mathbf{R}$ can be embedded in the field F. Therefore, $G/C_G(C_2/C_1)$ is isomorphic to some subgroup of $GL(r, F)$. By Corollary 3.8 [7], $G/C_G(C_2/C_1)$ is soluble. Every element of $C_G(C_2/C_1)$ acts trivially on every factor of the series $\langle 0 \rangle = C_0 \le C_1 = C \le C_2 = A$. It follows that $C_G(C_2/C_1)$ is abelian. Therefore, G is a soluble group. □

Lemma 9. *Let A be an $\mathbf{R}G$-module, G a locally soluble group, and let $r_p(G)$ be infinite for some $p \ge 0$. Suppose that the cocentralizer of M in A is a noetherian $\mathbf{R}$-module for every proper subgroup M such that $r_p(M)$ is infinite. If G is not soluble, then G is perfect.*

Proof. Note that if H is a proper normal subgroup of G of finite index, then $r_p(H)$ is infinite, and therefore the cocentralizer of H in A is a noetherian $\mathbf{R}$-module. It follows from Lemma 8 that H is soluble and from Lemma 2 that G/H is abelian. Thus, G is soluble, a contradiction. Hence, if $G \ne G'$ then G/G' is divisible. It follows that G contains a normal subgroup H such that $G/H \simeq C_{q^\infty}$ for some prime q. Then $r_p(H)$ is infinite, hence the cocentralizer of H in A is a noetherian $\mathbf{R}$-module. By Lemma 8, H is soluble. Therefore, G is soluble, a contradiction. □

By $d(G)$ we denote the derived length of a soluble group G, and by $T(G)$ we denote the maximal normal periodic subgroup of G.

Theorem 4 ([2])**.** *Let p be a prime and let G be a locally soluble group such that $r_p(G) = r$. Then $G/T(G)$ is a soluble group such that $d(G/T(G)) \leq s_p(r)$ and $G/T(G)$ has finite special rank at most $f_p(r)$.*

Lemma 10. *Let A be an $\mathbf{R}G$-module, G be a locally soluble group, $r_p(G)$ be infinite for some $p \geq 0$. Suppose that the cocentralizer of M in A is a noetherian $\mathbf{R}$-module for every proper subgroup M such that $r_p(M)$ is infinite. If G is not soluble, H is a normal subgroup of G and $r_p(H)$ is finite, then $H/T(H)$ is G-central.*

Proof. Let $r_p(H) = r$. If $p = 0$, then we apply Lemma 2.12 [9], and if $p > 0$, then we apply Theorem 4 to H, and we see that $H/T(H)$ is soluble and has finite special rank, which is a function of r. Let $n = r_0(H/T(H))$, which is dependent upon r. Then H has a finite series of G-invariant subgroups

$$T(H) = H_0 \ \leq \ H_1 \ \leq \cdots \leq \ H_d = H,$$

where all factors are abelian.

Note that $H_1/T(H)$ is torsion-free and has finite rank at most n, and so $\mathrm{Aut}(H_1/T(H))$ is isomorphic to a subgroup of $GL(n, \mathbb{Q})$. Hence, $G/C_G(H_1/T(H))$ is a locally soluble group isomorphic to a subgroup of $GL(n, \mathbb{Q})$. It follows from Corollary 3.8 [7] that $G/C_G(H_1/T(H))$ is soluble and trivial, by Lemma 9. Thus, $[G, H_1] \leq T(H)$.

Suppose inductively that $[G, H_{d-1}] \leq T(H)$. Then $H_{d-1}/T(H) \leq Z(G/T(H))$. Therefore, $H/T(H)$ is nilpotent of class at most 2. Let $K/T(H) = Z(H/T(H))$. Then, as above, since $K/T(H)$ and H/K are torsion-free abelian and have rank at most n, we have $[G, K] \leq T(H)$ and $[G, H] \leq K$. It follows from the three subgroups lemma and Lemma 9 that $[G, H] = [G, G, H] \leq T(H)$ and the result follows by induction on d. □

Lemma 11. *Let A be an $\mathbf{R}G$-module, G be an insoluble, locally soluble group, and let $r_p(G)$ be infinite for some $p \geq 0$. Suppose that the cocentralizer of M in A is a noetherian $\mathbf{R}$-module for every proper subgroup M such that $r_p(M)$ is infinite. Then G contains a proper normal subgroup V such that if U is a normal subgroup of G and $V \leq U \leq G$, $U \neq G$ then U is soluble and the cocentralizer of U in A is a noetherian $\mathbf{R}$-module.*

Proof. Let $T = T(G)$ and first suppose that $T \neq G$ and $r_p(T)$ is finite (if $p = 0$ then these conditions are automatically satisfied). By Lemma 9, G/T is not soluble and hence is not simple by Corollary 1 to Theorem 5.27 [10]. Hence, G contains a proper normal subgroup $L \geq T$, $L \neq T$. If $r_p(L)$ is finite, then Lemma 10 implies that L/T is G-central, and so G/T contains a nontrivial maximal normal abelian subgroup V/T. Certainly $V \neq G$. If U is a normal subgroup of G and $V \leq U \leq G$, $V \neq U, U \neq G$, then $r_p(U)$ is infinite by Lemma 10 and, by hypothesis, the cocentralizer of U in A is a noetherian $\mathbf{R}$-module. Then U is soluble by Lemma 8. If there is no such a subgroup L, then we set $V = T$ and, as above, V has the required property.

Next suppose that $p > 0$. First suppose that $r_p(T)$ is infinite. If $T \neq G$, then the cocentralizer of T in A is a noetherian $\mathbf{R}$-module. We deduce from Lemma 8 that T is soluble. Then G/T is non-soluble and non-simple by Corollary 1 of Theorem 5.27 [10]. If U is a normal subgroup of G and $T \leq U \leq G$, $U \neq G$, then $r_p(U)$ is infinite. Hence, the cocentralizer of U in A is a noetherian $\mathbf{R}$-module in this case.

Now let $T = G$. First suppose that all Sylow p-subgroups of G are of finite section p-rank. Then G has the minimal condition on p-subgroups, by Lemma 3.1 [8]. We have a contradiction, since in that case p-subgroups are Chernikov groups and of finite, bounded ranks. Thus, G contains some p-subgroup P of infinite rank, and so G contains an infinite elementary abelian p-subgroup A_0 by Corollary 2 of Theorem 6.36 [10]. Certainly A_0 is a proper subgroup of G, and if U is a normal subgroup of G, then $UA_0 \neq G$; otherwise G/U is abelian, that contradicts Lemma 9. Thus, UA_0 is a proper subgroup of G of infinite section p-rank, and so the cocentralizer of UA_0 in A is a noetherian $\mathbf{R}$-module. Consequently, the cocentralizer of U in A is a noetherian $\mathbf{R}$-module. By Lemma 8, U is soluble. We set $V = 1$ in this case. □

Theorem 5. *Let A be an $\mathbf{R}G$-module, let G be a locally soluble group, and let $r_p(G)$ be infinite for some $p \geq 0$. Suppose that the cocentralizer of M in A is a noetherian $\mathbf{R}$-module for every proper subgroup M such that $r_p(M)$ is infinite. Then G is soluble.*

Proof. Suppose that G is non-soluble. By Lemma 11, G contains a normal subgroup V with the following property: if U is a normal subgroup of G and $V \leq U \leq G$, $U \neq G$, then U is soluble and the cocentralizer of U in A is a noetherian $\mathbf{R}$-module. Set $V = U_0$ and $d(U_0) = d_0$. Assume that we have constructed normal soluble subgroups $U_0 \leq U_1 \leq \cdots \leq U_n$ such that $d(U_i) = d_i$ for $i = 0, 1, ..., n$, and $d_i < d_{i+1}$ for $i = 0, 1, ..., n-1$. Since G is not soluble, there exists a normal subgroup U_{n+1} containing U_n such that $d(U_{n+1}) = d_{n+1} > d(U_n)$ and we obtain an ascending chain of soluble normal subgroups of increasing derived lengths. Let $W = \bigcup_{n\geq 1} U_n$. By construction, W is not soluble and $V \leq W$. It follows that $W = G$.

Now let $C_A(U_n) = C_n$ for all $n \in \mathbf{N}$. Since U_n is a normal subgroup of G, C_n is an $\mathbf{R}G$-submodule for all n. Since the cocentralizer of U_n in A is a noetherian $\mathbf{R}$-module, A/C_n is a noetherian $\mathbf{R}$-module. A/C_n can be considered as a $\mathbf{R}(G/C_G(A/C_n))$-module. Thus, $G/C_G(A/C_n)$ is soluble by Lemma 8. We deduce from Lemma 9 that $G = C_G(A_n)$ for all $n \in \mathbf{N}$. Since $G = \bigcup_{n\geq 1} U_n$, it follows that

$$C_A(G) = \bigcap_{n\geq 1} C_A(U_n) = \bigcap_{n\geq 1} C_n,$$

and so G acts trivially on the factors of the series $0 \leq C_A(G) \leq A$. In this case G is abelian. This contradiction proves that G is soluble. □

Using a method analogous to that of the proof of Theorem 2 and applying Theorem 5 we obtain the following result.

Theorem 6. *Let A be an $\mathbf{R}G$-module, and let G be a locally soluble group of infinite abelian section rank. Suppose that the cocentralizer of every proper subgroup of infinite abelian section rank in A is a noetherian $\mathbf{R}$-module. Then G is soluble.*

Theorem 7. *Let A be an $\mathbf{R}G$-module, and let G be a locally soluble group of infinite special rank. Suppose that the cocentralizer of every proper subgroup of infinite special rank in A is a noetherian $\mathbf{R}$-module. Then G is soluble.*

Proof. Let G be a counterexample to the theorem. If N is a proper normal subgroup of infinite special rank, then the cocentralizer of N in A is a noetherian $\mathbf{R}$-module, and it follows from Lemma 8 that N is soluble. If N has finite special rank, then by Lemma 10.39 [10],

N is hyperabelian. Let $\{N_\alpha\}$ be the family of all proper normal subgroups of G. Then the subgroup $J = \prod N_\alpha$ is also hyperabelian. Since a simple locally soluble group is cyclic, it follows that G is also hyperabelian. By Theorem 7.1 [11], G contains a subgroup K which is either an elementary abelian q-subgroup of infinite special rank for some prime q or a torsion-free abelian subgroup of infinite special rank. Let N be a proper normal subgroup of G of finite special rank. By Lemma 10.39 [10], there exists a positive integer d such that $N^{(d)}$ is a direct product of Chernikov p-groups for distinct primes p. If $N^{(d)}K \neq G$, then N is soluble. If $N^{(d)}K = G$, then let r be some prime such that $r \neq q$, and let X be a Sylow $\{q, r\}'$-subgroup of $N^{(d)}$. Then $XK \neq G$, and as above, X and N are soluble. Thus, every proper normal subgroup of G is soluble, and its cocentralizer is a noetherian $\mathbf{R}$-module. The proof proceeds as that of Theorem 5. □

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