УДК 519.4

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KALEIDOSCOPICAL CONFIGURATIONS IN GROUPS

I. V. Protasov, S. Slobodianiuk. *Kaleidoscopical configurations in groups*, Mat. Stud. **36** (2011), 115–118.

A subset A of a group G is called a kaleidoscopical configuration if there exists a surjective coloring $\chi\colon X \to \varkappa$ such that the restriction $\chi|gA$ is a bijection for each $g \in G$. We give two topological constructions of kaleidoscopical configurations and show that each infinite subset of an Abelian group contains an infinite kaleidoscopical configuration.

И. В. Протасов, С. Слободянюк. *Калейдоскопические конфигурации на группах* // Мат. Студiї. – 2011. – Т.36, №2. – C.115–118.

Подмножество A группы G называется калейдоскопической конфигурацией, если существует сюръективное раскрашивание $\chi\colon X \to \varkappa$, такое что ограничение $\chi|gA$ является биекцией для всех $g \in G$. Предложено две топологические конструкции калейдоскопических конфигураций, доказано что каждое бесконечное подмножество абелевой группы содержит бесконечную калейдоскопическую конфигурацию.

Let X be a set and let $\mathfrak{F}$ be a family of subsets of X. Following [2], we say that the hypergraph $(X, \mathfrak{F})$ is *kaleidoscopical* if there exists a coloring $\chi\colon X \to \varkappa$ (i.e. a mapping of G onto a cardinal $\varkappa$) such that the restriction $\chi|F$ is a bijection for each $F \in \mathfrak{F}$.

A subset A of a group G is said to be a *kaleidoscopical configuration* if the hypergraph $(G, \{gA\colon g \in G\})$ is kaleidoscopical. Each kaleidoscopical configuration A in G is *complemented*, i.e. there exists a subset B of G such that $G = AB$ and the multiplication mapping $\mu\colon A \times B \to G$, $\mu(a, b) = ab$, is a bijection [2, Corollary 1.3]. If G is Abelian then the converse statement holds [2, Corollary 1.5]. We note also that if A is kaleidoscopical in some subgroup of G then A is kaleidoscopical in G.

In this note, we give two topological constructions of kaleidoscopical configurations and show that each infinite subset of an Abelian group contains an infinite kaleidoscopical configuration.

For a hypergraph $(X, \mathfrak{F})$, $x \in X$ and $A \subset X$, we put

$$\mathrm{St}(x, \mathfrak{F}) = \bigcup\{F \in \mathfrak{F}\colon x \in F\},\ \mathrm{St}(A, \mathfrak{F}) = \bigcup\{\mathrm{St}(a, \mathfrak{F})\colon a \in A\}.$$

Proposition 1. *A hypergraph $(X, \mathfrak{F})$ is kaleidoscopical provided that, for some cardinal $\varkappa$, the following two conditions are satisfied:*

(1) $|\mathfrak{F}| \le \varkappa$ and $|F| = \varkappa$ for each $F \in \mathfrak{F}$;

(2) for any subfamily $\mathfrak{A} \subset \mathfrak{F}$ of cardinality $|\mathfrak{A}| < \varkappa$ and any subset $B \subset X \setminus \bigcup\mathfrak{A}$ of cardinality $|B| < \varkappa$, the intersection $\mathrm{St}(B, \mathfrak{F}) \cap (\bigcup\mathfrak{A})$ has cardinality less than $\varkappa$.

2010 *Mathematics Subject Classification:* 05B45, 05C15, 20K01.
Keywords: kaleidoscopical configuration, T-sequence, rigid subset.

Proof. [2, Proposition 1.10]. □

We say that a subset A of a group G is *rigid* if for each $g \in G \setminus A$, the set $g^{-1}A \cap A^{-1}A$ is finite.

Proposition 2. *If A is a countable rigid subset of a group G then A is a kaleidoscopical configuration.*

Proof. We may suppose that G is countable. To apply Proposition 1, it suffices to show that, for all $c \in G$, $b \in G \setminus cA$, the set

$$Y = \bigcup \{yA \cap cA \colon y \in G, b \in yA\}$$

is finite. Clearly, $Y \subseteq bA^{-1}A \cap cA$. We put $g = c^{-1}b$ and note that $g \in G \setminus A$. By the assumption, the set $A^{-1}g \cap A^{-1}A$ is finite, so Y is finite. □

An injective sequence $(a_n)_{n\in\omega}$ of elements of an infinite group G is called a *T-sequence* [3] if there exists a Hausdorff group topology on G in which $(a_n)_{n\in\omega}$ converges to the identity e of G. By [3, Theorem 3.2.2], a countable group G admits a non-discrete Hausdorff group topology if and only if there is a T-sequence in G. By [3, Theorem 2.1.8], each infinite subset of an Abelian group G contains a T-sequence if and only if, for all $g \in G \setminus \{0\}$ and $k \in \mathbb{N}$, the set $\{x \in G \colon kx = g\}$ is finite.

Theorem 1. *For every T-sequence $(a_n)_{n<\omega}$ in a group G, the set $A = \{e, a_n, a_n^{-1} \colon n \in \omega\}$ is a kaleidoscopical configuration.*

Proof. In view of Proposition 2, it suffices to show that the set $A = \{e, a_n, a_n^{-1}\}$ is rigid. We assume the contrary and pick $g \in G \setminus A$, an injective sequence $(b_n)_{n\in\omega}$ in A and two sequences $(c_n)_{n\in\omega}$, $(d_n)_{n\in\omega}$ in A such that $b_n g = c_n d_n$ for each $n \in \omega$. Let τ be a Hausdorff group topology on G in which $(a_n)_{n\in\omega}$ converges to e. Since, at least one of the sequences $(c_n)_{n\in\omega}$ and $(d_n)_{n\in\omega}$ must take infinitely many values, passing to limit in τ, we get $g \in A$. □

Corollary 1. *For every T-sequence $(a_n)_{n<\omega}$ in a group G, the set $A = \{e, a_n, a_n^{-1} \colon n \in \omega\}$ is complemented.*

Question 1. Let $(a_n)_{n<\omega}$ be a T-sequence in a group G. Are the sets $\{e, a_n \colon n \in \omega\}$, $\{a_n \colon n \in \omega\}$ kaleidoscopical?

Remark 1. Let G be an infinite Abelian group, and let A be an infinite subset of G such that $A \neq A^{-1}$ and $A \cap A^{-1}$ is infinite. Then A is not rigid because the set $A^{-1}a_0^{-1} \cap A^{-1}A$ is infinite for each $a_0 \in A \setminus A^{-1}$. Thus, the sets $\{a_n \colon n \in \omega\}$ and $\{e, a_n \colon n \in \omega\}$ in Question 1 need not be rigid. Indeed, let G be a direct product $G = \bigotimes_{i\in\omega}\langle a_i\rangle$, where $\langle a_0\rangle$ is a cyclic group of order 3, $\langle a_i\rangle$, $i > 0$ are cyclic groups of order 2.

For $n \in \mathbb{N}$, we denote by $G^n[x]$ the set of all group words in the alphabet $G \cup \{x\}$ containing $\leq n$ letters x, x^{-1}. The family of subsets of G of the form

$$\{g \in G \colon w(g) \neq e, w(x) \in F\},$$

where F is a finite subset of $G^n[x]$, forms a base for the Zariski topology $\zeta_n(G)$ (see [2]). By [4], $\zeta_2(G)$ is non-discrete for every infinite group G.

Theorem 2. *Let G be a countable group, and let S be a subset of G such that $e \notin S$ but e belongs to the closure of S in $\zeta_2(G)$. Then S contains an infinite sequence $(a_n)_{n\in\omega}$ such that $\{a_n, a_n^{-1} : n \in \omega\}$ is a kaleidoscopical configuration in G.*

Proof. We enumerate $G = \{g_n : n \in \omega\}$, $g_0 = e$, construct inductively an injective sequence $(a_n)_{n\in\omega}$ in S such that the set $A = \{a_n, a_n^{-1} : n \in \omega\}$ is rigid and apply Proposition 2.

We put $b_0 = e$, $B_0 = \{b_0\}$, choose an arbitrary element $a_0 \in S$ and put $A_0 = \{a_0, a_0^{-1}\}$. Suppose that we have chosen $A_n = \{a_0, a_0^{-1}, ..., a_n, a_n^{-1}\}$, $B_n = \{b_0, b_0^{-1}, ..., b_n, b_n^{-1}\}$, $A_n \cap B_n = \varnothing$. We take the first element $g_m \in G \setminus (A_n \cup B_n)$ and put $b_{n+1} = g_m$, $B_{n+1} = B_n \cup \{b_{n+1}, b_{n+1}^{-1}\}$. Then we consider the finite system of relations

$$x^{-1}b \neq a^{-1}x,\ x^{-1}b \notin A_n^{-1}A_n,\ A_n^{-1}b \cap x^{-1}A_n = \varnothing,\ A_n^{-1}b \cap A_n^{-1}x = \varnothing,$$

where $a \in A_n$ $b \in B_{n+1}$. Since e is in the closure of S in $\zeta_2(G)$ and e satisfies the inequalities $x^{-1}b \neq a^{-1}x$, this system has a solution $a_{n+1} \in S \setminus (A_n \cup B_{n+1})$. We put $A_{n+1} = A_n \cup \{a_{n+1}, a_{n+1}^{-1}\}$.

After ω steps, we put $A = \{a_n, a_n^{-1} : n \in \omega\}$, $B = \{b_n, b_n^{-1} : n \in \omega\}$ and note that $B = G \setminus A$. By the construction, for each b_n, we have

$$A^{-1}b_n \cap A^{-1}A \subset A_n^{-1}a_n \cap A^{-1}A,$$

so A is rigid. □

Theorem 3. *Every infinite subset S of an Abelian group G contains an infinite kaleidoscopical configuration.*

Proof. We may suppose that G is countable and hence can be enumerated as $G = \{g_n : n \in \omega\}$, $g_0 = e$.

The case when $\{s^2 : s \in S\}$ is infinite. We put $b_0 = e$, $B_0 = \{b_0\}$, chose an arbitrary element $a_0 \in S \setminus \{e\}$ and put $A_0 = \{a_0\}$. Assume that we have chosen the subsets $A_n = \{a_0, ..., a_n\}$, $B_n = \{b_0, ..., b_n\}$, $A_n \cap B_n = \varnothing$. We take the first element $g_m \in G \setminus (A_n \cup B_n)$, put $b_{n+1} = g_m$, $B_{n+1} = B_n \cup \{b_{n+1}\}$ and consider the system of relations

$$x^{-1}b \neq a^{-1}x,\ x^{-1}b \notin A_n^{-1}A_n,\ A_n^{-1}b \cap x^{-1}A_n = \varnothing,\ A_n^{-1}b \cap A_n^{-1}x = \varnothing,\ a \in A_n,\ b \in B_{n+1}.$$

Since the set $\{s^2 : s \in S\}$ is infinite, this system has a solution $a_{n+1} \in S \setminus (A_n \cup \{B_{n+1}\})$. We put $A_{n+1} = A_n \cup \{a_{n+1}\}$.

After ω steps, we put $A = \{a_n : n \in \omega\}$, $B = \{b_n : n \in \omega\}$ and note that $B = G \setminus A$. By the construction, for each b_n, we have:

$$A^{-1}b_n \cap A^{-1}A \subset A_n^{-1}b_n \cap A^{-1}A,$$

so A is rigid and we can apply Proposition 2.

The case when $\{s^2 : s \in S\}$ is finite. We may suppose that $s^2 = c$ for some $c \in G$ and every $s \in S$. Choose an arbitrary $s_0 \in S$ and put $S' = s_0^{-1}S$. Then $s^2 = e$ for each $s \in S'$. We denote by H the subgroup of G generated by S', consider H as a linear space over $\mathbb{Z}_2$ and choose a countable linearly independent subset A' of S'. It is easy to see that A' is rigid in H and, by Proposition 2, A' is kaleidoscopical in H, so $s_0A' \subseteq S$ and s_0A' is a desired kaleidoscopical configuration in G. □

Corollary 2. *Every infinite subset of an Abelian group contains an infinite complemented subset.*

Remark 2. Let G be a group with presentation

$$\langle x_m, y_m \colon x_m^2 = y_m^2 = e, x_n x_m x_n = y_m, m < n < \omega \rangle.$$

To see that $x_m \neq x_n$, $y_m \neq y_n$, $m < n < \omega$, $x_i \neq x_j$ for all $i, j \in \omega$, we can use some homomorphisms from G onto the semidirect product

$$(\langle a_0 \rangle \times \langle b_0 \rangle) \leftthreetimes (\langle a_1 \rangle \times \langle b_1 \rangle),$$

where $a_0^2 = b_0^2 = a_1^2 = b_1^2 = e$ and a_1, b_1 acts on $\langle a_0 \rangle \times \langle b_0 \rangle$ as the swapping coordinates. Since $x_m x_n = x_n y_m$, $m < \omega$, we see that the subsets $X = \{x_n \colon n \in \omega\}$ has no infinite rigid subsets.

Question 2. Does every infinite subset of an infinite group contain an infinite kaleidoscopical (complemented) subset?

Acknowledgement. We thank Taras Banakh for a couple of technical and TEXnical remarks.

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Received 06.09.2011
Revised 13.10.2011