
A subset $A$ of a group $G$ is called a kaleidoscopical configuration if there exists a surjective coloring $\chi: X \to \kappa$ such that the restriction $\chi|gA$ is a bijection for each $g \in G$. We give two topological constructions of kaleidoscopical configurations and show that each infinite subset of an Abelian group contains an infinite kaleidoscopical configuration.

In this note, we give two topological constructions of kaleidoscopical configurations and show that each infinite subset of an Abelian group contains an infinite kaleidoscopical configuration.

For a hypergraph $(X, \mathcal{F})$, $x \in X$ and $A \subset X$, we put
\[
\St(x, \mathcal{F}) = \bigcup \{ F \in \mathcal{F} : x \in F \}, \quad \St(A, \mathcal{F}) = \bigcup \{ \St(a, \mathcal{F}) : a \in A \}.
\]

**Proposition 1.** A hypergraph $(X, \mathcal{F})$ is kaleidoscopical provided that, for some cardinal $\kappa$, the following two conditions are satisfied:

1. $|\mathcal{F}| \leq \kappa$ and $|F| = \kappa$ for each $F \in \mathcal{F}$;
2. for any subfamily $\mathcal{A} \subset \mathcal{F}$ of cardinality $|\mathcal{A}| < \kappa$ and any subset $B \subset X \setminus \bigcup \mathcal{A}$ of cardinality $|B| < \kappa$, the intersection $\St(B, \mathcal{F}) \cap (\bigcup \mathcal{A})$ has cardinality less than $\kappa$.

2010 Mathematics Subject Classification: 05B45, 05C15, 20K01.

Keywords: kaleidoscopical configuration, T-sequence, rigid subset.

© I. V. Protasov, S. Slobodianiuk, 2011
Proof. [2, Proposition 1.10].

We say that a subset $A$ of a group $G$ is rigid if for each $g \in G \setminus A$, the set $g^{-1}A \cap A^{-1}A$ is finite.

Proposition 2. If $A$ is a countable rigid subset of a group $G$ then $A$ is a kaleidoscopical configuration.

Proof. We may suppose that $G$ is countable. To apply Proposition 1, it suffices to show that, for all $c \in G$, $b \in G \setminus cA$, the set

$$Y = \bigcup \{ yA \cap cA : y \in G, b \in yA \}$$

is finite. Clearly, $Y \subseteq bA^{-1}A \cap cA$. We put $g = c^{-1}b$ and note that $g \in G \setminus A$. By the assumption, the set $A^{-1}g \cap A^{-1}A$ is finite, so $Y$ is finite.

An injective sequence $(a_n)_{n \in \omega}$ of elements of an infinite group $G$ is called a $T$-sequence [3] if there exists a Hausdorff group topology on $G$ in which $(a_n)_{n \in \omega}$ converges to the identity $e$ of $G$. By [3, Theorem 3.2.2], a countable group $G$ admits a non-discrete Hausdorff group topology if and only if there is a $T$-sequence in $G$. By [3, Theorem 2.1.8], each infinite subset of an Abelian group $G$ contains a $T$-sequence if and only if, for all $g \in G \setminus \{0\}$ and $k \in \mathbb{N}$, the set $\{ x \in G : kx = g \}$ is finite.

Theorem 1. For every $T$-sequence $(a_n)_{n < \omega}$ in a group $G$, the set $A = \{ e, a_n, a_n^{-1}, n \in \omega \}$ is a kaleidoscopical configuration.

Proof. In view of Proposition 2, it suffices to show that the set $A = \{ e, a_n, a_n^{-1} : n \in \omega \}$ is rigid. We assume the contrary and pick $g \in G \setminus A$, an injective sequence $(b_n)_{n \in \omega}$ in $A$ and two sequences $(c_n)_{n \in \omega}$, $(d_n)_{n \in \omega}$ in $A$ such that $b_ng = c_nd_n$ for each $n \in \omega$. Let $\tau$ be a Hausdorff group topology on $G$ in which $(a_n)_{n \in \omega}$ converges to $e$. Since, at least one of the sequences $(c_n)_{n \in \omega}$ and $(d_n)_{n \in \omega}$ must take infinitely many values, passing to limit in $\tau$, we get $g \in A$.

Corollary 1. For every $T$-sequence $(a_n)_{n < \omega}$ in a group $G$, the set $A = \{ e, a_n, a_n^{-1}, n \in \omega \}$ is complemented.

Question 1. Let $(a_n)_{n < \omega}$ be a $T$-sequence in a group $G$. Are the sets $\{ e, a_n : n \in \omega \}$, $\{ a_n : n \in \omega \}$ kaleidoscopical?

Remark 1. Let $G$ be an infinite Abelian group, and let $A$ be an infinite subset of $G$ such that $A \neq A^{-1}$ and $A \cap A^{-1}$ is infinite. Then $A$ is not rigid because the set $A^{-1}a_0^{-1} \cap A^{-1}A$ is infinite for each $a_0 \in A \setminus A^{-1}$. Thus, the sets $\{ a_n : n \in \omega \}$ and $\{ e, a_n : n \in \omega \}$ in Question 1 need not be rigid. Indeed, let $G$ be a direct product $G = \bigotimes_{i \in \omega} \langle a_i \rangle$, where $\langle a_0 \rangle$ is a cyclic group of order 3, $\langle a_i \rangle$, $i > 0$ are cyclic groups of order 2.

For $n \in \mathbb{N}$, we denote by $G^n[x]$ the set of all group words in the alphabet $G \cup \{ x \}$ containing $\leq n$ letters $x$, $x^{-1}$. The family of subsets of $G$ of the form

$$\{ g \in G : w(g) \neq e, w(x) \in F \},$$

where $F$ is a finite subset of $G^n[x]$, forms a base for the Zariski topology $\zeta_n(G)$ (see [2]). By [4], $\zeta_2(G)$ is non-discrete for every infinite group $G$. 


Theorem 2. Let $G$ be a countable group, and let $S$ be a subset of $G$ such that $e \notin S$ but $e$ belongs to the closure of $S$ in $\zeta_2(G)$. Then $S$ contains an infinite sequence $(a_n)_{n \in \omega}$ such that $\{a_n, a_n^{-1} : n \in \omega\}$ is a kaleidoscopical configuration in $G$.

Proof. We enumerate $G = \{g_n : n \in \omega\}$, $g_0 = e$, construct inductively an injective sequence $(a_n)_{n \in \omega}$ in $S$ such that the set $A = \{a_n, a_n^{-1} : n \in \omega\}$ is rigid and apply Proposition 2.

We put $b_0 = e$, $B_0 = \{b_0\}$, choose an arbitrary element $a_0 \in S$ and put $A_0 = \{a_0, a_0^{-1}\}$. Suppose that we have chosen $A_n = \{a_0, a_0^{-1}, \ldots, a_n, a_n^{-1}\}$, $B_n = \{b_0, b_0^{-1}, \ldots, b_n, b_n^{-1}\}$, $A_n \cap B_n = \emptyset$. We take the first element $g_m \in G \setminus (A_n \cup B_n)$ and put $b_{n+1} = g_m$, $B_{n+1} = B_n \cup \{b_n, b_n^{-1}\}$. Then we consider the finite system of relations

$$x^{-1}b \neq a^{-1}x, \quad x^{-1}b \notin A_n^{-1}A_n, \quad A_n^{-1}b \cap x^{-1}A_n = \emptyset, \quad A_n^{-1}b \cap A_n^{-1}x = \emptyset,$$

where $a \in A_n$, $b \in B_{n+1}$. Since $e$ is in the closure of $S$ in $\zeta_2(G)$ and $e$ satisfies the inequalities $x^{-1}b \neq a^{-1}x$, this system has a solution $a_{n+1} \in S \setminus (A_n \cup B_{n+1})$. We put $A_{n+1} = A_n \cup \{a_{n+1}, a_{n+1}^{-1}\}$.

After $\omega$ steps, we put $A = \{a_n, a_n^{-1} : n \in \omega\}$, $B = \{b_n, b_n^{-1} : n \in \omega\}$ and note that $B = G \setminus A$. By the construction, for each $b_n$, we have

$$A_n^{-1}b_n \cap A_n^{-1}A_n \subset A_n^{-1}a_n \cap A_n^{-1}A_n,$$

so $A$ is rigid.

Theorem 3. Every infinite subset $S$ of an Abelian group $G$ contains an infinite kaleidoscopical configuration.

Proof. We may suppose that $G$ is countable and hence can be enumerated as $G = \{g_n : n \in \omega\}$, $g_0 = e$.

The case when $\{s^2 : s \in S\}$ is infinite. We put $b_0 = e$, $B_0 = \{b_0\}$, choose an arbitrary element $a_0 \in S \setminus \{e\}$ and put $A_0 = \{a_0\}$. Assume that we have chosen the subsets $A_n = \{a_0, \ldots, a_n\}$, $B_n = \{b_0, \ldots, b_n\}$, $A_n \cap B_n = \emptyset$. We take the first element $g_m \in G \setminus (A_n \cup B_n)$, put $b_{n+1} = g_m$, $B_{n+1} = B_n \cup \{b_{n+1}\}$ and consider the system of relations

$$x^{-1}b \neq a^{-1}x, \quad x^{-1}b \notin A^{-1}A_n, \quad A_n^{-1}b \cap x^{-1}A_n = \emptyset, \quad A_n^{-1}b \cap A_n^{-1}x = \emptyset, \quad a \in A_n, \quad b \in B_{n+1}.$$

Since the set $\{s^2 : s \in S\}$ is infinite, this system has a solution $a_{n+1} \in S \setminus (A_n \cup \{B_{n+1}\})$. We put $A_{n+1} = A_n \cup \{a_{n+1}\}$.

After $\omega$ steps, we put $A = \{a_n : n \in \omega\}$, $B = \{b_n : n \in \omega\}$ and note that $B = G \setminus A$. By the construction, for each $b_n$, we have:

$$A^{-1}b_n \cap A^{-1}A_n \subset A^{-1}a_n \cap A^{-1}A_n,$$

so $A$ is rigid and we can apply Proposition 2.

The case when $\{s^2 : s \in S\}$ is finite. We may suppose that $s^2 = c$ for some $c \in G$ and every $s \in S$. Choose an arbitrary $s_0 \in S$ and put $S' = s_0^{-1}S$. Then $s^2 = e$ for each $s \in S'$. We denote by $H$ the subgroup of $G$ generated by $S'$, consider $H$ as a linear space over $\mathbb{Z}$, and choose a countable linearly independent subset $A'$ of $S'$. It is easy to see that $A'$ is rigid in $H$ and, by Proposition 2, $A'$ is kaleidoscopical in $H$, so $s_0A' \subseteq S$ and $s_0A'$ is a desired kaleidoscopical configuration in $G$. 

\qed
Corollary 2. Every infinite subset of an Abelian group contains an infinite complemented subset.

Remark 2. Let $G$ be a group with presentation

$$\langle x_m, y_m : x_m^2 = y_m^2 = e, x_n x_m x_n = y_m, m < n < \omega \rangle.$$ 

To see that $x_m \neq x_n, y_m \neq y_n, m < n < \omega, x_i \neq x_j$ for all $i, j \in \omega$, we can use some homomorphisms from $G$ onto the semidirect product

$$\langle (a_0) \times (b_0) \rangle \times \langle (a_1) \times (b_1) \rangle,$$

where $a_0^2 = b_0^2 = a_1^2 = b_1^2 = e$ and $a_1, b_1$ acts on $(a_0) \times (b_0)$ as the swapping coordinates. Since $x_m x_n = x_n y_m, m < \omega$, we see that the subsets $X = \{x_n : n \in \omega\}$ has no infinite rigid subsets.

Question 2. Does every infinite subset of an infinite group contain an infinite kaleidoscopical (complemented) subset?

Acknowledgement. We thank Taras Banakh for a couple of technical and \TeX\nic\al remarks.

REFERENCES

5. E. Zelenyuk, Topologization of groups, J. Group Theory, 10 (2007), 235–244.

Faculty of Cybernetics,
Kyiv Taras Shevchenko University,
i.v.protasov@gmail.com

Faculty of Mechanics and Mathematics,
Kyiv Taras Shevchenko University,
slobodianiuk@yandex.ru

Received 06.09.2011
Revised 13.10.2011