УДК 517.956.4

M. M. Mamedov

SOLVABILITY AND COMPLETENESS OF SOLUTIONS OF PARABOLIC DIFFERENTIAL-OPERATOR EQUATIONS

M. M. Mamedov. *Solvability and completeness of solutions of parabolic differential-operator equations*, Mat. Stud. **36** (2011), 77–85.

We consider an abstract Cauchy problem for parabolic differential-operator equations in Hilbert spaces. Initial boundary value problems for parabolic equations are reduced to the Cauchy problem for a system of parabolic differential equations. It is proved that the solution of an initial boundary value problem for partial parabolic equation can be approximated by linear combinations of elementary solutions. Completeness of elementary solutions is also proved for differential-operator equations in abstract Hilbert spaces. The obtained abstract results are applied to differential equations.

М. М. Мамедов. *Разрешимость и полнота решений параболических дифференциально-операторных уравнений* // Мат. Студії. – 2011. – Т.36, №1. – C.77–85.

Рассматривается абстрактная задача Коши для параболических дифференциально-операторных уравнений в гильбертовом пространстве. Краевая задача для параболического уравнения сводится к задаче Коши для системы параболических дифференциальных уравнений. Доказывается, что решение для краевой задачи для параболического уравнения в частных производных можно аппроксимировать линейной комбинацией элементарных решений. Для параболического дифференциально-операторного уравнения в абстрактном гильбертовом пространстве доказывается полнота элементарных решений. Полученные абстрактные результаты применяются к дифференциальным уравнениям.

1. Introduction. The issues of solvability of problems for differential operator equations were investigated by many authors. The papers devoted to these issues can be divided into two groups. The papers where the solvability conditions are formulated in the terms of operator coefficients refer to the first group. The papers where the conditions are formulated in terms of location of the spectrum and growth of the resolvent of an appropriate operator bundle refer to the second group.

If in the papers of the first group the conditions are acceptable in the solvability issues and spectral issues, the conditions of the second group that are acceptable in the solvability issues, are not acceptable in the spectral issues. Notice that the these results are reflected in S. Y. Yakubov monograph [1].

Notice that the results obtained the second group are reflected in S. G. Krein [4] and S. Y. Yakubov and Ya. Yakubov's monographs [5]. In recent papers A. Ashyralyev, Y. Sozen and P. E. Sobolevskii [2], V. B. Shakmurov [7] the existence of solution is investigated.

In this paper in contrast to we consider an abstract Cauchy problem for parabolic differential-operator equations in Hilbert spaces.

2010 *Mathematics Subject Classification:* 34G10, 35L20, 35L35.

It is proved that the solution of an initial boundary value problem for partial parabolic equation can be approximated by linear combinations of elementary solutions. Completeness of elementary solutions is also proved for differential-operator equations in abstract Hilbert spaces. The obtained abstract results are applied to differential equations.

2. The existence of a solution of the Cauchy problem in a Hilbert space and approximation by an elementary solution. We consider a Cauchy problem for a perturbed homogeneous equation of first order

$$u'(t) = Au(t) + Bu(t), \tag{1}$$

$$u(0) = \varphi_0 \tag{2}$$

in a Hilbert space H. We find the conditions providing approximation of solutions of Cauchy problem (1)–(2) by linear combinations of elementary solutions of equation (1). By λ_j we denote eigenvalues of the operator A+B with regard to algebraic multiplicity. If $u_{j0}, ..., u_{jk_j}$ is a chain of root vectors of the operator A+B corresponding to λ, then the function

$$u_j(t) = e^{\lambda_j t}\left(\frac{t^{k_j}}{k_j!}u_{j0} + \frac{t^{k_j-1}}{(k_j-1)!}u_{j1} + ... + \frac{t}{1!}u_{j,k_j-1} + u_{jk_j}\right) \tag{3}$$

is a solution of equation (1) and is said to be elementary solution of equation (1).

We giving some definitions. $\sigma_p(H)$, $p > 0$ set of operators A, which compact acting in H and

$$\|A\|_p^2 = \sum_{n=1}^{\infty} S_n^p(A, H) < \infty.$$

Hear $S_n(A, H)$ s-number completed continuously operator A, i.e. eigenvalue of operator $T = (A^*A)^{1/2}$. At $p = 2$ $\sigma_2(H)$ is called classes of Hilbert-Shmidt $C([0, T], H(A))$ — space of continuously functions in interval $[0, T]$, with value in $H(A)$, and

$$\|f\|_{C([0,T],H(A))} = \max_{[0,T]} \|f(x)\|_{H(A)},$$

where $H(A) = \{u\colon u \in D(A),\ \|u\|_{H(A)} = (\|u\|_H + \|Au\|_H)^{1/2}\}$.

The space $C^1([0, T], H(A), H)$ defined as follows:

$$C^1([0,T], H(A), H) = \Big\{f\colon f \in C([0,T], H(A)) \cap C^1([0,T], H(A)),$$
$$\|f\|_{C^1([0,T],H(A))} = \max_{[0,T]} \|f(x)\|_H + \max_{[0,T]} \|f'(x)\|_H\Big\}.$$

Theorem 1. *Let*

1. *the operator A in H have a dense domain of definition;*
2. *for some $p > 0$ and $\lambda_0 \in \rho(A)$ the operator $R(\lambda_0, A)$ belong to $\sigma_p(H)$;*
3. *there exist rays $\ell_k(a)$ with angles between the neighboring rays, at most $\frac{\pi}{p}$, and numbers $\alpha > 0$, $\beta \in (0, 1]$ such that*

$$\|R(\lambda, A)\| \le C|\lambda|^{-\beta},\ |\arg\lambda| \le \frac{\pi}{2} + \alpha, \lambda \in \ell_k(a),\ |\lambda| \to \infty;$$

4. *B be an operator in H, $D(B) \supset D(A)$and for any $\varepsilon > 0\|Bu\|_H \leq \varepsilon\|Au\|_H^{\beta}\|u\|_H^{1-\beta} + C(\varepsilon)\|u\|_H,\ u \in D(A)$; $C(\varepsilon) > 0$ some constant, depending only at ε.*
5. *$\varphi_0 \in D(A)$.*

Then problem (1)–(2) *has a unique solution*

$$u \in C([0,T],H) \cap C^1([0,T],H(A),H)$$

and there exist numbers C_{jn} such that

$$\lim_{n\to\infty} \max_{t\in[0,T]} \left\| u(t) - \sum_{j=1}^{n} C_{jn} u_j(t) \right\|_H = 0,$$

$$\lim_{n\to\infty} \sup_{t\in[0,T]} t\left\| u'(t) - \sum_{j=1}^{n} C_{jn} u_j'(t) \right\|_H + \left\| Au(t) - \sum_{j=1}^{n} C_{jn} Au_j(t) \right\|_H = 0,$$

where $u(t)$ is a solution of problem (1)–(2) *and $u_j(t)$ are elementary solutions of equation* (1).

Proof. Since the system of root vectors of the operator $A+B$ is complete in the space $H(A)$, there exist the numbers C_{jn} such that

$$\lim_{n\to\infty} \left(\left\| \varphi_0 - \sum_{j=1}^{n} C_{jn} u_{jk_j} \right\|_H + \left\| A\varphi_0 - \sum_{j=1}^{n} C_{jn} Au_{jk_j} \right\|_H \right) = 0. \tag{4}$$

On the other hand, the estimation

$$\left\| u(t) - \sum_{j=1}^{n} C_{jn} u_j(t) \right\| \leq C\left(\left\| \varphi_0 - \sum_{j=1}^{n} C_{jn} u_{jk_j} \right\|_H + \left\| A\varphi_0 - \sum_{j=1}^{n} C_{jn} Au_{jk_j} \right\|_H \right) \tag{5}$$

$$\begin{aligned} &\left\| u'(t) - \sum_{j=1}^{n} C_{jn} u_j'(t) \right\| + \left\| Au(t) - \sum_{j=1}^{n} C_{jn} Au_j(t) \right\|_H \leq \\ &\leq Ct^{-1}\left(\left\| \varphi_0 - \sum_{j=1}^{n} C_{jn} u_{jk_j} \right\|_H + \left\| A\varphi_0 - \sum_{j=1}^{n} C_{jn} Au_{jk_j} \right\|_H \right) \end{aligned} \tag{6}$$

follows from theorem 3.3 (see [1], p.113), for $f = 0$, $u_0 = \varphi_0 - \sum_{j=1}^{n} c_{jn} u_{jk_j}$. The affirmation of the theorem follows from (5) and (6), by the virtue of (4). □

3. The estimate operator $R(\lambda, A)$ and completeness of elementary solutions in a Hilbert space. In the Hilbert space H we consider the Cauchy problem

$$u'(t) = Au(t) + Bu(t) + f(t),\ \ u(0) = u_0. \tag{7}$$

$(H, H(A))_{1-\frac{1}{p},p}$-interpolation space between $H(A)$ and H, which defined as follows:

$$(H,H(A))_{\theta,p} = \left\{ u\colon u \in H,\ \|u\|_{(H,H(A))_{\theta,p}} = \left(\int_0^1 t^{(1-\theta)p-1} \left\| Ae^{-tA}u \right\|^p dt \right)^{1/p} + \|u\|_H < \infty \right\}.$$

At $\theta = 1 - \frac{1}{p}$, $\|u\|_{(H,H(A))_{1-\frac{1}{p},p}} = \left(\int_0^1 \|Ae^{-tA}u\|^p\, dt \right)^{1/p} + \|u\|_H < \infty$.

Theorem 2. *Let*

1. *the operator A in H have a dense domain of definition and*

$$\|R(\lambda, A)\| \leq C|\lambda|^{-1}, \ |\arg\lambda| < \frac{\pi}{2}, \quad |\lambda| \to \infty;$$

2. *B be an operator in H, $D(B) \supset D(A)$and for any $\varepsilon > 0$*

$$\|Bu\|_H \leq \varepsilon\|Au\|_H + C(\varepsilon)\|u\|_H, \ u \in D(A).$$

Then the operator $L\colon u \to (u'(t) - Au(t) - Bu(t),\, u(0))$ from $W_p^1((0,T), H(A), H)$ onto $L_p((0,T), H) \times (H, H(A))_{1-\frac{1}{p},p}$ for $p \in (1, \infty)$ is an isomorphism.

Proof. Based on the Banach theorem it suffices to prove that the operator L is an algebraic isomorphism. By the definition of the space $W_p^1((0,T), H(A), H)$, it follows from $u \in W_p^1((0, T), H(A), H)$ that $u' - Au - Bu \in L_p((0,T), H)$, and by the theorem on traces (see H. Triebel [6], p.46) it follows from $u \in W_p^1((0,T), H(A), H)$ that $u(0) \in (H, H(A))_{1-\frac{1}{p},p}$.
The following estimation is true (see for example [7], p.32, Lemma 11)

$$\|R(\lambda, A + B)\| \leq C|\lambda|^{-1}, \ |\arg\lambda| < \frac{\pi}{2}, \ |\lambda| \to \infty.$$

The mapping L is injective. The injective of mapping h is consequence from theorem 3.2 (see [1], p.100). Prove that it is surjective, i.e. for any collection

$$(f,\, u_0) \in L_p((0,T), H) \times (H, H(A))_{1-\frac{1}{p},p}$$

there exists a solution of Cauchy problem (11), belonging to $W_p^1((0,T), H(A), H)$. For $s \in C$ the solution of the problem is represented in the form of the sum $u(t) = u_1(t) + u_2(t)$ where $u_1(t)$ is a contraction on $[0, T]$ of the solution of the equation

$$u_1'(t) - (A + B - sI)u_1(t) = \tilde{f}(t), \ \ t \in R, \tag{8}$$

where $\tilde{f}(t) = f(t)$ for $t \in [0, T]$, $\tilde{f}(t) = 0$ for $t\bar{\in}[0, T]$, and $\varphi_2(x)$ is a solution of the problem

$$u_2'(t) - (A + B - sI)u_2(t) = 0, \ \ u_2(0) = u_0 - u_1(0). \tag{9}$$

It follows from the conditions that, for sufficiently large $s \to \infty$

$$\|[\lambda I - (A + B - sI)]^{-1}\| \leq C|\lambda|^{-1}, \quad |\arg\lambda| \leq \frac{\pi}{2} \tag{10}$$

the equation (8) has a unique solition $u_1 \in W_p^1(R, H(A), H)$.

By (10) there exists the semi-group $e^{t(A+B-sI)}$.

On the other hand, by the theorem on traces $u_1(0) \in (H, H(A))_{1-\frac{1}{p},p}$.

Now it is easy to observe that the function

$$u_2(t) = e^{t(A+B-sI)}(u_0 - u_1(0))$$

belongs to the space $W_p^1((0,T),H(A),H)$ and it is a solution of Cauchy problem (9). Indeed, by ([6], p.109) we have

$$\|u_2\|^p_{W_p^1((0,T),H(A+B-sI),H)} \le$$
$$\le C\int_0^T \|(A+B-sI)e^{t(A+B-sI)}(u_0-u_1(0))\|^p_{W_p^1((0,T),H(A+B-sI),H)}dt$$
$$\le C\|u_0-u_1(0)\|^p_{(H,H(A-B+sI))_{1-\frac{1}{p},p}}.$$

Now it suffices to take into account, that $H(-A-B+sI)=H(A)$. Using the substitution $u(t)=e^{-st}V(t)$, we also establish the existence of the solution of problem (7). □

Theorem 3. *Let*

1. *the operator A in H have a dense domain of definition $D\ (A)$;*
2. *For some $q>0$ and $\lambda_0\in\rho(A)$ we have $R(\lambda_0,A)\in\sigma_p(H)$;*
3. *there exist rays $\ell_k(A)$ with angles between neighboring rays, at most $\frac{\pi}{q}$, such that*

$$R(\lambda,A)\le C|\lambda|^{-1},\ |\arg\lambda|\le\frac{\pi}{2},\ \lambda\in\ell_k(A),\ \ |\lambda|\to\infty;$$

4. *B be an operator in H, $D(B)\supset D(A)$ and the operator $BR(\lambda_0,A)$ in H be compact;*
5. *$\varphi_0\in(H,H(A))_{1-\frac{1}{p},p}$ for some $p\in(1,\infty)$.*

Then problem (1)–(2) has a unique solution $u\in W_p^1((0,T),H(A),H)$ and there exist the numbers C_{jn} such that

$$\lim_{n\to\infty}\int_0^T\left(\left\|u'(t)-\sum_{j=1}^n C_{jn}u_j'(t)\right\|_H^p+\left\|Au(t)-\sum_{j=1}^n C_{jn}Au_j(t)\right\|_H^p\right)dt=0,$$

where $u(t)$ is a solution of problem (1)–(2), and $u_j(t)$, $j=1,\dots,n$, are the elementary solutions of equation (1).

Proof. A linear span of the root vectors $A+B$ is dense in the space $H(A)$. On the other hand, by H. Triebel ([6], p.40), the set $H(A)$ is dense in the space $((H,H(A))_{\cdot\theta,p}$. So, the linear span of root vectors of the operator $A+B$ is dense in the space $((H,H(A))_{\cdot\theta,p}$. Then by Theorem 2, we have

$$\left\|u(\cdot)-\sum_{j=1}^n C_{jn}u_j(\cdot)\right\|_{W_p^1((0,T),H(A),H)}\le C\left\|\varphi_0-\sum_{j=1}^n C_{jn}u_{jk_j}\right\|_{(H,H(A))_{1-\frac{1}{p},p}},$$

whence the affirmation of the theorem follows. □

4. Abstract results applied to partial differential equations. Consider the initial-boundary value problem for the parabolic equation

$$Lu=\frac{\partial u(t,x)}{\partial t}+a(t,x)\frac{\partial^{2m}u(t,x)}{\partial x^{2m}}+\sum_{j=1}^{2m-1}a_j(t,x)\frac{\partial^j u(t,x)}{\partial x^j}=f(t,x), \tag{11}$$

$$L_\nu u = \alpha_\nu u^{(m_\nu)}(t,0) + \beta_\nu u_x^{(m_\nu)}(t,1) + \sum_{s=1}^{N_\nu} \delta_{\nu s}^{(n_\nu)} u_x^{(m_\nu)}(t, x_{\nu s}) + T_\nu u(t,\cdot) = 0 \qquad (12)$$
$$\nu = 1,\dots,2m,$$

$$u(0,x) = u_0(x), \qquad (13)$$

where $x_{\nu s} \in (0,1)$, $m_\nu \le 2m-1$.

Let $W_q^{m_\nu}(0,1)$ — space of Sobolev, which defined as follows

$$W_q^{m_\nu}(0,1) = \{u\colon u,\, u^{m_\nu} \in L_q(0,1)\}, \quad \|u\|_{W_q^{m_\nu}(0,1)} = \{\|u\|_{L_q(0,1)} + \|u^{m_\nu}\|_{L_q(0,1)}\}^{1/q}.$$

Theorem 4. *Let*

1. $a \in C([0,T]\times[0,1])$, $a(t,x) \ne 0$, *for* $(t,x) \in [0,T]\times[0,1]$,
$$a_j \in C([0,T],\, L_2(0,1)),\ j = 0,\dots,2m-1;$$
2. *For some* $\delta > 0$ $|\arg a(t,x)| \le \frac{\pi}{2} - \delta$, *if m is even and* $|\arg a(t,x)| \ge \frac{\pi}{2} + \delta$, *if m is odd;*
3. *boundary conditions* (16) *be m-regular, i.e.*
$$\theta = \begin{vmatrix} \alpha_1\omega_1^{m_1} \dots \alpha_1\omega_m^{m_1} & & \beta_1\omega_{m+1}^{m_1} \dots \beta_1\omega_{2m}^{m_1} \\ - & - & - \\ \alpha_{2m}\omega_1^{m_{2m}} \dots \alpha_{2m}\omega_m^{m_{2m}} & & \beta_{2m}\omega_{m+1}^{m_{2m}} \dots \beta_{2m}\omega_{2m}^{m_{2m}} \end{vmatrix} \ne 0$$
where $\omega_1 = 1$, $\omega_2 = e^{i\frac{\pi}{m}}, \dots, \omega_{2m} = e^{i\frac{\pi}{m}(2m-1)}$;
4. *for some* $q \in [1,\infty)$ *the functionals* T_ν *in* $W_q^{m_\nu}(0,I)$ *be continuous.*

Then the operator $\mathcal{L}\colon u \to (Lu,\, u(0,x))$ *from* $W_{(p,2)}^{(1,2m)}((0,T)\times(0,1), L_\nu u = 0, \nu = 1,\dots,2m)$ *onto* $L_{(\nu,2)}((0,T)\times(0,1)\times(L_2(0,1), W_2^{2m}((0,1), L_\nu u = 0, \nu = 1,\dots,2m)_{1-\frac{1}{p},p}))$ *is an isomorphism for* $p \in (1,\infty)$.

Proof. We reduce problem (11)–(13) to the Cauchy problem for an abstract parabolic equation of first order.

By A(t) the denote an operator in $L_2(0,1)$ with domain of definition $D(A(t)) = = D(A) = W_2^{2m}((0,1), L_\nu u = 0,\ \nu = 1,\dots,2m)$ independent of $t \in [0,T]$ and law of action $A(t)u = -a(t,x)u^{(2m)}(x)$. By $B(t)$ we denote an operator in $L_2(0,1)$ with domain of definition $D(B(t)) = W_2^{2m}(0,1)$ and law of action

$$B(t)u = -\sum_{j=0}^{2m-1} a_j(t,x)u^{(j)}(x).$$

Then problem (11)–(13) can be written in the form

$$u'(t) = A(t)u(t) + B(t)u(t) + f(t), \quad u(0) = u_0, \qquad (14)$$

where $u(t) = u(t;\cdot)$, $f(t) = f(t,\cdot)$ $u = u_0(\cdot)$ are functions with values in the Hilbert space $H = L_2(0,1)$. The operator $A(t)$ satisfies the estimation $\|R(\lambda, A(t))\| \le C|\lambda|^{-1}$, $|\arg\lambda| \le \frac{\pi}{2}$, $|\lambda| \to \infty$.

By condition 1, the operator $B(t)$ is completely subjected to the operator $A(t)$ uniformly with respect to $t \in [0,T]$. Indeed, applying the estimation

$$\|u^{(j)}\|_{L_\infty(0,1)} \le \varepsilon\|u^{(2m)}\|_{L_2(0,1)} + C(\varepsilon)\|u\|_{L_2(0,1)},\ j < 2m$$

we get

$$\|B(t)u\|_{L_2(0,1)} \le \sum_{j=0}^{2m-1}\left(\int_0^1 |a_j(t,x)|^2|u^{(j)}(x)|^2dx\right)^{\frac{1}{2}} \le \sum_{j=0}^{2m-1}\|a_j(t,\cdot)\|_{L_2(0,1)}\|u^{(j)}\|_{L_\infty(0,1)} \le$$
$$\le \varepsilon\|u^{(2m)}\|_{L_2(0,1)} + C(\varepsilon)\|u\|_{L_2(0,1)} \le \varepsilon\|A(t)u\|_{L_2(0,1)} + C(\varepsilon)\|u\|_{L_2(0,1)},$$

where $C(\varepsilon) > 0$ finite number at any $\varepsilon > 0$. So, all the conditions of Theorem 3.8 of the paper ([1], p.139) are satisfied, whence the affirmation of Theorem 4 follows. □

Consider the initial-boundary value problem

$$\frac{\partial u(t,x)}{\partial t} + a(t)\frac{\partial^{2m}u(t,x)}{\partial x^{2m}} + \sum_{k=0}^{2m-1} a_k(x)\frac{\partial^k u(t,x)}{\partial x^k} = 0. \tag{15}$$

$$\begin{aligned} &L_\nu u = \alpha_\nu u_x^{(m_\nu)}(t,x) + \beta_\nu u_x^{(m_\nu)}(t,1) + \sum_{p=1}^{N_\nu}\delta_{\nu p}u_x^{(m_\nu)}(t,x_{\nu p}) + T_\nu u(t,\cdot) = 0,\\ &\nu = 1,\dots,2m, \end{aligned} \tag{16}$$

$$u(0,x) = \varphi_0(x), \tag{17}$$

where $x_{\nu p} \in (0,1)$, $m_\nu \le 2m-1$, and appropriate spectral problem

$$\lambda u(x) + a(x)u^{(2m)}(x) + \sum_{k=0}^{2m-1} a_k(x)u^{(k)}(x) = 0, \tag{18}$$

$$L_\nu u = 0, \quad \nu = 1,\dots,2m. \tag{19}$$

The function of the form

$$u_j(t,x) = e^{\lambda_j t}\left(\frac{t^{k_j}}{k_j!}u_{j0}(x) + \frac{t^{k_j-1}}{(k_j-1)!}u_{j1}(x) + ... + u_{jk_j}(x)\right)$$

becomes an elementary solution of problem (15)–(17), if and only if the system of functions $u_{j0}(x), u_{j1}(x), ..., u_{jk_j}(x)$ is a chain of root functions of problem (18)–(19) corresponding to the eigenvalue λ_j.

Theorem 5. *Let*

1. $a(x) \ne 0$, $a \in C[0,1]$, $a_k \in L_2(0,1)$, $k = 0,\dots,2m-1$;
2. $|\arg a(x)| < \frac{\pi}{2}$, *if m is even;* $|\arg a(x)| > \frac{\pi}{2}$, *if m is odd;*
3. $\theta = \begin{vmatrix} \alpha_1\omega_1^{m_1} & \cdots & \alpha_1\omega_m^{m_1}\beta_1\omega_{m+1}^{m_1} & \cdots & \beta_1\omega_{2m}^{m_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{2m}\omega_1^{m_{2m}} & \cdots & \alpha_{2m}\omega_m^{m_{2m}}\beta_{2m}\omega_{m+1}^{m_{2m}} & \cdots & \beta_{2m}\omega_{2m}^{m_{2m}} \end{vmatrix} \ne 0,$
   where $\omega_1 = 1$, $\omega_2 = e^{i\frac{\pi}{m}}, \dots, \omega_{2m} = e^{i\frac{\pi}{m}(2m-1)}$;
4. *for some* $q \in [1,\infty)$ *the functions* T_ν *in* $W_q^{m_\nu}(0,1)$ *be continuous.*
5. $\varphi_0 \in W_2^{2m}((0,1), L_\nu u = 0,\ \nu = 1,\dots,2m)$.

Then the problem (15)–(17) *has a unique solution*

$$u \in C'([0,T], W_2^{2m}(0,1), L_2(0,1))$$

and there exist numbers C_{jn} *such, that*

$$\lim_{n\to\infty} \max_{t\in[0,T]} \left(\left\| u'_t(t,\cdot) - \sum_{j=1}^{n} C_{jn} u'_{jt}(t,\cdot) \right\|_{L_2(0,1)} + \left\| u(t,\cdot) - \sum_{j=1}^{n} C_{jn} u_j(t,\cdot) \right\|_{W_2^{2m}(0,1)} \right) = 0, \quad (20)$$

where $u(t,x)$ *is a solution of problem* (15)–(17), *and* $u_j(t,x)$ *are elementary solutions of problem* (15)–(17).

Proof. We apply Theorem 1 to problem (15)–(17). In $H = L_2(0,1)$ we consider the operators A and B given by the equalities.

$$D(A) = W_2^{2m}((0,1),\ L_\nu u = 0,\ \nu = 1,\dots,2m), Au = -a(x)u^{(2m)}(x); \quad (21)$$

$$D(B) = W_2^{2m}(0,1), \ \ Bu = -\sum_{k=0}^{2m-1} a_k(x) u^{(k)}(x). \quad (22)$$

Then problem (15)–(17) can be written in the form

$$u'(t) = Au(t) + Bu(t), \quad (23)$$

$$u(0) = \varphi_0. \quad (24)$$

The operator A satisfies the first condition of Theorem 2. For some $\alpha > 0$, $R(\lambda, A) \le C|\lambda|^{-1}$, $|\arg \lambda| \le \frac{\pi}{2} + \alpha$, $|\lambda| \to \infty$. By H. Triebel ([6], p.437)

$$s_j(J,\, W_2^{2m}(0,1), L_2(0,1)) \sim j^{-2m}.$$

So, for $p > \frac{1}{2m}$ we have $J \in \sigma_p(W_2^{2m}(0,1), L_2(0,1))$. Since $W_2^{2m}((0,1), L_\nu u = 0,\ \nu = 1,\dots,2m)$ is a subspace of $W_2^{2m}(0,1)$, one has that $J \in \sigma_p(W_2^{2}(0,1),\ L_\nu u = 0,\ \nu = 1,\dots,2m,\ L_2(0,1))$.

Then for $p > \frac{1}{2m}$ and $\lambda_0 \in \rho(A)$ we have $R(\lambda_0 A) \in \sigma_p(L_2(0,1))$. Consequently condition 2 of Theorem 2 is satisfied. When proving Theorem 5 we establish the estimation

$$\|Bu\|_{L_2(0,1)} \le \varepsilon \|Au\|_{L_2(0,1)} + C(\varepsilon)\|u\|_{L_2(0,1)}, \quad u \in D(A), \ \ \varepsilon > 0,$$

i.e. condition of Theorem 2 holds.

Therefore, applying Theorem 1 to problem (23)–(24), we prove Theorem 5. □

Acknowledgements. The author would like to express their thanks to Prof. V. S. Guliyev, Prof. T. S. Gadjiev and Prof. G. I. Aslanov for many helpful discussions on this subject.

REFERENCES

1. Yakubov S.Y. Linear differential-operator equations and their applications. – Baku, Elm, 1985.
2. Ashyralyev A., Sozen Y., Sobolevskii P.E. A note on the parabolic differential and difference equations, Abstr. Appl. Anal. – 2007, Art. ID 61659. – 16p.
3. Amann H. Linear and quasilinear parabolic problems, Birkhauser – Basel, 1995.
4. Krein S.G. Linear differential equations in Banach space – M. Nauka, 1967.
5. Yakubov S.Y., Yakubov Y.Y. Differential-Operator Equations, Ordinary and Partial Differential Equations, – Chapmen and Hall /CRC, Boca Raton, 2000.
6. Triebel H. Interpolation theory, function spaces, differential operators, VEB Deutscher Verlag der Wissenschaften – Berlin, 1978. – 528p.
7. Shakhmurov V.B. *Coercive boundary value problems for regular degenerate differential-operator equations*// J. Math. Anal. Appl. – 2004. – V.292, №2. – P. 605–620.

Sumgait State University, Azerbaijan
E-mail: mamadali@yahoo.com

Received 25.02.2011
Revised 06.06.2011