УДК 517.574

Yu. S. Protsyk, Ya. V. Vasyl'kiv

ON GROWTH MAJORANTS OF SUBHARMONIC FUNCTIONS

Yu. S. Protsyk, Ya. V. Vasyl'kiv. *On growth majorants of subharmonic functions*, Mat. Stud. **36** (2011), 73–76.

We describe growth majorants of subharmonic in $\mathbb{R}^m$ $(m \geq 2)$ functions. To do this, we exceptionally reduce the problem to problems in the theory of positive monotonous functions.

Ю. С. Процык, Я. В. Василькив. *О мажорантах роста субгармонических функций* // Мат. Студії. – 2011. – Т.36, №1. – С.73–76.

Описаны мажоранты роста субгармонических в $\mathbb{R}^m$ $(m \geq 2)$ функций. При этом проблема исследования мажорант роста сведена исключительно к задачам из теории положительных монотонных функций.

1. Introduction. Statements of the main results. This communication should be considered as an addition to [1]. So, we follow the notation from [1]. Recall the following definitions briefly.

Let λ be a nonnegative, continuous, nondecreasing, and unbounded function on $(0, +\infty)$, named the growth function and $\lambda(0) = 0$. The set of nonnegative Borel measures μ in $\mathbb{R}^m$ $(m \geq 2)$, $0 \notin \operatorname{supp}\mu$, such that $N(r, \mu) \leq a\lambda(br)$ for some $a, b > 0$ and all $r > 0$ is denoted by $\mathcal{M}_\lambda^m$. Here

$$N(r;\mu) = \begin{cases} (m-2)\int\limits_0^r n(t;\mu)t^{1-m}dt, & m \geq 3; \\ \int\limits_0^r n(t;\mu)t^{-1}dt, & m = 2; \end{cases} \qquad n(t;\mu) = \mu\left(\{y\colon |y| \leq t\}\right).$$

By S_λ^m we denote the set of subharmonic functions u in $\mathbb{R}^m$ $(m \geq 2)$, $u(0) = 0$ whose Riesz measure μ_u belongs to $\mathcal{M}_\lambda^m$.

Definition 1. Let λ be a growth function. A δ-subharmonic function w in $\mathbb{R}^m$ $(m \geq 2)$, $w(0) = 0$, $0 \notin \operatorname{supp}\mu_w$, is said to be a *function of finite λ-type* if there are constants a and b such that $T(r, w) \leq a\lambda(br)$ for all $r > 0$, where $T(r, w)$ is the Nevanlinna characteristic of w [2].

The class of such functions is denoted by $\Lambda_\delta^m(\lambda)$, and by $\Lambda_S^m(\lambda)$ we denote the subclass of subharmonic functions of finite λ-type.

We recall the following definitions from [1].

2010 *Mathematics Subject Classification:* 31A05.

Definition 2. A class $\Lambda_\delta^m(\lambda)$ admits a *canonical representation* if each function w from $\Lambda_\delta^m(\lambda)$ can be represented as a difference $w = u - v$ of subharmonic functions u, v from $\Lambda_S^m(\lambda)$ such that $\mu_u = \mu_w^+,\ \mu_v = \mu_w^-$. Here $\mu_w^+,\ \mu_w^-$ are positive and negative variations of the Riesz measure μ_w.

Definition 3. A growth function $\widetilde{\lambda}$ is called a *growth majorant for* S_λ^m if for an arbitrary measure μ from $\mathcal{M}_\lambda^m$ there exists a subharmonic function u from $\Lambda_S^m(\widetilde{\lambda})$ such that $\mu_u = \mu$.

Definition 4. A growth function $\widehat{\lambda}$ is called a *minimal growth majorant for* S_λ^m if it is a growth majorant and for each growth majorant $\widetilde{\lambda}$ for S_λ^m there exist constants $a, b > 0$ such that $\widehat{\lambda}(r) \le a\widetilde{\lambda}(br)$ for all $r > 0$.

The following results were obtained in [1].

Theorem A. *Let λ be a growth function such that the function $r^{m-1}\lambda'(r)$ $(m \ge 2)$ is nondecreasing on $(0, +\infty)$, where $\lambda'(r)$ denotes the right-hand derivative.*

i) $\Lambda_\delta^m(\lambda)$ admits a canonical representation if and only if λ is the minimal growth majorant for S_λ^m;

ii) λ is the minimal growth majorant for S_λ^m if and only if

$$\int_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}}\,dt \le ak^l\left(\frac{\lambda(br_1)}{r_1^k} + \frac{\lambda(br_2)}{r_2^k}\right), \qquad k \in \mathbb{N} \tag{1}$$

for some $a,\ b,\ l$ and for arbitrary $r_1,\ r_2$, such that $0 < r_1 < r_2$.

Lemma B. *Let $m \in \mathbb{N} \cap [2, +\infty)$. If a growth function $\lambda(r)$ is such that the function $r^{m-1}\lambda'(r)$ $(m \ge 2)$ is nondecreasing on $(0, +\infty)$ and $\widetilde{\lambda}(r)$ is a growth majorant for S_λ^m, then*

1. *$\lambda(r) \le a\widetilde{\lambda}(br)$ for some $a, b > 0$ and all $r > 0$;*
2. *each function w from $\Lambda_\delta^m(\lambda)$ is representable as a difference $w = u - v$ of subharmonic functions u, v from $\Lambda_S^m(\widetilde{\lambda})$ such that $\mu_u = \mu_w^+,\ \mu_v = \mu_w^-$.*

From Theorem A it follows that for an arbitrary growth function λ the class $\Lambda_\delta^m(\lambda)$ need not admit a canonical representation. Lemma B lets us to set a more general statement on solvability of the problem on a canonical representation. The following theorem describes the growth majorants.

Theorem 1. *Let λ be a growth function such that function $r^{m-1}\lambda'(r)$ $(m \ge 2)$ is nondecreasing on $(0, +\infty)$. Then a growth function $\widetilde{\lambda}$ is a growth majorant for S_λ^m if and only if*

$$\int_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}}\,dt \le ak^l\left(\frac{\widetilde{\lambda}(br_1)}{r_1^k} + \frac{\widetilde{\lambda}(br_2)}{r_2^k}\right),\ k \in \mathbb{N} \tag{2}$$

for some $a,\ b,\ l$ and for arbitrary $r_1,\ r_2$, such that $0 < r_1 < r_2$.

The proof of this theorem is similar (mutatis mutandis) to that of Theorem A so, we omitted it.

The following result follows from Theorem 1.

Theorem 2. *Let λ be a growth function such that function $r^{m-1}\lambda'(r)$ $(m \geq 2)$ is nondecreasing on $(0, +\infty)$ and $q(t)$ is nondecreasing, positive, integer function such that the integral $\int\limits_r^{+\infty} \left(\frac{r}{t}\right)^{q(t)} \frac{\lambda(t)}{t}\, dt$ is finite for any $r > 0$. Then the function*

$$\widetilde{\lambda}(r) = \int\limits_0^r \left(\frac{r}{t}\right)^{q(t)-1} \frac{\lambda(t)}{t}\, dt + \int\limits_r^{+\infty} \left(\frac{r}{t}\right)^{q(t)} \frac{\lambda(t)}{t}\, dt \tag{3}$$

is the growth majorant for S_λ^m.

2. Proof of Theorem 2. Let us show that $\widetilde{\lambda}(r)$ from Theorem 1 satisfies condition (2).

Let $\{n_1, n_2, \ldots, n_i, \ldots\}$, $n_i \in \mathbb{N}$ be the set of values for the function $q(t)$. Note that the elements are placed in the ascending order. We set $y_i = \inf\{t\colon\ q(t) = n_i\}$.

For arbitrary $r_1,\ r_2$ $(0 < r_1 < r_2)$, $i \in \mathbb{N}$, we have (a) there exists $n_i \in \mathbb{N}$ such that $r_1 \leq y_i \leq r_2$ or (b) such n_i does not exist.

For the case (a), we have

$$\int\limits_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}} dt = \int\limits_{r_1}^{y_j} \frac{\lambda(t)}{t^{k+1}} dt + \sum_{i=j}^{l-1} \int\limits_{y_i}^{y_{i+1}} \frac{\lambda(t)}{t^{k+1}} dt + \int\limits_{y_l}^{r_2} \frac{\lambda(t)}{t^{k+1}} dt, \tag{4}$$

where $j = \min\{i \in \mathbb{N}\colon\ r_1 \leq y_i\}$, $l = \max\{i \in \mathbb{N}\colon\ y_i \leq r_2\}$. Now we estimate the first summand from the right-hand side of (4). For $k \geq n_{j-1}$, we get

$$\int\limits_{r_1}^{y_j} \frac{\lambda(t)}{t^{k+1}} dt = \frac{1}{r_1^k} \int\limits_{r_1}^{y_j} \left(\frac{r_1}{t}\right)^k \frac{\lambda(t)}{t} dt \leq \frac{1}{r_1^k} \int\limits_{r_1}^{y_j} \left(\frac{r_1}{t}\right)^{n_{j-1}} \frac{\lambda(t)}{t} dt = \frac{1}{r_1^k} \int\limits_{r_1}^{y_j} \left(\frac{r_1}{t}\right)^{q(t)} \frac{\lambda(t)}{t} dt.$$

For $k \leq n_{j-1} - 1$, we obviously have

$$\int\limits_{r_1}^{y_j} \frac{\lambda(t)}{t^{k+1}} dt = \frac{1}{r_2^k} \int\limits_{r_1}^{y_j} \left(\frac{r_2}{t}\right)^k \frac{\lambda(t)}{t} dt \leq \frac{1}{r_2^k} \int\limits_{r_1}^{y_j} \left(\frac{r_2}{t}\right)^{n_{j-1}-1} \frac{\lambda(t)}{t} dt = \frac{1}{r_2^k} \int\limits_{r_1}^{y_j} \left(\frac{r_2}{t}\right)^{q(t)-1} \frac{\lambda(t)}{t} dt.$$

Therefore,

$$\int\limits_{r_1}^{y_j} \frac{\lambda(t)}{t^{k+1}} dt \leq \frac{1}{r_1^k} \int\limits_{r_1}^{y_j} \left(\frac{r_1}{t}\right)^{q(t)} \frac{\lambda(t)}{t} dt + \frac{1}{r_2^k} \int\limits_{r_1}^{y_j} \left(\frac{r_2}{t}\right)^{q(t)-1} \frac{\lambda(t)}{t} dt. \tag{5}$$

As above, for the other integrals in the right-hand side of (4), we obtain

$$\int\limits_{y_j}^{y_{j+1}} \frac{\lambda(t)}{t^{k+1}} dt \leq \frac{1}{r_1^k} \int\limits_{y_j}^{y_{j+1}} \left(\frac{r_1}{t}\right)^{q(t)} \frac{\lambda(t)}{t} dt + \frac{1}{r_2^k} \int\limits_{y_j}^{y_{j+1}} \left(\frac{r_2}{t}\right)^{q(t)-1} \frac{\lambda(t)}{t} dt,\ j \leq i \leq l-1, \tag{6}$$

$$\int\limits_{y_l}^{r_2} \frac{\lambda(t)}{t^{k+1}} dt \leq \frac{1}{r_1^k} \int\limits_{y_l}^{r_2} \left(\frac{r_1}{t}\right)^{q(t)} \frac{\lambda(t)}{t} dt + \frac{1}{r_2^k} \int\limits_{y_l}^{r_2} \left(\frac{r_2}{t}\right)^{q(t)-1} \frac{\lambda(t)}{t} dt. \tag{7}$$

Using (4)–(7), we get

$$\int_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}}dt \le \frac{1}{r_1^k}\int_{r_1}^{r_2}\left(\frac{r_1}{t}\right)^{q(t)}\frac{\lambda(t)}{t}dt + \frac{1}{r_2^k}\int_{r_1}^{r_2}\left(\frac{r_2}{t}\right)^{q(t)-1}\frac{\lambda(t)}{t}dt \le$$

$$\le \frac{1}{r_1^k}\int_{r_1}^{+\infty}\left(\frac{r_1}{t}\right)^{q(t)}\frac{\lambda(t)}{t}dt + \frac{1}{r_2^k}\int_{0}^{r_2}\left(\frac{r_2}{t}\right)^{q(t)-1}\frac{\lambda(t)}{t}dt \le \frac{\widetilde{\lambda}(r_1)}{r_1^k} + \frac{\widetilde{\lambda}(r_2)}{r_2^k}.$$

Similarly, for the case (b), we have

$$\int_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}}dt \le \frac{1}{r_1^k}\int_{r_1}^{r_2}\left(\frac{r_1}{t}\right)^{q(r_1)}\frac{\lambda(t)}{t}dt + \frac{1}{r_2^k}\int_{r_1}^{r_2}\left(\frac{r_2}{t}\right)^{q(r_1)-1}\frac{\lambda(t)}{t}dt =$$

$$= \frac{1}{r_1^k}\int_{r_1}^{r_2}\left(\frac{r_1}{t}\right)^{q(t)}\frac{\lambda(t)}{t}dt + \frac{1}{r_2^k}\int_{r_1}^{r_2}\left(\frac{r_2}{t}\right)^{q(t)-1}\frac{\lambda(t)}{t}dt \le$$

$$\le \frac{1}{r_1^k}\int_{r_1}^{+\infty}\left(\frac{r_1}{t}\right)^{q(t)}\frac{\lambda(t)}{t}dt + \frac{1}{r_2^k}\int_{0}^{r_2}\left(\frac{r_2}{t}\right)^{q(t)-1}\frac{\lambda(t)}{t}dt \le \frac{\widetilde{\lambda}(r_1)}{r_1^k} + \frac{\widetilde{\lambda}(r_2)}{r_2^k}.$$

Theorem 2 is proved.

REFERENCES

1. Yu.S. Protsyk, *Growth majorants and canonical representation of δ-subharmonic functions*, Mat. Stud. **20** (2003), №1, 40–52. (in Ukrainian)
2. M.G. Arsove, *Function representable as differences of subharmonic functions*, Trans. Amer. Math. Soc. **75** (1953), 327–365.

Technological Faculty,
Ukrainian National Forestry University,
yu.protsyk@gmail.com

Faculty of Mechanics and Mathematics,
Lviv National University,
yavvasylkiv@gmail.com

Received 16.05.2006
Revised 18.03.2011