ON GROWTH MAJORANTS OF SUBHARMONIC FUNCTIONS


We describe growth majorants of subharmonic in $\mathbb{R}^m$ ($m \geq 2$) functions. To do this, we exceptionally reduce the problem to problems in the theory of positive monotonous functions.

1. Introduction. Statements of the main results. This communication should be considered as an addition to [1]. So, we follow the notation from [1]. Recall the following definitions briefly.

Let $\lambda$ be a nonnegative, continuous, nondecreasing, and unbounded function on $(0, +\infty)$, named the growth function and $\lambda(0) = 0$. The set of nonnegative Borel measures $\mu$ in $\mathbb{R}^m$ ($m \geq 2$), $0 \notin \text{supp} \mu$, such that $N(r, \mu) \leq a\lambda(br)$ for some $a, b > 0$ and all $r > 0$ is denoted by $\mathcal{M}_\lambda^\infty$. Here

$$N(r; \mu) = \begin{cases} (m - 2) \int_0^r n(t; \mu)t^{1-m}dt, & m \geq 3; \\ \int_0^r n(t; \mu)t^{-1}dt, & m = 2; \end{cases}$$

$$n(t; \mu) = \mu(\{y : |y| \leq t\}).$$

By $S^\infty_m$ we denote the set of subharmonic functions $u$ in $\mathbb{R}^m$ ($m \geq 2$), $u(0) = 0$ whose Riesz measure $\mu_u$ belongs to $\mathcal{M}_\lambda^\infty$.

Definition 1. Let $\lambda$ be a growth function. A $\delta$-subharmonic function $w$ in $\mathbb{R}^m$ ($m \geq 2$), $w(0) = 0$, $0 \notin \text{supp} \mu_w$, is said to be a function of finite $\lambda$-type if there are constants $a$ and $b$ such that $T(r, w) \leq a\lambda(br)$ for all $r > 0$, where $T(r, w)$ is the Nevanlinna characteristic of $w$ [2].

The class of such functions is denoted by $\Lambda^\infty_\delta(\lambda)$, and by $\Lambda^m_\delta(\lambda)$ we denote the subclass of subharmonic functions of finite $\lambda$-type.

We recall the following definitions from [1].

2010 Mathematics Subject Classification: 31A05.
Definition 2. A class $\Lambda^m_S(\lambda)$ admits a canonical representation if each function $w$ from $\Lambda^m_S(\lambda)$ can be represented as a difference $w = u - v$ of subharmonic functions $u, v$ from $\Lambda^m_S(\lambda)$ such that $\mu_u = \mu^+_w$, $\mu_v = \mu^-_w$. Here $\mu^+_w$, $\mu^-_w$ are positive and negative variations of the Riesz measure $\mu_w$.

Definition 3. A growth function $\lambda$ is called a growth majorant for $S^m_\lambda$ if for an arbitrary measure $\mu$ from $M^m_\lambda$ there exists a subharmonic function $u$ from $\Lambda^m_S(\lambda)$ such that $\mu_u = \mu$.

Definition 4. A growth function $\lambda$ is called a minimal growth majorant for $S^m_\lambda$ if it is a growth majorant and for each growth majorant $\tilde{\lambda}$ for $S^m_\lambda$ there exist constants $a, b > 0$ such that $\tilde{\lambda}(r) \leq a\lambda(br)$ for all $r > 0$.

The following results were obtained in [1].

Theorem A. Let $\lambda$ be a growth function such that the function $r^{m-1}\lambda'(r)$ ($m \geq 2$) is nondecreasing on $(0, +\infty)$, where $\lambda'(r)$ denotes the right-hand derivative.

i) $\Lambda^m_S(\lambda)$ admits a canonical representation if and only if $\lambda$ is the minimal growth majorant for $S^m_\lambda$;

ii) $\lambda$ is the minimal growth majorant for $S^m_\lambda$ if and only if

$$\int_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}} dt \leq ak^l \left( \frac{\lambda(br_1)}{r_1^k} + \frac{\lambda(br_2)}{r_2^k} \right), \quad k \in \mathbb{N}$$

(1)

for some $a, b, l$ and for arbitrary $r_1, r_2$, such that $0 < r_1 < r_2$.

Lemma B. Let $m \in \mathbb{N} \cap [2, +\infty)$. If a growth function $\lambda(r)$ is such that the function $r^{m-1}\lambda'(r)$ ($m \geq 2$) is nondecreasing on $(0, +\infty)$ and $\tilde{\lambda}(r)$ is a growth majorant for $S^m_\lambda$, then

1. $\lambda(r) \leq a\tilde{\lambda}(br)$ for some $a, b > 0$ and all $r > 0$;

2. each function $w$ from $\Lambda^m_S(\lambda)$ is representable as a difference $w = u - v$ of subharmonic functions $u, v$ from $\Lambda^m_S(\lambda)$ such that $\mu_u = \mu^+_w$, $\mu_v = \mu^-_w$.

From Theorem A it follows that for an arbitrary growth function $\lambda$ the class $\Lambda^m_S(\lambda)$ need not admit a canonical representation. Lemma B lets us to set a more general statement on solvability of the problem on a canonical representation. The following theorem describes the growth majorants.

Theorem 1. Let $\lambda$ be a growth function such that function $r^{m-1}\lambda'(r)$ ($m \geq 2$) is nondecreasing on $(0, +\infty)$. Then a growth function $\lambda$ is a growth majorant for $S^m_\lambda$ if and only if

$$\int_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}} dt \leq ak^l \left( \frac{\tilde{\lambda}(br_1)}{r_1^k} + \frac{\tilde{\lambda}(br_2)}{r_2^k} \right), \quad k \in \mathbb{N}$$

(2)

for some $a, b, l$ and for arbitrary $r_1, r_2$, such that $0 < r_1 < r_2$.

The proof of this theorem is similar (mutatis mutandis) to that of Theorem A so, we omitted it.

The following result follows from Theorem 1.
**Theorem 2.** Let \( \lambda \) be a growth function such that function \( r^{m-1}\lambda'(r) \) \((m \geq 2)\) is nondecreasing on \((0, +\infty)\) and \(q(t)\) is nondecreasing, positive, integer function such that the integral 
\[
\int_0^r \left( \frac{t}{r} \right)^{q(t) - 1} \frac{\lambda(t)}{t} \, dt
\]
is finite for any \( r > 0 \). Then the function 
\[
\tilde{\lambda}(r) = \int_0^r \left( \frac{t}{r} \right)^{q(t) - 1} \frac{\lambda(t)}{t} \, dt + \int_r^{+\infty} \left( \frac{t}{r} \right)^{q(t) - 1} \frac{\lambda(t)}{t} \, dt
\]
is the growth majorant for \( S^m_{\lambda} \).

**2. Proof of Theorem 2.** Let us show that \( \tilde{\lambda}(r) \) from Theorem 1 satisfies condition (2).

Let \( \{n_1, n_2, \ldots, n_i, \ldots\}, n_i \in \mathbb{N} \) be the set of values for the function \( q(t) \). Note that the elements are placed in the ascending order. We set \( y_i = \inf \{ t : q(t) = n_i \} \).

For arbitrary \( r_1, r_2 \) \((0 < r_1 < r_2)\), \( i \in \mathbb{N} \), we have (a) there exists \( n_i \in \mathbb{N} \) such that \( r_1 \leq y_i \leq r_2 \) or (b) such \( n_i \) does not exist.

For the case (a), we have
\[
\int_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}} \, dt = \int_{r_1}^{y_j} \left( \frac{r_1}{r} \right)^k \frac{\lambda(t)}{t} \, dt + \sum_{i=j}^{l-1} \int_{y_i}^{y_{i+1}} \left( \frac{r_1}{r} \right)^{n_i-1} \frac{\lambda(t)}{t} \, dt + \int_{y_l}^{r_2} \left( \frac{r_1}{r} \right)^{n_{j-1}} \frac{\lambda(t)}{t} \, dt,
\]
where \( j = \min \{ i \in \mathbb{N} : r_1 \leq y_i \} \), \( l = \max \{ i \in \mathbb{N} : y_i \leq r_2 \} \). Now we estimate the first summand from the right-hand side of (4). For \( k \geq n_{j-1} \), we get
\[
\int_{r_1}^{y_j} \frac{\lambda(t)}{t^{k+1}} \, dt = \frac{1}{r_1^k} \int_{r_1}^{y_j} \left( \frac{r_1}{r} \right)^k \frac{\lambda(t)}{t} \, dt \leq \frac{1}{r_1^k} \int_{r_1}^{y_j} \left( \frac{r_1}{r} \right)^{n_j-1} \frac{\lambda(t)}{t} \, dt = \frac{1}{r_1^k} \int_{r_1}^{y_j} \left( \frac{r_1}{r} \right)^{q(t)-1} \frac{\lambda(t)}{t} \, dt.
\]
For \( k \leq n_{j-1} - 1 \), we obviously have
\[
\int_{r_1}^{y_j} \frac{\lambda(t)}{t^{k+1}} \, dt = \frac{1}{r_1^k} \int_{r_1}^{y_j} \left( \frac{r_2}{r} \right)^k \frac{\lambda(t)}{t} \, dt \leq \frac{1}{r_1^k} \int_{r_1}^{y_j} \left( \frac{r_2}{r} \right)^{n_j-1} \frac{\lambda(t)}{t} \, dt = \frac{1}{r_1^k} \int_{r_1}^{y_j} \left( \frac{r_2}{r} \right)^{q(t)-1} \frac{\lambda(t)}{t} \, dt.
\]

Therefore,
\[
\int_{r_1}^{y_j} \frac{\lambda(t)}{t^{k+1}} \, dt \leq \frac{1}{r_1^k} \int_{r_1}^{y_j} \left( \frac{r_1}{r} \right)^{q(t)-1} \frac{\lambda(t)}{t} \, dt + \frac{1}{r_1^k} \int_{y_j}^{y_l} \left( \frac{r_2}{r} \right)^{q(t)-1} \frac{\lambda(t)}{t} \, dt \quad (5)
\]
As above, for the other integrals in the right-hand side of (4), we obtain
\[
\int_{y_j}^{y_{j+1}} \frac{\lambda(t)}{t^{k+1}} \, dt \leq \frac{1}{r_1^k} \int_{y_j}^{y_{j+1}} \left( \frac{r_1}{r} \right)^{q(t)-1} \frac{\lambda(t)}{t} \, dt + \frac{1}{r_1^k} \int_{y_j}^{y_{j+1}} \left( \frac{r_2}{r} \right)^{q(t)-1} \frac{\lambda(t)}{t} \, dt, \quad j \leq i \leq l - 1, \quad (6)
\]
\[
\int_{y_j}^{r_2} \frac{\lambda(t)}{t^{k+1}} \, dt \leq \frac{1}{r_1^k} \int_{y_j}^{r_2} \left( \frac{r_1}{r} \right)^{q(t)-1} \frac{\lambda(t)}{t} \, dt + \frac{1}{r_1^k} \int_{y_j}^{r_2} \left( \frac{r_2}{r} \right)^{q(t)-1} \frac{\lambda(t)}{t} \, dt. \quad (7)
\]
Using (4)–(7), we get
\[
\int_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}} dt \leq \frac{1}{r_1^k} \int_{r_1}^{r_2} \left( \frac{r_1}{t} \right)^{q(t)} \frac{\lambda(t)}{t} dt + \frac{1}{r_2^k} \int_{r_1}^{r_2} \left( \frac{r_2}{t} \right)^{q(t)} \frac{\lambda(t)}{t} dt \leq \frac{\tilde{\lambda}(r_1)}{r_1^k} + \frac{\tilde{\lambda}(r_2)}{r_2^k}.
\]

Similarly, for the case (b), we have
\[
\int_{r_1}^{r_2} \frac{\lambda(t)}{t^{k+1}} dt \leq \frac{1}{r_1^k} \int_{r_1}^{r_2} \left( \frac{r_1}{t} \right)^{q(r_1)} \frac{\lambda(t)}{t} dt + \frac{1}{r_2^k} \int_{r_1}^{r_2} \left( \frac{r_2}{t} \right)^{q(r_1)} \frac{\lambda(t)}{t} dt =
\]
\[
= \frac{1}{r_1^k} \int_{r_1}^{r_2} \left( \frac{r_1}{t} \right)^{q(t)} \frac{\lambda(t)}{t} dt + \frac{1}{r_2^k} \int_{r_1}^{r_2} \left( \frac{r_2}{t} \right)^{q(t)} \frac{\lambda(t)}{t} dt \leq \frac{\tilde{\lambda}(r_1)}{r_1^k} + \frac{\tilde{\lambda}(r_2)}{r_2^k}.
\]

Theorem 2 is proved.

REFERENCES

1. Yu.S. Protsyk, *Growth majorants and canonical representation of $\delta$-subharmonic functions*, Mat. Stud. 20 (2003), №1, 40–52. (in Ukrainian)