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NEVANLINNA CHARACTERISTICS OF SEQUENCES OF MEROMORPHIC FUNCTIONS AND JULIA'S EXCEPTIONAL FUNCTIONS

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Uniformly convergent sequences of meromorphic functions in the Carathéodory-Landau sense on annuli are considered. We prove that the sequences of their Nevanlinna type characteristics converge uniformly on intervals. The result is applied to the study of the Nevanlinna characteristics of Julia's exceptional functions.

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Рассматриваются равномерно сходящиеся в смысле Каратеодори-Ландау последовательности мероморфных функций в кольцах. Доказано, что последовательности их характеристик типа Неванлинны сходятся равномерно на промежутках. Результат применен к изучению характеристик Неванлинны исключительных функций в смысле Жюлиа.

1. Introduction. Notation and preliminaries. Researches concerning the impact of various aspects on the value distribution of meromorphic functions had appeared yet before the fundamental works of Nevanlinna. These researches have dealt mainly with strengthening and generalization of the famous Picard's theorem (1879). Among them, there were mostly papers of Julia, Montel, Ostrowski. In particular, Julia's works on theory of asymptotic values of meromorphic functions and rays Ostrowski called [1] "besonders schöne und überraschende Fortschritte". These researches collected in [2].

The following useful generalization of the notion of uniform convergence for sequences of meromorphic functions was introduced by C. Carathéodory and E. Landau in 1911 [3].

Definition 1 ([3])**.** Let f_ν, $\nu \in \mathbb{N}$, be meromorphic functions in a domain G. A sequence $\{f_\nu(z)\}$ is said to be *uniformly convergent* to $f(z)$ on G *in the Carathéodory-Landau sense* if for any point $z_0 \in G$ there exists a disk $K(z_0)$ centered at this point such that $K(z_0) \subset G$ and

$$(\forall \varepsilon > 0)(\exists \nu_0 \in \mathbb{N})(\forall \nu > \nu_0)(\forall z \in K(z_0))\colon |f_\nu(z) - f(z)| < \varepsilon,$$

whenever $f(z_0) \neq \infty$, or

$$\left| \frac{1}{f_\nu(z)} - \frac{1}{f(z)} \right| < \varepsilon,$$

whenever $f(z_0) = \infty$.

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We will use the symbol $\overset{\text{C-L}}{\rightrightarrows}$ to denote the uniform convergence in the Carathéodory-Landau sense. Note that this convergence is equivalent to the convergence in the spherical metric.

We consider functions identically equal to 0 or ∞ as meromorphic and assume $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$.

To investigate Nevanlinna characteristics of meromorphic functions as well as Julia's exceptional functions we need the following auxiliary results.

We put

$$\mathfrak{M}(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta.$$

Lemma A([1]). *Let f_ν, $\nu \in \mathbb{N}$, be meromorphic functions on the annulus $\{z\colon r_1 < |z| < r_2\}$ and let*

$$f_\nu(z) \overset{C\text{-}L}{\rightrightarrows} f(z)$$

on $\{z\colon r_1 < |z| < r_2\}$ as $\nu \to \infty$. Then for any r_1', r_2' such that $r_1 < r_1' < r_2' < r_2$, the sequence $\mathfrak{M}(\rho, f_\nu)$ converges to $\mathfrak{M}(\rho, f)$ uniformly on $[r_1', r_2']$. Here $\mathfrak{M}(\rho, 0) = -\infty$, $\mathfrak{M}(\rho, \infty) = +\infty$.

The following theorem is a consequence of Rouché's theorem (see, e.g., [4, p.426]).

Theorem A (Hurwitz). *Let $f_\nu(z)$, $\nu \in \mathbb{N}$ be analytic functions in a domain G. Suppose the sequence $\{f_\nu(z)\}$ converges uniformly to $f(z)$ on G, $f(z) \not\equiv 0$. Then for any closed curve γ in G that passes through no zero of f there exists $\nu_0 \in \mathbb{N}$ such that for all $\nu > \nu_0$ the functions f_ν and f have the same number of zeros inside γ.*

Lemma 1.

$$(\forall x > 0)(\forall y > 0)\colon |\log^+ x - \log^+ y| \le |x - y|.$$

The proof is trivial.

Lemma 2. *Let $\{\varphi_\nu(t)\}$, $\nu \in \mathbb{N}$, be a sequence of nondecreasing functions on $[a, b]$ and let $\varphi_\nu(t) \to \varphi(t)$ as $\nu \to \infty$, $t \in [a, b] \setminus \{t_1, ..., t_m\}$, where $t_1, ..., t_m \in (a, b)$. Suppose $\varphi(t)$ is continuous and nondecreasing on $[a, b]$. Then the sequence $\{\varphi_\nu(t)\}$ converges to $\varphi(t)$ uniformly on $[a, b]$.*

Proof. The function $\varphi(t)$ is uniformly continuous on $[a, b]$, that is

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall t', t'' \in [a, b])\colon |t' - t''| < \delta \Rightarrow |\varphi(t') - \varphi(t'')| < \varepsilon. \tag{1}$$

For any $t \in (a, b)$ we choose $t', t'' \in [a; b]$ such that

$$1)\ t' < t < t'';\ \ 2)\ t'' - t' < \delta;\ \ 3)\ t' \neq t_k,\ t'' \neq t_k,\ k = 1, 2, \ldots, m. \tag{2}$$

Since $\varphi_\nu(t') \to \varphi(t')$ and $\varphi_\nu(t'') \to \varphi(t'')$ as $\nu \to \infty$, we see that for any $\varepsilon > 0$ and for all $\nu > \nu_0$ the following inequalities hold

$$\varphi_\nu(t') > \varphi(t') - \varepsilon,\ \varphi_\nu(t'') < \varphi(t'') + \varepsilon \tag{3}$$

It follows from (3) and the monotonicity of φ, φ_ν that

$$\varphi_\nu(t) - \varphi(t) > \varphi_\nu(t') - \varphi(t'') > \varphi(t') - \varphi(t'') - \varepsilon,$$

$$\varphi_\nu(t)-\varphi(t)<\varphi_\nu(t'')-\varphi(t')<\varphi(t'')-\varphi(t')+\varepsilon,$$

for all $\nu>\nu_0$.

Finally, using (1) and (2) we obtain

$$|\varphi_\nu(t)-\varphi(t)|<2\varepsilon \quad (\forall \nu>\nu_0 \text{ and } \forall t\in(t',t'')).$$

We have thus proved that for any $t\in(a,b)$ there exists an open interval $(t',t'')\subset[a,b]$ such that none of its endpoints coincides with any of $t_1,...,t_m$ and $\{\varphi_\nu(t)\}$ converges to $\varphi(t)$ uniformly on (t',t''). Simultaneously, the union of these intervals forms a cover of $[a+\frac{\delta}{2},b-\frac{\delta}{2}]$, for any $\delta>0$. Therefore there exists a finite subcover of $[a+\frac{\delta}{2},b-\frac{\delta}{2}]$. If necessary, we choose δ so that none of the points $a+\frac{\delta}{2}$, $b-\frac{\delta}{2}$ coincides with any of $t_1,...,t_m$. Reasoning as above, we see that $\{\varphi_\nu(t)\}$ converges uniformly also on the intervals $[a,a+\frac{\delta}{2}]$ and $[b-\frac{\delta}{2},b]$, and thus on the whole interval $[a,b]$. □

2. Nevanlinna's characteristics of sequences of meromorphic function. Let $\{g_\nu(z)\}$ be a sequence of functions meromorphic on the annulus $\{z\colon R_1<|z|<R_2\}$. Suppose $\{g_\nu(z)\}$ converges uniformly on this annulus in the Carathéodory-Landau sense. We consider the functions

$$f_\nu(z){:=}g_\nu\left(\frac{z}{\sqrt{R_1R_2}}\right),\ \nu\in\mathbb{N};\ f(z){:=}g\left(\frac{z}{\sqrt{R_1R_2}}\right).$$

Clearly, $f_\nu(z)\overset{\text{C-L}}{\rightrightarrows} f(z)$ as $\nu\to\infty$ on the annulus $\{z\colon \frac{1}{R_0}<|z|<R_0\}$, where $R_0=\sqrt{\frac{R_2}{R_1}}$. Therefore without loss of generality, we can consider sequences of meromorphic functions on "symmetric" annulus $A_{R_0}=\{z\colon \frac{1}{R_0}<|z|<R_0\}$.

A Nevanlinna type characteristic for the functions meromorphic on A_{R_0}, was introduced in the paper [5]. Namely,

$$T_0(r,f)=m_0(r,f)+N_0(r,f),\ 1<r<R_0,$$

where

$$m_0(r,f)=m(r,f)+m\left(\frac{1}{r},f\right)-2m(1,f), \tag{4}$$

$$m(t,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(te^{i\theta})|d\theta,\ \frac{1}{R_0}<t<R_0,\ N_0(r,f)=\int_1^r\frac{n_0(t,f)}{t}dt,$$

$n_0(t,f)$ is the number of poles of f in the annulus $\{z\colon \frac{1}{t}<|z|\le t\}$ with regard to their multiplicity.

The function $T_0(r,f)$ is nonnegative, nondecreasing, continuous and convex with respect to $\log r$ on $[1,R_0)$ ([5]). By the definition we have $T_0(r,0)=0$. Put $T_0(r,\infty)=+\infty$.

Our main result is the following theorem.

Theorem 1. *Let $f_\nu(z)$, $\nu\in\mathbb{N}$, be meromorphic functions on the annulus*

$$A_{R_0}=\left\{z\colon \frac{1}{R_0}<|z|<R_0\right\},\quad 1<R_0\le+\infty.$$

Suppose $f_\nu(z) \overset{C\text{-}L}{\rightrightarrows} f(z)$ *on* A_{R_0} *as* $\nu \to \infty$ *and* f *is nonidentically* ∞*. Assume that the function* f *does not have neither zeros nor poles on the unit circle* $|z| = 1$*. Then*

$$(\forall R \in (1, R_0))\colon T_0(r, f_\nu) \to T_0(r, f)$$

as $\nu \to \infty$ *uniformly on* $[1, R]$*.*

3. Proof of Theorem 1. Suppose there are no poles of f on the circle $|z| = \tau$. First we show that

$$m(\tau, f_\nu) \to m(\tau, f)$$

as $\nu \to \infty$.

Indeed, in this case $f_\nu(z) \to f(z)$ uniformly on the circle $|z| = \tau$ as $\nu \to \infty$. Given $\varepsilon > 0$ there is ν_0 such that $|f_\nu(z) - f(z)| < \varepsilon$ for all $\nu > \nu_0$ and for all $z(|z| = \tau)$.

Using this and Lemma 1, we get

$$|m(\tau, f_\nu) - m(\tau, f)| \le \frac{1}{2\pi}\int_0^{2\pi} |\log^+|f_\nu(\tau e^{i\theta})| - \log^+|f(\tau e^{i\theta})||d\theta \le$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} ||f_\nu(\tau e^{i\theta})| - |f(\tau e^{i\theta})||d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f_\nu(\tau e^{i\theta}) - f(\tau e^{i\theta})|d\theta < \varepsilon$$

whenever $\nu > \nu_0$.

Thus, by (4), if there are no poles of f on the circles $|z| = r$, $|z| = \frac{1}{r}$, $|z| = 1$, we obtain that

$$m_0(r, f_\nu) \to m_0(r, f)$$

as $\nu \to \infty$.

If no zeros of f lie on the unit circle $|z| = 1$, then as above we have $m(1, \frac{1}{f_\nu}) \to m(1, \frac{1}{f})$ as $\nu \to \infty$. Applying Lemma A and the identity

$$\log^+|f| - \log^+\frac{1}{|f|} = \log|f|$$

we obtain that

$$m(1, f_\nu) \to m(1, f)$$

as $\nu \to \infty$.

Now we prove that if the circles $|z| = r$, $|z| = \frac{1}{r}$ do not contain poles of f then

$$N_0(r, f_\nu) \to N_0(r, f)$$

as $\nu \to \infty$.

In fact, if we represent $N_0(r, f)$ in the form

$$N_0(r, f) = \int_1^r \frac{\mu(t, f) - \mu(\frac{1}{t}, f)}{t}dt = \int_1^r \frac{\mu(t, f)}{t}dt - \int_{1/r}^1 \frac{\mu(t, f)}{t}dt,$$

where $\mu(t,f)$ is the distribution function of poles b_j of the function f on A_{R_0},

$$\mu(t_2,f)-\mu(t_1,f)=\sum_{t_1<|b_j|\le t_2}1$$

then integrating by parts, we get

$$N_0(r,f)=\int\limits_1^r \mu(t,f)d\log t-\int\limits_{1/r}^1 \mu(t,f)d\log t=$$

$$=\left(\mu(r,f)+\mu\left(\frac{1}{r},f\right)\right)\log r-\sum_{1<|b_j|<r}\log|b_j|+\sum_{\frac{1}{r}<|b_j|\le 1}\log|b_j|.$$

Similarly,

$$N_0(r,f_\nu)=\left(\mu(r,f_\nu)+\mu\left(\frac{1}{r},f_\nu\right)\right)\log r-\sum_{1<|b_j^{(\nu)}|<r}\log|b_j^{(\nu)}|+\sum_{\frac{1}{r}<|b_j^{(\nu)}|\le 1}\log|b_j^{(\nu)}|.$$

If $f_\nu \overset{\text{C-L}}{\rightrightarrows} f$, then also $\frac{1}{f_\nu} \overset{\text{C-L}}{\rightrightarrows} \frac{1}{f}$ as $\nu\to\infty$ [6]. Therefore by Theorem A there exists ν_0 such that for all $\nu>\nu_0$ the functions f_ν and f have the same number of poles in A_r. Moreover,

$$b_j^{(\nu)}\to b_j \text{ as } \nu\to\infty. \tag{5}$$

We have

$$|N_0(r,f_\nu)-N_0(r,f)|\le\sum_{\frac{1}{r}<|b_j|\le 1}|\log|b_j^{(\nu)}|-\log|b_j||+\sum_{1<|b_j|<r}|\log|b_j|-\log|b_j^{(\nu)}||.$$

Applying Lemma 1 and Lagrange's theorem, we obtain

$$|N_0(r,f_\nu)-N_0(r,f)|\le\sum_{\frac{1}{r}<|b_j|<r}R||b_j^{(\nu)}|-|b_j||, \tag{6}$$

for all $\nu>\nu_0$. Then (6) and (5) imply that $N_0(r,f_\nu)\to N_0(r,f)$ as $\nu\to\infty$.

Hence, if the function f has no poles on the circles $|z|=r$, $|z|=\frac{1}{r}$ and either there are no poles or there are no zeros of f on the unit circle $|z|=1$, then we have $T_0(r,f_\nu)\to T_0(r,f)$ as $\nu\to\infty$. To complete the proof we apply Lemma 2 to the sequence of characteristics $\{T_0(r,f_\nu)\}$. The case $f(z)\equiv 0$ is trivial.

Remark 1. The function $N_0(r,f)$ is nondecreasing and continuous on $[1,R_0)$. Therefore if $f_\nu \overset{\text{C-L}}{\rightrightarrows} f$ on A_{R_0}, then by Lemma 2 $N_0(r,f_\nu)\underset{\nu\to\infty}{\longrightarrow} N_0(r,f)$ uniformly on every interval $[1,R]$, $R<R_0$. Thus under the assumptions of Theorem 1 we have $m_0(r,f_\nu)\underset{\nu\to\infty}{\longrightarrow} m_0(r,f)$ uniformly on every interval $[1,R]$, $R<R_0$.

Remark 2. The question of whether Theorem 1 is true in the case when the function f has both zeros and poles on the unit circle remains open.

4. Nevanlinna characteristics of Julia's exceptional functions.

Definition 2 ([1])**.** A function $f(z)$ meromorphic in $\mathbb{C}$ is said to be a *J-exceptional function* if for any sequence of complex numbers σ_n, $\sigma_n \to \infty$, as $n \to \infty$, there exists a subsequence σ_{n_k} such that the sequence $\{f(\sigma_{n_k} z)\}$ converges uniformly on $\mathbb{C}\setminus\{0\}$ in the Carathéodory-Landau sense as $k \to \infty$.

O. Lehto and K. Virtanen [7] have proved that the function f is J-exceptional if and only if the spherical derivative $\rho(f(z))$ satisfies

$$\rho(f(z)) = O\left(1/|z|\right), \quad z \to \infty.$$

It follows that the Ahlfors-Shimizu characteristics $\overset{\circ}{T}(r, f)$ (see, e.g. [8, p.30]) satisfy

$$\overset{\circ}{T}(r, f) = O(\log^2 r), r \to \infty.$$

Hence [8, p.33], this is also true for the Nevanlinna characteristics $T(r, f) = O(\log^2 r)$, $r \to \infty$.

Applying Theorem 1 we now establish the following property of the characteristics $T(r, f)$ of J-exceptional function.

Theorem 2. *Let f be a J-exceptional function and let the sequence $\{f(\sigma_{n_k} z)\}$ be uniformly convergent on $\mathbb{C}\setminus\{0\}$ in the Carathéodory-Landau sense as $k \to \infty$ to a function $g(z)$ meromorphic in $\mathbb{C}\setminus\{0\}$. Suppose that g is not identically 0 or ∞ and have either no zeros or no poles on the unit circle. If $|\sigma_{n_k}| = \gamma_k$ then*

$$T(\gamma_k r, f) + T\left(\frac{\gamma_k}{r}, f\right) - 2T(\gamma_k, f) \longrightarrow T_0(r, g) \tag{7}$$

as $k \to \infty$ uniformly on every interval $[1, R]$.

Proof. By Theorem 1 the sequence $\{T_0(r, f_k)\}$, where $f_k(z) = f(\sigma_{n_k} z)$ converges to $T_0(r, g)$ uniformly in r on every interval $[1, R]$.

Since [5]

$$T_0(r, f_k) = T(r, f_k) + T\left(\frac{1}{r}, f_k\right) - 2T(1, f_k)$$

the theorem is proved. □

We now give a geometric interpretation of Theorem 2.

Denote

$$\log r = \rho, \quad \log \gamma_k = x_k, \quad T(r, f) = \psi(\rho).$$

Since the Nevanlinna characteristics $T(r, f)$ are nondecreasing and convex with respect to $\log r$, we obtain that $\psi(\rho)$ is nondecreasing and convex. Using (7) we have

$$\frac{\psi(x_k + \rho) - \psi(x_k)}{(x_k + \rho) - x_k} - \frac{\psi(x_k) - \psi(x_k - \rho)}{x_k - (x_k - \rho)} \to \frac{T_0(e^\rho, g)}{\rho} \tag{8}$$

This means that the increment of chord's slope on the sequence x_k converges to the value on the right-hand side of (8).

A simple example of J-exceptional function is given in [1], [6, p.144]

$$f(z)=\frac{\prod_{k=0}^{+\infty}\left(1-\frac{z}{q^k}\right)}{\prod_{k=0}^{+\infty}\left(1+\frac{z}{q^k}\right)},\ q>1.$$

This function satisfies

$$f(qz)=\frac{1-qz}{1+qz}f(z). \tag{9}$$

We denote $f_n(z)=f(q^nz)$ then, using (9), we obtain

$$f_n(z)=\frac{(1-qz)(1-q^2z)\cdots(1-q^nz)}{(1+qz)(1+q^2z)\cdots(1+q^nz)}f(z).$$

If $n=2m$, $m\in\mathbb{N}$, we have

$$f_n(z)=\frac{\left(1-\frac{1}{qz}\right)\left(1-\frac{1}{q^2z}\right)\cdots\left(1-\frac{1}{q^nz}\right)}{\left(1+\frac{1}{qz}\right)\left(1+\frac{1}{q^2z}\right)\cdots\left(1+\frac{1}{q^nz}\right)}f(z),\ \ z\neq 0. \tag{10}$$

The sequence given by the fraction in front of $f(z)$ in (10) converges uniformly on $\mathbb{C}\setminus\{0\}$ to the function

$$\frac{z+1}{z-1}f\left(\frac{1}{z}\right).$$

Therefore the sequence

$$\left\{\frac{1+z}{1-z}f_{2m}(z)\right\}$$

converges uniformly $\mathbb{C}\setminus\{0\}$ as $m\to\infty$ to the function

$$g(z)=-\left(\frac{z+1}{z-1}\right)^2 f\left(\frac{1}{z}\right)f(z),$$

The function g has neither zeroes nor poles on the unit circle. Thus, applying Theorem 2 and using elementary properties of the characteristics $T_0(r,f)$ [5], [9], we see that for sufficiently large r and for all m greater than some positive integer m_0

$$T(q^{2m}r,f)+T\left(\frac{q^{2m}}{r},f\right)-2T(q^{2m},f)=2T(r,f)+O(\log r).$$

If $n=2m-1$, $m\in\mathbb{N}$, then

$$\left\{\frac{1+z}{1-z}f_{2m-1}(z)\right\}$$

converges uniformly on $\mathbb{C}\setminus\{0\}$ as $m\to\infty$ to the function $-g(z)$. Hence, for sufficiently large r and for all $n>n_0$

$$T(q^nr,f)+T\left(\frac{q^n}{r},f\right)-2T(q^n,f)=2T(r,f)+O(\log r).$$

REFERENCES

1. Ostrowski A. *Über Folgen analytischer Funktionen und einige Verschärfungen des Picardschen satzes*// Mathematische Zeitschrift. – 1925. – V.24, №1. – P. 215–258.
2. Julia G. Leçons sur les fonctions uniformes à point singulier essentiel isolé. – Gauthier-Villars, Paris, 1924.
3. Carathéodory C., Landau E. *Beiträge zur Konvergenz von Funktionenfolgen*// Sitzungsber. Kon. Preuss. Akad. Wiss. Berlin. – 1911. – P. 587–613.
4. Markushevich A.I. Theory of analytic functions. V.1. – Moscow: Nauka, 1967. (in Russian)
5. Khrystiyanyn A.Ya., Kondratyuk A.A. *On the Nevanlinna Theory for meromorphic functions on annuli. I*// Mat. Stud. – 2005. – V.23, №1. – P. 19–30.
6. Montel P. Leçons sur les familles normales de fonctions analytiques et leurs applications, Gauthier-Villars, Paris, 1927.
7. Lehto I., Virtanen K.I. *On the behaviour of meromorphic functions in the neigbourhood of an isolated singularity*// Ann. Acad. Sci. Fenn. – 1957. – V.240. – P. 1–9.
8. Goldberg A.A., Ostrovski I.V. Distibution of values of meromorphic functions. – Moscow: Nauka, 1970. (in Russian)
9. Kondratyuk A., Laine I. *Meromorphic functions in multiply connected domains*// Fourier series methods in complex analysis (Mekrijärvi, 2005), Univ. Joensuu Dept. Math. Rep. – 2006. – V.10. – P. 9–111.

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