УДК 519.156

O. S. Bondarenko

REGULAR SUBGRAPHS OF LINEAR EXTENSION GRAPHS

O. S. Bondarenko. *Regular subgraphs of linear extension graphs*, Mat. Stud. **36** (2011), 6–11.

The paper deals with the properties of regular subgraphs of linear extension graphs. We obtain a necessary and sufficient condition for regularity of linear extension graphs, prove a proposition on cardinality and degree partition of special linear extension graphs boundaries, give a characterization of a linear extension graphs class with 0 or 1 bump numbers linear extensions.

А. С. Бондаренко. *Регулярные подграфы графов линейных расширений* // Мат. Студії. – 2011. – Т.36, №1. – С.6–11.

В работе рассматриваются свойства регулярных подграфов графов линейных расширений. Получены необходимые и достаточные условия регулярности графа линейных расширений, доказано утверждение о мощности и степенном разбиении границ графов линейных расширений специального вида, дана характеризация класса графов линейных расширений, содержащих линейные расширения с дополнением числа скачков, равным 0 или 1.

1. Introduction. Every finite poset $\mathcal{P}$ is associated to its *linear extension graph* $G(\mathcal{P})$, in which the vertices correspond to linear extensions (LE) of $\mathcal{P}$ and any two of them are adjacent exactly if they differ by one adjacent transposition.

A degree partition of the graph $G(\mathcal{P})=(V,E)$ is a family of subsets of V $\{B_1,\dots,B_m\}$, such that $\forall b\in B_i$ and $\forall c\in B_j$ the following holds:

- $i=j$ exactly if deg(b) = deg(c),
- $i\neq j$ exactly if deg(b) $\neq$ deg(c),

i.e. the least by cardinality partition of $G(\mathcal{P})$ on regular subgraphs, induced by the sets B_i, $\forall i\in[m](=\{1,\dots,m\})$. With any such a degree partition we associate the degree set, i.e. the set, which contains the values of degrees that vertices in $G(\mathcal{P})$ have.

Let $L=x_1x_2\dots x_n$ be a linear extension of some poset. We say that two adjacent elements x_ix_{i+1} of L separated by bump (respectively, jump) [7], if $x_i<_{\mathcal{P}}x_{i+1}$ ($x_i||_{\mathcal{P}}x_{i+1}$). Bump and jump number problems have been studied in the literature. The former problem is polynomially solvable, but the latter one is $\mathcal{NP}$-hard. Evidently, adjacent transposition of elements, separated by bump, takes out of the given $G(\mathcal{P})$, i.e. we obtain a permutation which is not a LE of $\mathcal{P}$. Conversely, by adjacent transposition of elements, separated by jump, we obtain another permutation that contains the given $\mathcal{P}$. Since the results are formulated in the terms of graph theory, it will be convenient to talk about bump (jump) number in linear extension as about outer (inner) degree of the vertices in $G(\mathcal{P})$ [2]. Maximum outer (inner) degree we denote by Δ_o (Δ_i), minimum outer (inner) degree we denote by δ_o (δ_i).

2010 *Mathematics Subject Classification:* 06A07.

For outer (inner) degrees we reserve $\deg_{\mathrm{o}}$ ($\deg_{\mathrm{i}}$). Consecutively, we will say about outer (inner) degree sets.

In several papers the authors investigated such $G(\mathcal{P})$ properties as hamiltonicity [1], connectivity [2], diameter [3]. Moreover, some LE graphs generalizations [4] were studied. One of the directions for research is the investigation of $G(\mathcal{P})$ degree partitions.

The *aim* of the paper is to characterize some classes of LE graphs by their degree sets.

2. Weak orders. A weak order (WO) on $[n]$ is an irreflexive and transitive binary relation with the additional condition: $a||b$, $b||c$, then $a||c$ (transitivity of incomparability). By $\mathbf{k}$ we denote a k-element chain, for example: $\mathbf{3}$ — 3-element chain.

Proposition 1 ([6])**.** *Let $\mathcal{P} = (P, <)$ be some poset. Then the following statements are equivalent:*

1. *$\mathcal{P}$ is a weak order;*
2. *$\mathcal{P}$ does not contain $\mathbf{1}+\mathbf{2}$ as a subposet;*
3. *P can be partitioned into antichains $A_1, A_2, \ldots, A_h$ so that if $x \in A_i$ and $y \in A_j$ with $i < j$, then $x <_{\mathcal{P}} y$.*

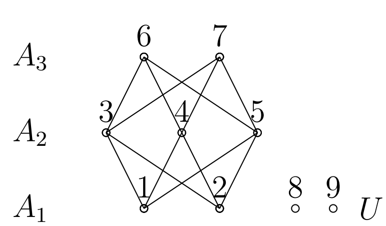
It is known that all linear extensions of weak orders have the same bump (jump) number, although, as far as the author knows, the proof of necessity and sufficiency of this fact does not exist.

Proposition 2. *$G(\mathcal{P})$ is regular exactly if $\mathcal{P}$ is a weak order.*

Proof. Let $\mathcal{P}$ be some poset, $x, y \in \mathcal{P}$ and $x \lessdot_{\mathcal{P}} y$ (y covers x in $\mathcal{P}$), $L_{\mathcal{P}}$ — a LE of $\mathcal{P}$, $x \lessdot_{L_{\mathcal{P}}} y$ (y covers x in $L_{\mathcal{P}}$).
Necessity. Suppose graph $G(\mathcal{P})$ is regular, and $\mathcal{P}$ is not a weak order. Then, by Proposition 1, poset $\mathcal{P}$ contains $\mathbf{1}+\mathbf{2}$. This implies that in $\mathcal{P}$ there exists such z that $x \lessdot_{L'_{\mathcal{P}}} z \lessdot_{L'_{\mathcal{P}}} y$ in some $L'_{\mathcal{P}} \neq L_{\mathcal{P}}$, i.e. the existence of the element z in $\mathcal{P}$ allows to change LE outer degree by insertion z between x and y. Hence, graph $G(\mathcal{P})$ is not regular.
Sufficiency. Suppose graph $G(\mathcal{P})$ is not regular, and $\mathcal{P}$ is a weak order. Then $\exists L_{\mathcal{P}}, L'_{\mathcal{P}}$ such that $x \lessdot_{L_{\mathcal{P}}} y$, $x \lessdot_{L'_{\mathcal{P}}} z \lessdot_{L'_{\mathcal{P}}} y$ and $L'_{\mathcal{P}} \neq L_{\mathcal{P}}$, i.e. $\mathcal{P}$ contains $\mathbf{1}+\mathbf{2}$. Hence, $\mathcal{P}$ is not a weak order. □

3. Generalized weak orders. In this section we consider the class of posets, which got the name of generalized weak orders (GWO) in [7]. Posets, that belong to that class, can be defined as: we fix, for $r \geqslant 1, s \geqslant 0$ on the ground set of some poset, a partition $\{A_1, \ldots, A_r, U\}$, where $A_i, \forall i$ and U are antichains, and $|U| = s$. Then GWO can be presented as: $(A_1 \oplus \cdots \oplus A_r) + U$.

Figure 1: Hasse diagram for GWO



For GWO it is known [7] that the least outer degree equals $\max\{\mathrm{r}-1-\mathrm{s},0\}$. In its turn, the maximum outer degree does not exceed $r-1$.

Now we consider one property of a subset of the class GWO. This property concerns the question of the regular subgraph cardinality. There are some papers on LE with minimum jump number counting, see for example [8].

By the generalized star (GS) we will call an order $A_1 \oplus A_2 + U$, where $|A_1| = 1$ or $|A_2| = 1$. By a boundary element of a set H we mean any $h \in H$, for which $\deg_o(\mathrm{h}) > 0$. We denote the boundary set of $G(\mathcal{P})$ by $G_B(\mathcal{P})$. We will describe GS with $a_1, \dots, a_i < (>) a_{i+1}a_{i+2}$, where the elements $a_1, \dots, a_i$ are pairwise incomarable, less (more) than a_{i+1} and a_{i+2} and a_{i+1} less than a_{i+2}.

Proposition 3. *Let $\mathcal{P} = (P, \leqslant)$ be a poset. The cardinality of $G_B(\mathcal{P})$ for $\mathcal{P}$, presented by the generalized star, equals $(n-1)!$, where $|P| = n$.*

Proof. Consider a family of $G(\mathcal{P})$, induced by the relation $a_1, \dots, a_i < (>) a_{i+1}$ and $A \subseteq [n-1]$. Boundary sets of these graphs can be partitioned into i blocks, presented by the relation $a_1, \dots, a_{i-1} < a_i a_{i+1}$ (1). Such relations cover $G_B(\mathcal{P})$ and do not intersect. As long as the relations in the partition have the same comparability graph, they induce $G(\mathcal{P})$ of equal order. Such $G_B(\mathcal{P})$ can be described as the union of the following kind: $\cup_{i \in A} a_1, \dots, a_{i-1} < a_i a_{i+1}$. So we have some partition $\Pi = \{\pi_1, \dots, \pi_k\}$ of $G_B(\mathcal{P})$. We are going to evaluate the size of Π and the size of its elements. At first, we can denote the sizes of two parts of GS, which induce $G(\mathcal{P})$, with the condition that $|A_1| = 1$ in all cases, as $|A_2| = k$, $|U| = s$. Here we have $1 + k + s = n$, where $|P| = n$. Since we can construct k GS of the form (1), which are the elements of Π, so Π will contain k elements. Further, all the elements of Π will have the following sizes of the A_2 and U: $|A_2| = k-1$ and $|U| = s$. Then $|G(\pi_i)| = (k-1)!(n-1)^{\underline{s}}$, $\forall i \in [k]$, where $n^{\underline{s}}$ means the falling factorial in the notation of [9]. Now we can express the size of $G_B(\mathcal{P})$ through the sizes of its partition elements:

$$|G_B(\mathcal{P})| = (k-1)!(n-1)^{\underline{s}}k = k!\,\underbrace{(n-1)(n-2)\dots(n-s)}_{s=n-k-1}.$$

From what we conclude that the cardinality of $G_B(\mathcal{P})$ equals $(n-1)!$. □

Whereas for any LE of $\mathcal{P} = \mathrm{GS}$ transposition of only one pair of elements allows one to go out of the given set, viz.: $a_j a_{i+1}$, $\forall j \in A$, for all v from $G_B(\mathcal{P})$ $\deg_o(\mathrm{v}) = 1$.

4. Proper generalized weak orders. The statement we want to prove is on the equivalence of height 2 proper GWO (PGWO) and LE graphs with outer degree set $\{0, 1\}$. Our proof strategy is the following one: at first, we investigate identities on outer and inner degrees under the action of a disjoint sum. Then we define the set of functions whose action can give us every connected bipartite order which is not a weak order. Then by combination of different definitions and results on the identities we are going to prove the statement. For any vertice of $G(\mathcal{P})$ the following identities hold:

$$\delta_i + \Delta_o = n - 1,$$

$$\delta_o + \Delta_i = n - 1.$$

Let us consider the change in the degrees under the disjoint sum with k-chain, C_n, and k-antichain, A_n. For any poset the following statement on the maximum outer degree holds:

$$\Delta_o(\mathcal{P}) = \Delta_o(\mathcal{P} + C_k) - k - 1, \tag{1}$$

$$\Delta_o(\mathcal{P}) = \Delta_o(\mathcal{P} + A_k). \tag{2}$$

By duality, we can derive the identities for minimum inner degrees:

$$\delta_i(\mathcal{P}) = \delta_i(\mathcal{P} + C_k) - 1,$$

$$\delta_i(\mathcal{P}) = \delta_i(\mathcal{P} + A_k) - k.$$

The following identities concern δ_o and Δ_i. At the beginning, we consider the identities on the maximum inner degrees:

$$\Delta_i(\mathcal{P}) = \Delta_i(\mathcal{P} + C_k) - \begin{cases} 2k, & k < \delta_o(\mathcal{P}), \\ \delta_o(\mathcal{P}) - k, & \delta_o \leqslant k \leqslant |\mathcal{P}| + 1, \\ \delta_o(\mathcal{P}) - |\mathcal{P}| - 1, & |\mathcal{P}| + 1 < k; \end{cases}$$

$$\Delta_i(\mathcal{P}) = \Delta_i(\mathcal{P} + A_k) - \begin{cases} 2k, & k < \delta_o(\mathcal{P}), \\ \delta_o(\mathcal{P}) - k, & \delta_o(\mathcal{P}) \leqslant k; \end{cases}$$

We can describe the identities for the minimum outer degrees completely analogically:

$$\delta_i(\mathcal{P}) = \delta_i(\mathcal{P} + C_k) + \begin{cases} k, & k < \delta_o(\mathcal{P}), \\ \delta_o(\mathcal{P}), & \delta_o \leqslant k \leqslant |\mathcal{P}| + 1, \\ \delta_o(\mathcal{P}) - |\mathcal{P}| - 1 - k, & |\mathcal{P}| + 1 < k; \end{cases}$$

$$\delta_i(\mathcal{P}) = \delta_i(\mathcal{P} + A_k) + \begin{cases} k, & k < \delta_o(\mathcal{P}), \\ \delta_o(\mathcal{P}), & \delta_o(\mathcal{P}) \leqslant k. \end{cases}$$

In the sequel we will need the following obvious fact.

Lemma 1. *A poset contains a k-chain exactly if it has a vertice with* $\mathrm{deg_o} = \mathrm{k} - 1$.

This means we can exclude from consideration posets of heights $k > 2$. Moreover, identities 1 and 2 imply that

- disjoint union of any $\mathcal{P}$ with A_k do not change $\Delta_o(\mathcal{P} + A_k)$ comparing to $\Delta_o(\mathcal{P})$;
- disjoint union of any $\mathcal{P}$ with C_k increases $\Delta_o(\mathcal{P} + C_k)$ comparing to $\Delta_o(\mathcal{P})$.

So, analogically, we can exclude disconnected orders. After deriving these identities and stating Lemma 1 we need to define the set of functions which would give us all the height 2 bipartite orders $\mathcal{P}'_{BO}$ that are not connected and do not contain weak orders. We define the set $\mathcal{F}$ of functions $f \colon \mathcal{P}'_{BO} \to \mathcal{P}'_{BO}$. By a function from $\mathcal{F}$ we will mean an including of one additional element into the given order such that the resultant order is connected and bipartite. So, this element can be compared only with elements from one partition and it cannot be compared to any element of the initial order. Having defined such a family we can formulate the following lemma.

Lemma 2. *By the application of an arbitrary function from $\mathcal{F}$ to the order that is not a weak order we cannot obtain a weak order.*

Thus, $\mathcal{P}'_{BO}$ is closed under $\mathcal{F}$. We have one requirement left not discussed on obtaining by $\mathcal{F}$ *all* the orders from $\mathcal{P}'_{BO}$. This is true only if we apply f to thoroughly chosen $\mathcal{P}$. It has to differ as less as possible from a weak order, a disconnected order and be of minimum size. A natural candidate for these conditions is the N order. This order is the least order which is bipartite, is not a weak order and is not disconnected. This means that by a finite number of f applications we can obtain any member of $\mathcal{P}'_{BO}$. By proper GWO we mean posets which are GWO but are not WO. Another fact which we need to establish is the following one.

Lemma 3. *The application of any f from $\mathcal{F}$ to an arbitrary $\mathcal{P}$ either do not change Δ_o or increases it by 1.*

Further, the main result of this section follows.

Proposition 4. *The orders from $\mathcal{P}'_{BO}$ have*

$$\Delta_o > 1.$$

Proof. The proof proceeds by induction on the size of $\mathcal{P}$. The base of induction is the order N. We make sure that $\Delta_o(N) = 2 > 1$. Next, let us suppose that for n the statement is true. Then for $n+1$ the validity of the statement follows from Lemma 3. The proof is completed. □

Corollary 1. *Some poset belongs to the class of height 2 proper GWO exactly if its outer degree set is $\{0,1\}$.*

Proof. All the bipartite orders consist of

- GWO,
- $\mathcal{P}'_{BO}$, and
- disconnected orders which are not included in GWO and $\mathcal{P}'_{BO}$.

That classification implies that the only orders with $\{0,1\}$ are proper GWO of height 2. □

REFERENCES

1. West D.B. *Generating linear extensions by adjacent transpositions*// Douglas Brent West, J. Comb. Theory Ser. B. – 1993. – V.58. – №1. – P. 58–64.
2. Naatz M. *The graph of linear extensions revisited*// SIAM J. Disc. Math. – 2000. – V.13. – №3. – P. 354–369.
3. Felsner S., Massow M. *Linear Extension Diameter of Downset Lattices of 2-Dimensional Posets*// Electronic Notes in Discrete Mathematics. – 2009. – V.34, №1. – P. 313–317.
4. Naatz M. Acyclic Orientations of Mixed Graphs, Thesis. – Berlin, 2001. – 125 p.
5. Trotter W.T., Graham R.L., Grötschel M., Lovász L., Partially ordered sets, in: Handbook of Combinatorics, Elsevier, Amsterdam, 1995, p. 433–480.

6. Keller M.T. Some Results on Linear Discrepancy for Partially Ordered Sets. Thesis. – Atlanta, 2010. – 69 p.
7. Habib M., Möhring R.H., Steiner G. *Computing the Bump Number is Easy*// Order. – 1988. – V.5. – P. 107–129.
8. Jung H.C. *Lower bounds of the number of jump optimal linear extensions: Products of some posets*// Bull. Korean Math. Soc. – 1995. – V.32. – №2. – P. 171–177.
9. Knuth D.E. *Two notes on notation*// American Math. Monthly. – 1992. – V.99. – №5. – P. 403–422.

Zaporizhzhya National University
buenasdiaz@gmail.com

Received 24.02.2010