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ON THE RELATION BETWEEN THE LEBESGUE INTEGRAL MEANS AND NEVANLINNA CHARACTERISTIC OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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The best possible asymptotic estimates for Lebesgue integral means $m_p(r, \log f), 1 \le p < +\infty$ of logarithms of analytic functions $f(z)$ in the unit disc in terms of their Nevanlinna characteristic $T(r, f)$ are obtained. We get sharp relation between the order of $T(r, f)$ and the order of $m_p(r, \log f)$ for an analytic function $f(z)$ of finite order $\alpha(f)$. This generalizes well-known results of L. R. Sons and C. N. Linden.

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Получены неулучшаемые асимптотические оценки лебеговых интегральных средних $m_p(r, \log f)$, $1 \le p < +\infty$ логарифмов аналитических в единичном круге функций в терминах их неванлинновских характеристик $T(r, f)$. В качестве следствия для аналитических функций $f(z)$ конечного порядка получены точные соотношения между порядками в терминах $T(r, f)$ и порядками в терминах $m_p(r, \log f)$, обобщающие известные теоремы L. R. Sons и C. N. Linden.

1. Introduction. Let $f(z)$ be an analytic function in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, $f(0) = 1$. By definition, we put

$$\log f(z) := \int_0^z \frac{f'(\zeta)}{f(\zeta)} d\zeta \tag{1}$$

where $z \in \mathbb{D} \setminus \bigcup_{\nu=1}^{+\infty} [a_\nu, a_\nu/|a_\nu|)$ and (a_ν) be all zeroes of the function f, i.e. $f^{-1}(0) = \{a_\nu \colon \nu \in \mathbb{N}\}$,

$$T(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad \log M(r, f) := \max_{|z|=r} \log |f(z)|.$$

We define Lebesgue integral means by

$$m_p(r, \log f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \log f(re^{i\theta}) \right|^p d\theta \right)^{1/p}$$

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if $1 \le p < +\infty$, $0 \le r < 1$. Notice that

$$m_p(r, \log |f|) \le m_p(r, \log f)$$

for all $1 \le p < +\infty$, $0 \le r < 1$.

Let us introduce the following notation

$$\alpha(f) = \limsup_{r \nearrow 1} \frac{\log^+ T(r,f)}{-\log(1-r)}, \quad \beta(f) = \limsup_{r \nearrow 1} \frac{\log^+ \log^+ M(r,f)}{-\log(1-r)},$$
$$\alpha_p(\log f) = \limsup_{r \nearrow 1} \frac{\log^+ m_p(r, \log f)}{-\log(1-r)}, \quad \alpha_p(\log |f|) = \limsup_{r \nearrow 1} \frac{\log^+ m_p(r, \log |f|)}{-\log(1-r)}.$$

From the relation ([1, p.54])

$$T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(r,f), \quad 0 \le r < R < 1$$

we obtain the following sharp inequality

$$\alpha(f) \le \beta(f) \le \alpha(f) + 1.$$

And since

$$T(r,f) \le m_1(r, \log |f|) \le 2T(r,f), \tag{2}$$

we have

$$\alpha(f) = \alpha_1(\log |f|).$$

L. R. Sons in [6] obtained the following sharp inequalities

$$\alpha(f) \le \alpha_2(\log |f|) \le \alpha(f) + 1/2, \tag{3}$$

when $\alpha(f) < +\infty$.

In [7], J. Miles and D. Shea showed that

$$m_2(r, \log |f|) \le T(R,f) \left\{ 1 + \frac{8\sqrt{\log 2}}{\sqrt{\log(R/r)}} \right\}, \ 0 < r < R, \tag{4}$$

for all meromorphic functions $f(z)$ in the disc $\overline{\mathbb{D}}_R = \{z \in \mathbb{C} \colon |z| \le R\}$ such that $f(0) = 1$. This proves the right-hand inequality in (3). In [8] the mentioned statement (4) is extended on logarithms $\log f$ of meromorphic functions $f(z)$, $f(0) = 1$; all estimates for $m_p(r, \log f)$ if $p \in [1; +\infty)$ are given in terms of $T(r,f)$, $0 < r < R$.

Theorem 1 in this paper improves the previous estimates of $m_p(r, \log f)$ for $1 \le p \le 2$ in case of logarithm $\log f$ of analytic function $f(z)$ in the unit disc, $f(0) = 1$.

In [9] these results are extended to functions $u(z)$, $\breve{u}(z)$ such that $\mathcal{F}(z) = u(z) + \breve{u}(z)$; the function $u(z)$ is subharmonic in $\mathbb{C}$, $u(z)$ is harmonic in some neighbourhood of $z = 0$, and $u(0) = 0$; the function $\breve{u}(z)$ is conjugate of the function $u(z)$. Let us remark that $\mathcal{F} = \log f$ if the following conditions are hold: $\log f(z)$ is defined as (1), $f(z)$ is analytic in $\mathbb{D}$, $f(0) = 1$, $u = \log |f|$. For more details, we refer the reader to [10].

From C. N. Linden's results (see [3], [4]) it follows that

$$\alpha(f) \le \alpha_p(\log|f|) \le \alpha(f) + 1 - 1/p \tag{5}$$

for all $p \in [1; +\infty)$. And these inequalities are sharp.

The aim of this paper is to get sharp asymptotic estimates for increasing of integral means $m_p(r, \log f)$ of an analytic function f in $\mathbb{D}$ such that $f(0) = 1$ for all $p \in [1; +\infty)$ in terms of Nevanlinna characteristic $T(R, f)$ whenever $2/3 \le r < R < 1$. Also we obtain an analogue for Theorem 2 from [9]. Using representation $\arg f := \operatorname{Im}\log f$ in terms of the Gilbert transformation for circle (Theorem A), the famous Marcel Riesz theorem (Theorem B), monotonicity of $m_p(r, \log f)$ with respect to p, convexity of $m_p(r, \log f)$ with respect to $\log p$ (see [12, p.48]), we extend (4) to $\alpha_p(\log f)$ when $\alpha(f) < +\infty$ and $r \nearrow 1$.

2. Preliminaries. The following theorems are needed for the sequel.

Theorem A (Ya. V. Vasyl'kiv, A. A. Kondratyuk, [10]). *Let $f(z)$ be an analytic function in $\mathbb{D}$, $f(0) = 1$. Then for any $0 < r < 1$*

$$\log f(re^{i\theta}) = \log|f(re^{i\theta})| + i\left(\widetilde{\log|f(re^{i\theta})|} - \widetilde{p(re^{i\theta})}\right)$$

holds for almost all $\theta \in [0, 2\pi]$ where

$$\widetilde{\log|f(re^{i\theta})|} = -i\sum_{k\in\mathbb{Z}} \operatorname{sign} k\, c_k(r, f)e^{ik\theta},$$

$$\operatorname{sign} k = \frac{k}{|k|},\ k \ne 0,\ \operatorname{sign} 0 = 0,\ c_k(r, f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}\log|f(re^{i\theta})|d\theta,$$

$$\widetilde{p(re^{i\theta})} = \int_0^r \left(\sum_{|a_\nu|\le t} \operatorname{Im}\frac{r + te^{i(\theta-\alpha_\nu)}}{r - te^{i(\theta-\alpha_\nu)}}\right)\frac{dt}{t},\ a_\nu \in \{f^{-1}(0)\},\ \alpha_\nu = \arg a_\nu.$$

Theorem B (M. Riesz, [11]). *If $q \in (1; +\infty)$ and $g(e^{i\theta}) \in L^1[0; 2\pi]$, then*

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |\widetilde{g}(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}} \le M(q)\left(\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^q d\theta\right)^{\frac{1}{q}}$$

where

$$M(q) = \begin{cases} \operatorname{tg}\frac{\pi}{2q}, & 1 < q \le 2; \\ \operatorname{ctg}\frac{\pi}{2q}, & 2 \le q < +\infty. \end{cases}$$

Theorem C (Ya. V. Vasyl'kiv, [8]). *Let $0 < r < R$, and let a function $f(z)$ be meromorphic in $\overline{\mathbb{D}}_R$, $f(0) = 1$. Then*

$$m_p(r, \log f) \le T(R, f)\left\{1 + 14\left(\frac{p}{\log\frac{R}{r}}\right)^{1-\frac{1}{p}}\right\},\ 2 \le p < +\infty,$$

$$m_p(r, \log f) \le \frac{T(R, f)}{\sqrt[p]{\log\frac{R}{r}}}\left\{\frac{15R}{r}\right\}^{\frac{2}{p}-1}\left\{\sqrt{\log\frac{R}{r}} + 14\sqrt{2}\right\}^{2\left(1-\frac{1}{p}\right)},\ 1 \le p \le 2.$$

3. Main results and proofs.

Theorem 1. *Let $f(z)$ be an analytic function in $\mathbb{D}$, $f(0) = 1$, $0 < \delta < 1$. Then there exists $r_0 \in [2/3; 1)$ such that for any $r_0 \le r < R < 1$, $1 \le p < +\infty$,*

$$m_p(r, \log f) \le C(p, \delta) \frac{T(R, f)}{(R - r)^{\delta\left(\frac{2}{p}-1\right)^+ + 1 - \frac{1}{p}}} \tag{6}$$

holds, where $C(p, \delta)$ is some positive constant, $C(\cdot, \delta) \to +\infty$ as $\delta \to +0$, and $C(p, \cdot) \to +\infty$ as $p \to +\infty$.

Proof. Let $2 \le p < +\infty$, $2/3 \le r < R < 1$. By Theorem C, since the inequality

$$\log(R/r) \ge R - r$$

holds, wee observe that

$$m_p(r, \log |f|) \le m_p(r, \log f) \le C_1(p) \frac{T(R, f)}{(R - r)^{1 - \frac{1}{p}}}, \tag{7}$$

where $C_1(p)$ is some positive constant, $C_1(p) \to +\infty$ as $p \to +\infty$.

Note that

$$m_2(r, \log |f|) \le m_2(r, \log f) \le C_1(2) \frac{T(R, f)}{\sqrt{R - r}} \tag{8}$$

for $2/3 \le r < R < 1$.

Now, convexity of $m_p(r, \log f)$ with respect to $\log p$ (see [12]) yields

$$m_p(r, \log f) \le (m_1(r, \log f))^{\frac{2}{p} - 1} (m_2(r, \log f))^{2\left(1 - \frac{1}{p}\right)}, \ 0 < r < 1. \tag{9}$$

Suppose $0 < \delta < 1$. Then Theorem A and Theorem B it implay that

$$m_1(r, \log f) \le m_{1+\delta}(r, \log |f|) + m_{1+\delta}(r, \widetilde{\log |f|}) + m_1(r, \widetilde{p}) \le$$

$$\le (1 + M(1 + \delta)) m_{1+\delta}(r, \log |f|) + m_1(r, \widetilde{p}), \ 0 < r < 1. \tag{10}$$

Similarly to (8) we obtain

$$m_{1+\delta}(r, \log |f|) \le (m_1(r, \log |f|))^{\frac{2}{1+\delta} - 1} (m_2(r, \log |f|))^{2(1 - \frac{1}{1+\delta})}. \tag{11}$$

Taking into account (11), (7), and (2) we get

$$m_{1+\delta}(r, \log |f|) \le C_2(\delta) \frac{T(R, f)}{(R - r)^{\frac{\delta}{1+\delta}}}, \ \frac{2}{3} < r < R < 1. \tag{12}$$

Put

$$n(t) := \sum_{|a_\nu| \le t} 1, \ a_\nu \in \{f^{-1}(0)\}, \ N(t) := \int_0^t \frac{n(t)}{t} dt.$$

Further, using the Fubini theorem

$$m_1(r,\widetilde{p(re^{i\theta})}) = \frac{1}{2\pi}\int_0^{2\pi} |\widetilde{p(re^{i\theta})}| d\theta \le \int_0^r \left(\sum_{|a_\nu|\le t} \frac{1}{2\pi}\int_0^{2\pi} \frac{2rt|\sin(\theta-\alpha_\nu)|}{r^2-2rt\cos(\theta-\alpha_\nu)+t^2} d\theta\right)\frac{dt}{t} =$$

$$= \frac{1}{\pi}\int_0^r \frac{n(t)}{t}dt \int_0^\pi \frac{2rt\sin\theta}{r^2-2rt\cos\theta+t^2}d\theta = \frac{2}{\pi}\int_0^r \log\frac{r+t}{r-t}\cdot\frac{n(t)}{t}dt.$$

In the sequel, $2/3 \le r < 1$. We change the order of integration in the last integral

$$I(r) = \int_0^r \log\frac{r+t}{r-t}\cdot\frac{n(t)}{t}dt = \int_0^r \frac{n(t)}{t}dt\int_{r-t}^{r+t}\frac{dx}{x} =$$

$$= \int_0^r \frac{dx}{x}\int_{r-x}^r \frac{n(t)}{t}dt + \int_r^{2r}\frac{dx}{x}\int_{x-r}^r \frac{n(t)}{t}dt := I_1(r) + I_2(r).$$

Firstly,

$$I_1(r) = \int_0^r \frac{dx}{x}\int_{r-x}^r \frac{n(t)}{t}dt = \int_0^{\frac{R-r}{r}} \frac{dx}{x}\int_{r-x}^r \frac{n(t)}{t}dt + \int_{\frac{R-r}{r}}^r \frac{dx}{x}\int_{r-x}^r \frac{n(t)}{t}dt := I_{1,1}(r) + I_{1,2}(r),$$

$$I_{1,1}(r) \le n(r)\int_0^{\frac{R-r}{r}} \frac{dx}{r-x} \le n(r)\frac{R-r}{r}\frac{1}{r-\frac{R-r}{r}} \le$$

$$\le 9n(r)(R-r) \le 9RN(R) \le 9T(R,f),$$

$$I_{1,2}(r) \le N(r)\log\frac{1}{R-r} \le N(R)\log\frac{r}{R-r} \le T(R,f)\log\frac{r}{R-r}.$$

Secondly,

$$I_2(r) = \int_r^{2r}\frac{dx}{x}\int_{x-r}^r \frac{n(t)}{t}dt \le N(r)\int_r^{2r}\frac{dx}{x} = N(r)\log 2.$$

Since

$$I_1(r) \le 9T(R,f) + T(R,f)\log\frac{r}{R-r}$$

and $I_2(r) \le N(r)\log 2 \le T(R,f)$, we have

$$I(r) \le T(R,f)\Big(10 + \log\frac{r}{R-r}\Big),\ \frac{2}{3} \le r < R < 1.$$

Thus,

$$m_1(r,\widetilde{p}) \le \frac{2}{\pi}T(R,f)\Big(10+\log\frac{r}{R-r}\Big),\ \frac{2}{3} \le r < R < 1. \tag{13}$$

Combining (10), (12), and (13), we see that

$$m_1(r,\log f) \le (1+M(1+\delta))C_3(1+\delta)T(R,f)\frac{1}{(R-r)^{\frac{\delta}{1+\delta}}} + \frac{2}{\pi}T(R,f)\left(10+\log\frac{r}{R-r}\right) \le$$
$$\le C_4(1+\delta)T(R,f)\frac{1}{(R-r)^{\frac{\delta}{1+\delta}}},\ \frac{2}{3}\le r<R<1.$$

From (9), we can estimate

$$m_p(r,\log f) \le C_5(p)T(R,f)\frac{1}{(R-r)^{\frac{\delta}{1+\delta}(\frac{2}{p}-1)}}\frac{1}{(R-r)^{1-\frac{1}{p}}} \le C_5(p)\frac{T(R,f)}{(R-r)^{1+\delta-\frac{1}{p}}}, \tag{14}$$

whenever $2/3 \le r < R < 1,\ \ 1 \le p \le 2$.

Using (7) and (14), we obtain

$$m_p(r,\log f) \le C_6(p,\delta)\frac{T(R,f)}{(R-r)^{\delta(\frac{2}{p}-1)^+ +1-\frac{1}{p}}}$$

for all $1 \le p < +\infty$. This completes the proof of Theorem 1. □

Corollary 1. *Suppose f satisfies the assumptions of Theorem 1. Let also $0<\epsilon(r)<1-r$, $\epsilon(r)\searrow 0$ as $r\nearrow 1$. Then for all $1\le p<+\infty$ holds*

$$m_p(r,\log f) \le C_1(p,\delta)\frac{T(\frac{r}{1-\epsilon(r)},f)}{(\epsilon(r))^{\delta(\frac{2}{p}-1)^+ +1-\frac{1}{p}}} \tag{15}$$

for $r_0 < r < 1$.

Proof. Put $R = r/\,(1-\epsilon(r))$ in Theorem 1. We take into account that

$$R-r = r\Big(\frac{1}{1-\epsilon(r)}-1\Big) = r\frac{\epsilon(r)}{1-\epsilon(r)} > \epsilon(r).$$

This concludes the proof. □

Corollary 2. *Suppose f satisfies the assumptions of Theorem 1 and $\alpha(f)<+\infty$. Then*

$$\alpha(f) \le \alpha_p(\log f) \le \alpha(f)+1-1/p,\ \ 1\le p<+\infty. \tag{16}$$

Proof. Put $R=(1+r)/2$ in (6). Then

$$m_p(r,\log f) \le \widetilde{C}(p,\delta)\frac{T(\frac{1+r}{2},f)}{(1-r)^{\delta(\frac{2}{p}-1)^+ +1-\frac{1}{p}}},\ \ 1\le p<+\infty.$$

Hence, $\alpha_p(\log f) \le \delta(2/p-1)^+ + \alpha(f)+1-1/p$. Let now δ tends to $+0$. We get

$$\alpha_p(\log f) \le \alpha(f)+1-1/p,\ \ 1\le p<+\infty.$$ □

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