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KALEIDOSCOPICAL GRAPHS AND SEMIGROUPS

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We give a semigroup characterization of kaleidoscopical graphs. A connected graph Γ (considered as a metric space with the path metric) is called kaleidoscopical if there is a vertex coloring of Γ which is bijective on each unit ball.

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Предложена полугрупповая характеризация калейдоскопических графов. Связный граф Γ (как метрическое пространство с метрикой кратчайших расстояний между вершинами) называется калейдоскопическим, если существует раскраска множества вершин Γ, биективная на каждом единичном шаре.

Let $\Gamma(V, E)$ be a connected graph with the set of vertices V and the set of edges E, d be the path metric on V, $B(v, r) = \{u \in V : d(v, u) \le r\}, v \in V, r \in \omega = \{0, 1, ...\}$.

A graph $\Gamma(V, E)$ is called *kaleidoscopical* [4] if there exists a coloring (a surjective mapping) $\chi : V \to \kappa$, κ is a cardinal, such that the restriction $\chi|B(v, 1) : B(v, 1) \to \kappa$ is a bijection on each unit ball $B(v, 1), v \in V$. For kaleidoscopical graphs see also [2, Chapter 6] and [3].

Let G be a group, X be a transitive G-space with the action $G \times X \to X$, $(g, x) \longmapsto gx$. A subset A of X, $|A| = \kappa$ is said to be a *kaleidoscopical configuration* [1] if there exists a coloring $\chi : X \to \kappa$ such that, for each $g \in G$, the restriction $\chi|gA : gA \to \kappa$ is a bijection.

We note that kaleidoscopical graphs and kaleidoscopical configurations can be considered as partial cases of kaleidoscopical hypergraphs defined in [2, p.5]. Recall that a *hypergraph* is a pair $(X, \mathfrak{F})$ where X is a set, $\mathfrak{F}$ is a family of subsets of X.

A hypergraph $(X, \mathfrak{F})$ is said to be *kaleidoscopical* if there exists a coloring $\chi : X \to \kappa$ such that, for each $F \in \mathfrak{F}$, the restriction $\chi \mid F : F \to \kappa$ is a bijection.

Clearly, a graph $\Gamma(V, E)$ is kaleidoscopical if and only if the hypergraph $(V, \{B(v, 1) : v \in V\})$ is kaleidoscopical. A subset A of a G-space X is kaleidoscopical if and only if the hypergraph $(X, \{g(A) : g \in G\})$ is kaleidoscopical.

We say that two hypergraphs $(X_1, \mathfrak{F}_1)$, $(X_2, \mathfrak{F}_2)$ with kaleidoscopical colorings $\chi_1 : X_1 \to \kappa$, $\chi_2 : X_2 \to \kappa$ are *kaleidoscopically isomorphic* if there is a bijection $f : X_1 \to X_2$ such that

- $\forall A \subseteq X_1 : A \in \mathfrak{F}_1 \iff f(A) \in \mathfrak{F}_2$;
- $\forall x \in X_1 : \chi_1(x) = \chi_2(f(x))$.

We describe an algebraic construction which up to isomorphisms gives all kaleidoscopical graphs.

The *kaleidoscopical semigroup* $KS(\kappa)$ is a semigroup in the alphabet κ determined by the relations $xx = x$, $xyx = x$ for all $x, y \in \kappa$. For our purposes, it is convenient to identify $KS(\kappa)$ with the set of all non-empty words in κ with no factors xx, xyx where $x, y \in \kappa$.

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For every $x \in \kappa$, the set $KG(\kappa, x)$ of all words from $KS(\kappa)$ with the first and the last letter x is a subgroup (with the identity x) of the semigroup $KS(\kappa)$. To obtain the inverse element to the word $w \in KG(\kappa, x)$ it suffices to write w in the inverse order. The group $KG(\kappa, x)$ is called the *kaleidoscopical group* in the alphabet κ with the identity x.

For finite cardinals κ, the following theorem is proved in [2, p.64–66] but corresponding arguments work for arbitrary κ.

Theorem 1. *For any cardinal κ, the following statements hold:*

- *The only idempotents of the semigroup $KS(\kappa)$ are the words x, xy where $x, y \in \kappa, x \neq y$.*
- *The kaleidoscopical group $KG(k, x)$ is a free group with the set of free generators*

$$\{xyzx\colon y, z \in \kappa \backslash \{x\}, y \neq z\}.$$

- *The kaleidoscopical semigroup $KS(\kappa)$ is isomorphic to the sandwich product $L(x) \times KG(\kappa, x) \times R(x)$ with the multiplication*

$$(l_1, g_1, r_1)(l_2, g_2, r_2) = (l_1, g_1 r_1 l_2 g_2, r_2),$$

where $L(x) = \{yx\colon y \in \kappa\}$, $R(x) = \{xy\colon y \in \kappa\}$.

We fix $x \in \kappa$, denote by æ(w) the first letter of the word $w \in KS(\kappa)$ and say that an equivalence $\sim$ on $KS(\kappa)$ is *kaleidoscopical* if, for all $w, w' \in KS(\kappa)$ and $y \in \kappa$,

$$w \sim w' \to \text{æ}(w) = \text{æ}(w') \wedge yw = yw',$$
$$w \sim w' \Longleftrightarrow wx \sim w'x.$$

Let $[w]$ be the class of equivalence $\sim$ containing $w \in KS(\kappa)$.

We put

$$S_x = [x] \cap KG(\kappa, x),$$

observe that S_x is a subgroup of $KG(\kappa, x)$ and show that $\sim$ is uniquely determined by S_x

$$w \sim w' \Longleftrightarrow \text{æ}(w) = \text{æ}(w') \wedge xwx \sim xw'x \Longleftrightarrow (xwx)^{-1}(xw'x) \in S_x.$$

We see also that any subgroup of $KG(\kappa, x)$ can be taken as S_x to determine a kaleidoscopical equivalence on $KS(\kappa)$.

A kaleidoscopical equivalence $\sim$ determines a graph $\Gamma(\kappa, \sim)$ with the set of vertices $KS(\kappa)/\sim$ and the set of edges E defined by the rule

$$(u, v) \in E \Longleftrightarrow \exists\ w \in u\ \exists\ y \in \kappa\colon \text{æ}(w) \neq y \wedge yw \in v.$$

A coloring $\chi\colon KS(\kappa)/\sim \to \kappa$ defined by $\chi([w]) = $ æ(w) shows that $\Gamma(\kappa, \sim)$ is kaleidoscopical.

Now let $\Gamma(V, E)$ be a kaleidoscopical graph with kaleidoscopical coloring $\chi\colon V \to \kappa$. We define a transitive action of the semigroup $KS(\kappa)$ on the set V as follows. Let $v \in V$, $x \in \kappa$. Pick $u \in B(v, 1)$ such that $\chi(u) = x$ and put $x(v) = u$. Then we extended the action onto $KS(\kappa)$ inductively. If $w = KS(\kappa)$, $w = xw'$, $w' \in KS(\kappa)$, $x \in \kappa$, we put $w(v) = x(w'(v))$. Given any $v_1, v_2 \in V$, the sequence of colors of the vertices on a path from v_1 to v_2 determines a word $w \in KS(\kappa)$ such that $w(v_1) = v_2$ so $KS(\kappa)$ acts on V transitively. Clearly, the group $KG(\kappa, x)$ acts transitively on the set $\chi^{-1}(x)$ of vertices of color x.

We fix $v \in V$ with $\chi(v) = x$, determine a kaleidoscopical equivalence $\sim$ on $KS(\kappa)$ by the rule $w \sim w' \iff w(v) = w'(v)$, and note that the graphs $\Gamma(V, E)$ and $\Gamma(\kappa, \sim)$ are kaleidoscopically isomorphic via bijection $f\colon V \to KS(\kappa)/\sim$, $f(u) = \{w \in KS(\kappa)\colon w(v) = u\}$.

All above considerations are focused in the following theorem.

Theorem 2. *For every kaleidoscopical graph $\Gamma(V, E)$ with kaleidoscopical coloring $\chi\colon V \to \kappa$, there exists a kaleidoscopical equivalence $\sim$ on the semigroup $KS(\kappa)$ such that $\Gamma(V, E)$ is kaleidoscopically isomorphic to $\Gamma(\kappa, \sim)$. Every kaleidoscopical equivalence $\sim$ on $KS(\kappa)$ is uniquely determined by some subgroup of the group $KG(\kappa, x)$.*

Every group G can be considered as a G-space with the left regular action $(g, x) \longmapsto yx$. Let A be a kaleidoscopical configuration in G. By [1, Corollary 1.3], A is complemented, i.e. there exists a subset B of G such that the multiplication $A \times B \to G$, $(a, b) \longmapsto ab$ is bijective.

Let A be a system of generators of a group G such that $A = A^{-1}$ and $e \in A$, e is the identity of G. We consider the Cayley graph $\mathrm{Cay}(G, A)$ with the set of vertices G and the set of edges E defined by the rule

$$(g, h) \in E \iff g^{-1}h \in A, g \neq h.$$

Clearly, $\mathrm{Cay}(G, A)$ is connected. Assume that $\mathrm{Cay}(G, A)$ is kaleidoscopical with kaleidoscopical coloring $\chi\colon G \to |A|$. Since $B(g, 1) = gA$ and χ is bijective on each ball $B(g, 1)$, we see that A is a kaleidoscopical configuration. On the other hand, if A is a kaleidoscopical configuration in G with kaleidoscopical coloring $\chi\colon G \to A$ then χ is bijective on each set gA so $\mathrm{Cay}(G, A)$ is kaleidoscopical. Thus, we get the following theorem.

Theorem 3. *Let G be a group, A be a system of generators of G such that $A = A^{-1}$ and $e \in A$. Then A is a kaleidoscopical configuration if and only if $\mathrm{Cay}(G, A)$ is kaleidoscopical.*

We conclude the paper with two open questions.

Question 1. *How to detect whether a kaleidoscopical hypergraph is kaleidoscopically isomorphic to a hypergraph of unit balls of some kaleidoscopical graph?*

Question 2. *How to detect whether a kaleidoscopical hypergraph is kaleidoscopically isomorphic to a hypergraph determined by a kaleidoscopical configuration in a G-space.*

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