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REGULAR EXTENSION OPERATORS FOR PARTIAL FUZZY (PSEUDO)METRICS

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Using the fuzzy modification of the Prokhorov metric on the set of probability measures we prove the theorem on existence of simultaneous extensions of partial fuzzy metrics (for the Łukasiewicz t-norm) defined on closed subsets of a compact metrizable space.

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Используя нечеткую модификацию метрики Прохорова на множестве вероятностных мер, мы доказываем теорему о существовании одновременного продолжения частичных нечетких метрик (для t-нормы Лукасевича), определенных на замкнутых подмножествах компактного метризуемого просторанства.

1. Introduction. The problem of extension of functions and (pseudo)metrics has a long history (see, e.g., the papers [1, 2, 11, 5]).

E. Tymchatyn and M. Zarichnyi [11] proved a counterpart of the theorem of H. Künzi and L. Schapiro on simultaneous extension of continuous functions [5]. In his previous publication, the author proved a counterpart of this theorem for the so called stationary fuzzy metrics.

The present paper is devoted to the problem of simultaneous extension of partial fuzzy (pseudo)metrics. We use the construction of fuzzy metric on the set of probability measures (fuzzy Prokhorov metric) to define an extension operator for the fuzzy (pseudometrics) defined on the nonempty closed subsets of compact spaces. Thus, the main result is a fuzzy version of the mentioned result by Tymchatyn and Zarichnyi.

Note that the notion of the fuzzy metric space is a modification of that of probabilistic metric space. In the latter, the values of the distance function are distributions. There are different notions of the fuzzy metric space; in the present paper we use that of George and Veeramani [4].

The notion of fuzzy metric space is now widely investigated; it finds numerous applications in different areas of mathematics.

2. Fuzzy metric spaces. We start with the definition of fuzzy metric spaces (see, e.g., [4]). A continuous t-norm is a continuous map $(x, y) \mapsto x * y\colon [0,1]\times[0,1]\to[0,1]$ which satisfies the following properties:

1. $(x*y)*z = x*(y*z)$;
2. $x*y = y*x$;

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3. $x * 1 = x$;
4. if $x \leq x'$ and $y \leq y'$, then $x * y \leq x' * y'$.

In other words, a continuous t-norm is a continuous Abelian monoid with unit 1 and with the monotonic operation. The following are examples of continuous t-norms:

1. $x * y = \min\{x, y\}$;
2. $x * y = \max\{0, x + y - 1\}$ (Łukasiewicz t-norm).

Definition 1. A *fuzzy metric space* is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set of $X \times X \times (0, \infty)$ (i.e. M is a map from $X \times X \times (0, \infty)$ to $[0, 1]$) satisfying the following properties:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, s) * M(y, z, t) \leq M(x, z, s + t)$;
(v) the function $M(x, y, -)\colon (0, \infty) \to [0, 1)$ is continuous.

We obtain the notion of a fuzzy pseudometric space if we replace condition (ii) from the above definition by the following condition:

(ii') $M(x, x, t) = 1$.

In a fuzzy metric space $(X, M, *)$, we say that the set

$$B_M(x, r, t) = \{y \in X \mid M(x, y, t) > 1 - r\},\ x \in X,\ r \in (0, 1),\ t \in (0, \infty),$$

is the *open ball* of radius $r > 0$ centered at x for t. It is proved in [4] that the family of all open balls is a base of a topology on X; this topology is denoted by τ_M.

If we speak on a fuzzy (pseudo)metric on a topological space, we always assume that this metric is compatible with the topology of this space.

3. Space of partial fuzzy (pseudo)metrics. Let $\exp X$ denote the hyperspace of X, i.e. the set of all nonempty compact subsets of X. This space is endowed with the Vietoris topology, i.e. the topology whose base consists of the sets of the form

$$\langle U_1, \dots, U_n \rangle = \{A \in \exp X \mid A \subset \cup_{i=1}^n A_i,\ A \cap U_i \neq \varnothing,\ i = 1, \dots, n\}.$$

The set $C((0, \infty), [0, 1])$ of all continuous functions from $(0, \infty)$ to $[0, 1]$ is endowed with the compact-open topology. Note that this space is metrizable (see, e.g., [3]).

A partial fuzzy metric M in X is a fuzzy metric on some $A \in \exp X$. We express this fact by writing either $A = \mathrm{dom}(M)$ or $M \in \mathcal{FM}(A)$. Denote by $\mathcal{FM}$ the set of all partial fuzzy metrics in X, $\mathcal{FM} = \bigcup\{\mathcal{FM}(A) \mid A \in \exp X\}$.

Every fuzzy metric M on a space X generates the map $v_M\colon X \times X \to C(0, \infty), [0, 1])$ defined by the formula $v_M(x, y)(t) = M(x, y, t)$. It follows from the exponential law that this map is continuous.

In order to topologize the set of all partial fuzzy (pseudo)metrics on a compact metrizable space X we identify every partial fuzzy metric M with the graph Γ_M of the map v_M,

$$\Gamma_M = \{(x, y, v_M(x, y)) \mid (x, y) \in \mathrm{dom}(M)^2\} \in \exp(\mathrm{dom}(M)^2 \times C(0, \infty), [0, 1]).$$

Then the topology on the set of all partial fuzzy (pseudo)metrics is that induced by the Vietoris topology.

4. Space of probability measures. Let X be a compact metrizable space. By $P(X)$ we denote the set of all probability measures on X endowed with the weak* topology. The support of $\mu \in P(X)$ is denoted by $\mathrm{supp}(\mu)$. For any $x \in X$, by δ_x we denote the Dirac measure concentrated at x.

In the sequel, by $*$ we denote the Łukasiewicz t-norm. Let $(X, M, *)$ be a compact fuzzy metric space.

For every $A \subset X$, $r \in (0,1)$, $t \in (0,\infty)$ define:

$$A^{r,t} = \cup\{B(x,r,t) \mid x \in A\} \subset X.$$

Define the function $\hat{M}\colon P(X) \times P(X) \times (0,\infty) \to [0,1]$ by the formula

$$\begin{aligned}\hat{M}(\mu,\nu,t) =& 1 - \inf\{r \in (0,1) \mid \mu(A) \le \nu(A^{r,t}) + r \text{ and } \nu(A) \le \mu(A^{r,t}) + r \\ & \text{for every Borel subset } A \subset X\}.\end{aligned}$$

It is proved in [7] that the function $\hat{M}$ is a fuzzy metric on $P(X)$ that generates the weak* topology.

If we replace the fuzzy metrics by the fuzzy pseudometrics, we obtain the spaces $\mathcal{FPM}(A)$ and $\mathcal{FPM}$ respectively.

5. Soft maps. The following notion is introduced by E.V. Shchepin [9]. A map $f\colon X \to Y$ is said to be *soft*, if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow f \\ Z & \xrightarrow[\psi]{} & Y \end{array} \tag{1}$$

such that A is a closed subset of a paracompact space Z there exists a map $\Phi\colon Z \to X$ such that $f\Phi = \psi$ and $\Phi|A = \varphi$.

From the Michael Selection Theorem ([6]) it follows that any open map of metrizable compacta whose preimages are convex subsets of a locally convex space (in particular, they are spaces of the form $P(Y)$) is soft.

6. Main result. An extension operator for partial fuzzy metrics on X is a map $e\colon \mathcal{FM} \to \mathcal{FM}(X)$ satisfying the property $e(M)|\mathrm{dom}(M)^2 = M$, for every $M \in \mathcal{FM}$.

Theorem 1. *Let X be a compact metrizable space. There exists a continuous extension operator for partial fuzzy (pseudo)metrics.*

Proof. Let $Y = \{(A,x) \in \exp X \times X \mid x \in A\} \subset \exp X \times X$, $K = \{(A,\mu) \in \exp X \times P(X) \mid \mathrm{supp}(\mu) \subset A\}$. Consider the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & K \\ \downarrow & & \downarrow f \\ \exp X \times X & \xrightarrow[\psi]{} & \exp X \end{array},$$

where $\varphi(A, x) = (A, \delta_x)$, ψ is the projection onto the first factor and f is the restriction of the projection $\exp X \times P(X) \to \exp X$. Let also $\pi\colon \exp(X) \times P(X) \to P(X)$ denote the projection onto the first factor.

Note that the map f is open and therefore f is soft. Thus, there exists a map $\Phi\colon \exp X \times X \to K$ such that $\Phi|Y = \varphi$ and $f\Phi = \psi$.

Now, given a partial fuzzy (pseudo)metric $M \in \mathcal{FM}$, define $e'(M)$ by the following formula:

$$e'(M)(x, y, t) = \hat{M}(\pi\Phi(\mathrm{dom}(M), x), \pi\Phi(\mathrm{dom}(M), y), t).$$

Note that the function $e'(M)$ is a fuzzy pseudometric on X. If $x, y \in \mathrm{dom}(M)$, then

$$e'(M)(x, y, t) = \hat{M}(\pi(\mathrm{dom}(M), \delta_x), \pi(\mathrm{dom}(M), \delta_y), t) = \hat{M}(\delta_x, \delta_y, t) = M(x, y, t)$$

(the latter equality follows from [7, Proposition 3.10]), and therefore e' is an extension operator.

Therefore, in the case of pseudometrics, one can define $e = e'$.

Now we consider the case of fuzzy metrics.

Let $\sim$ denote the following equivalence relation on the set $\exp X \times X$: $(A, a) \sim (B, b)$ if and only if either $(A, a) = (B, b)$ or $a, b \in A = B$. Let d be any compatible metric on the space $Z = (\exp X \times X)/\sim$. Denote by $q\colon \exp X \times X \to Z$ the quotient map. Define the function $N\colon (\exp X \times X)^2 \times (0, \infty) \to [0, 1]$ by the formula

$$N((A, a), (B, b), t) = \frac{t}{t + d(q(A, a), q(B, b))}.$$

Finally, define $e(M)(x, y, t) = e'(M)(x, y, t) * N(x, y, t)$. Clearly, $e(M)$ is a fuzzy pseudometric on the space X. We are going to show that $e(M)$ is a fuzzy metric on X. To this end, let $x, y \in X$, $x \neq y$, and $t \in (0, \infty)$. We consider the following two cases.

1) $x, y \in \mathrm{dom}(M)$. Then

$$e(M)(x, y, t) = M(x, y, t) * \frac{t}{t + d(q(\mathrm{dom}(M), x), (\mathrm{dom}(M), y))} = M(x, y, t) * 1 < 1.$$

2) $\{x, y\} \not\subset \mathrm{dom}(M)$. Then $q(\mathrm{dom}(M), x) \neq q(\mathrm{dom}(M), y)$ and therefore

$$e(M)(x, y, t) \leq \frac{t}{t + d(q(\mathrm{dom}(M), x), (\mathrm{dom}(M), y))} < 1.$$

That e is an extension operator is proved similarly as in the case of fuzzy pseudometrics. Thus, we obtain the required operator e. □

7. Nonmetrizable case.

Theorem 2. *For a compact Hausdorff space X the following are equivalent:*

1. *X is metrizable;*
2. *there exists a continuous extension operator $e\colon \mathcal{FPM} \to \mathcal{FPM}(X)$.*

Proof. We have only to prove that 2)⇒1).

For any $(x, y) \in X^2$, $x \neq y$, let M_{xy} denote the stationary fuzzy metric on X defined as follows: $M(x, y, t) = \frac{1}{2}$, for every $t \in (0, \infty)$.

Denote by Δ the diagonal of the product $X^2 = X \times X$. If there exists a continuous extension operator $e\colon \mathcal{FPM} \to \mathcal{FPM}(X)$, then define the map $\varphi\colon X^2 \setminus \Delta \to C(X)$ (as usual, the space of continuous functions on X) as follows: $\varphi(x,y)(z) = e(M)(x,z,1)$, $z \in X$. Then $\varphi(x,y)(x) = e(M)(x,x,1) = 1$, while $\varphi(x,y)(y) = e(M)(x,y,1) = \frac{1}{2}$. Then, by Stepanova's theorem (see [10]), X is metrizable. □

8. Remarks and open problems. The main open question is whether the results of this note can be extended for the case of the other t-norms. Another open problem is whether the extension operator preserves operations on the family of fuzzy metrics.

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