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ON NORMALITY OF SPACES OF SCATTEREDLY CONTINUOUS MAPS

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A map $f\colon X\to Y$ between topological spaces is called scatteredly continuous if for each non-empty subspace $A\subset X$ the restriction $f|_A$ has a point of continuity. By $SC_p(X)$ we denote the space of all scatteredly continuous real-valued functions on X endowed with the topology of pointwise convergence.

In this paper we focus on the normality of the space $SC_p(X)$. Particularly, it is proved that if the function space $SC_p(X)$ is normal, then all compact and all scattered subspaces of X are countable.

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Отображение $f\colon X\to Y$ между топологическими пространствами называют разреженно непрерывным, если для каждого непустого подпространства $A\subset X$ сужение $f|_A$ имеет точку непрерывности. Через $SC_p(X)$ обозначаем пространство всех разреженно непрерывных вещественных функций на пространстве X в топологии поточечной сходимости.

Исследуется нормальность пространства $SC_p(X)$. В частности, доказано, что если пространство $SC_p(X)$ нормально, то все компактные и все разреженные подпространства пространства X счетны.

1. Introduction. A map $f\colon X\to Y$ between topological spaces is called *scatteredly continuous* if for each non-empty subspace $A\subset X$ the restriction $f|_A$ has a point of continuity. By $SC_p(X)$ we denote the space of all scatteredly continuous real-valued functions on X endowed with the topology of pointwise convergence. Clearly, that the space of all continuous maps $C_p(X)$ is a subspace of the space $SC_p(X)$, and the function space $SC_p(X)$ is a subspace of the space $\mathbb{R}^X$. It is well known that the space $\mathbb{R}^X$ is normal if and only if X is countable. On the other hand, there are uncountable spaces X such that the function space $C_p(X)$ is normal, in particular if the network weight of X is countable. A natural question arises: under what conditions on a space X, is the space of all scatteredly continuous functions $SC_p(X)$ normal? In this paper we prove, in particular, that if the function space $SC_p(X)$ is normal, then all compact and all scattered subspaces of X are countable.

2. Terminology and notation. A “space” always means a “topological space”. By $\mathbb{R}$ and $\mathbb{Q}$ we denote the usual spaces of real and rational numbers, respectively; $\mathbb{N}$ stands for the set of integer positive numbers.

A standard base of neighborhoods of a function $f\colon X\to\mathbb{R}$ in the space $SC_p(X)$ consists of the sets of the form $W(f,x_1,...,x_k,\varepsilon)=\{g\in SC(X)\colon |g(x_i)-f(x_i)|<\varepsilon, i=1,\dots,k\}$ with $k\in\mathbb{N}, x_1,\dots,x_k\in X$ and $\varepsilon>0$.

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For a subset A of a topological space X by $cl_X(A)$ or $\overline{A}$ we denote the closure of A in X while $\mathrm{Int}(A)$ stands for the interior of A in X.

Recall that a space X is called *normal*, if it is a T_1-space and for an arbitrary pair of disjoint closed subsets F_1, F_2 of X there are open subsets U_1, U_2 of X such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \varnothing$.

All spaces encountered in this paper (unless stated otherwise) are assumed to be Hausdorff. The rest of the notation and terminology is standard and can be found in [1].

3. The restriction operator and the dual map. Let Y be a subspace of a space X. By $\pi_Y\colon SC_p(X) \to SC_p(Y)$ we denote the restriction operator from $SC_p(X)$ onto $SC_p(Y)$, that is $\pi_Y(f) = f\,|_Y$ for all $f \in SC_p(X)$. The definition of a scatteredly continuous map implies that $\pi_Y(SC_p(X))$ is a subspace of the space $SC_p(Y)$.

We say that a set A is *dividing* (see [2]), if there is a non-empty set F such that $\overline{A \cap F} = \overline{F \backslash A}$, and A is called *undividing* if $\overline{A \cap F} \neq \overline{F \backslash A}$ for arbitrary non-empty set F.

Obviously, all closed, open and scattered subsets of any topological space X are undividing. In [2] it is proved that if X is a hereditary Baire perfectly paracompact space, then a subset A of X is undividing if and only if A is an F_σ-set and G_δ-set in X.

Theorem 1 ([4])**.** *Let $f\colon X \to Y$ be a scatteredly continuous map from a topological space X to a regular topological space Y. Then each non-empty subspace $A \subset X$ contains an open (in A) dense subset $U \subset A$ such that the restriction $f\,|_A : A \to Y$ is continuous at every point of the set U.*

Proposition 1. *For an arbitrary subspace Y of a topological space X the following statements are true:*

1. *The operator $\pi_Y\colon SC_p(X) \to SC_p(Y)$ is continuous and $\overline{\pi_Y(SC_p(X))} \supset SC_p(Y)$;*
2. *The operator $\pi_Y\colon SC_p(X) \to SC_p(Y)$ is an open map from $SC_p(X)$ onto the subspace $\pi_Y(SC_p(X))$ of $SC_p(Y)$;*
3. *If Y is an undividing set in X, then $\pi_Y(SC_p(X)) = SC_p(Y)$;*
4. *If Y is a scattered subspace of a space X, then $\pi_Y(SC_p(X)) = \mathbb{R}^Y$;*
5. *The operator π_Y is injective if and only if $Y = X$.*

Proof. 1. Obviously, $\pi_Y\colon SC_p(X) \to SC_p(Y)$ is continuous. We prove that $\overline{\pi_Y(SC_p(X))} \supset SC_p(Y)$. Take an arbitrary $g \in SC_p(Y)$ and a standard neighborhood $W(g, y_1, ..., y_n, \varepsilon)$ of the point $g \in SC_p(Y)$. We define a function $f\colon X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0, & x \in X\backslash \{y_1, ..., y_n\}, \\ g(y_i), & x \in \{y_1, ..., y_n\}. \end{cases}$$

It is easy to check, that $f \in SC_p(X)$ and $\pi_Y(f) \in W(g, y_1, ..., y_n, \varepsilon)$.

2. Consider an arbitrary standard open set $W(f, x_1, ..., x_k, \varepsilon) \subset SC_p(X)$. Without loss of generality, we may assume that $x_1, ..., x_l \in Y$ and $x_{l+1}, ..., x_k \in X\backslash Y$ with $0 \le l \le k$. Obviously, $\pi_Y(W(f, x_1, ..., x_k, \varepsilon)) \subset W(\pi_Y(f), x_1, ..., x_l, \varepsilon) \cap \pi_Y(SC_p(X))$. We show that $\pi_Y(W(f, x_1, ..., x_k, \varepsilon)) = W(\pi_Y(f), x_1, ..., x_l, \varepsilon) \cap \pi_Y(SC_p(X))$, which implies that the set $\pi_Y(W(f, x_1, ..., x_k, \varepsilon))$ is open in the space $\pi_Y(SC_p(X))$. And this means that the operator $\pi_Y\colon SC_p(X) \to SC_p(Y)$ is open.

It remains to show that $\pi_Y(W(f,x_1,...,x_k,\varepsilon)) \supset W(\pi_Y(f),x_1,...,x_l,\varepsilon) \cap \pi_Y(SC_p(X))$. Let $g \in \pi_Y(SC_p(X))$ and $|g(x_i) - \pi(f)(x_i)| < \varepsilon$, $i = 1,...,l$. Since $g \in \pi_Y(SC_p(X))$, there is a map $g_1 \in SC_p(X)$ such that $g = \pi_Y(g_1)$. We fix a function $\varphi\colon X \to \mathbb{R}$ such that

$$\varphi(x) = \begin{cases} 0, & x \notin \{x_{l+1},...,x_k\} \\ f(x_i) - g_1(x_i), & x \in \{x_{l+1},...,x_k\}\,. \end{cases}$$

It is easy to check that the function φ is scatteredly continuous. Put $h = \varphi + g_1$. Obviously, $h \in W(f,x_1,...,x_k,\varepsilon)$ and $\pi_Y(h) = g$. Therefore, $g \in \pi_Y(W(f,x_1,...,x_k,\varepsilon))$ and the statement (2) is proved.

3. Let Y be some non-empty undividing set in X. We show that $\pi_Y(SC_p(X)) = SC_p(Y)$. Consider some map $g \in SC_p(Y)$. Define a function $f\colon X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} g(x), & x \in Y \\ 0, & x \notin Y. \end{cases}$$

It is easy to see that $\pi_Y(f) = g$. Now show that $f \in SC_p(X)$. Let A be an arbitrary non-empty subset of X. Put $P = A \cap Y$ and $Q = A \backslash Y$. According to Theorem 1, the space P contains an open (in P) dense subspace $U \subset P$ such that the restriction $g\,|_P$ is continuous at every point of the set U. Put $B = ((P \backslash \bar{Q}) \cap U) \cup (Q \backslash \bar{P})$. Since Y is an undividing set in X and U is dense in P, the set $B \neq \varnothing$. Obviously, the restriction $f|_A$ is continuous at every point of the set B.

4. Since every scattered subspace of a topological space is an undividing set, then statement 3 of this proposition implies that $\pi_Y(SC_p(X)) = SC_p(Y)$. Since Y is scattered, $SC_p(Y) = \mathbb{R}^Y$.

5. Assume that $Y \neq X$. Fix an arbitrary point $x_0 \in X \backslash Y$ and maps $f_1\colon X \to \mathbb{R}$ and $f_2\colon X \to \mathbb{R}$, which are defined as follows:

$$f_1(x) = \begin{cases} 0, & x \in X \backslash \{x_0\} \\ 1, & x = x_0 \end{cases}, \quad f_2(x) = \begin{cases} 0, & x \in X \backslash \{x_0\} \\ 2, & x = x_0 \end{cases}$$

Observe that $f_1, f_2 \in SC_p(X)$, $f_1 \neq f_2$, but $\pi_Y(f_1) = \pi_Y(f_2)$. Thus, the map π_Y is not injective. □

Proposition 2. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be scatteredly continuous maps and let Y be a regular space. Then the composition $g \circ f\colon X \to Z$ is a scatteredly continuous map.*

Proof. Let A be an arbitrary subspace of X. By Theorem 1, the space A contains an open (in A) dense subspace $U \subset A$ such that the restriction $f\,|_A$ is continuous at every point of the set U. Using the scattered continuity of g, we find a continuity point $y \in f(U)$ of the map $g\,\big|_{f(U)}$. Take an arbitrary point $x \in f^{-1}(y) \cap U$ and notice that the composition $g \circ f$ is continuous at the point x. □

The following example shows that the regularity of the space Y in the previous proposition is essential.

Example 1. Let $f\colon \mathbb{R} \to \mathbb{R}_{\mathbb{Q}}$ be the identity map from the real line equipped with the standard topology τ to the real line endowed with the topology generated by the subbase

$\tau \cup \{\mathbb{Q}\}$. Let $\chi_{\mathbb{Q}}\colon \mathbb{R} \to \{0;1\}$ denote the characteristic function of the set $\mathbb{Q}$. It is easy to show that the maps $f\colon \mathbb{R} \to \mathbb{R}_{\mathbb{Q}}$ and $\chi_{\mathbb{Q}}\colon \mathbb{R} \to \{0;1\}$ are scatteredly continuous while their composition $\chi_{\mathbb{Q}} \circ f\colon \mathbb{R} \to \{0;1\}$ is everywhere discontinuous (and hence fails to be scatteredly continuous).

Recall that a space X is called a *Preiss-Simon* space if for an arbitrary non-empty closed subset A of X and each point $x \in A$ there is a sequence $\{U_n\colon n \in \mathbb{N}\}$ of non-empty open subsets of A that converges to x in the sense that each neighborhood of x contains all but finitely many sets U_n.

Lemma 1. *Let f be a surjective map from a topological space X onto a topological space Y and g be a map from the space Y to some topological space Z. Then scattered continuity of the map $g \circ f$ implies the scattered continuity of the map g if one of the following conditions is satisfied:*

1) *the map $f\colon X \to Y$ is open;*
2) *the map $f\colon X \to Y$ is closed, the space X is perfectly paracompact, Y is a hereditary Baire Preiss-Simon space and Z is a regular space.*

Proof. 1. Assume that the map $\varphi = g \circ f$ is scatteredly continuous and let f be an open map. To show that g is scatteredly continuous, fix a non-empty subset $B \subset Y$ and let $A = f^{-1}(B)$. It follows that $f|_A\colon A \to B$ is an open map. Since the map φ is scatteredly continuous, the restriction $\varphi|_A\colon A \to Z$ has a continuity point $x_0 \in A$. We show that the map $g|_B : B \to Z$ is continuous at the point $y_0 = f(x_0)$. Assume that $O(g(y_0))$ is a neighborhood of the point $g(y_0)$ in Z. Since $\varphi(x_0) = g(y_0)$, there is a neighborhood $O(x_0)$ of the point x_0 in the subspace A such that $\varphi(O(x_0)) \subset O(g(y_0))$. Since the restriction $f|_A\colon A \to B$ is an open map, one has that $f(O(x_0))$ is a neighborhood of the point y_0. It is easy to deduce that $g(f(O(x_0))) = \varphi(O(x_0)) \subset O(g(y_0))$.

2. In [5], in particular, is proved that a map g from a hereditary Baire Preiss-Simon space Y to a regular space Z is scatteredly continuous if for any open subset in Z its preimage is a G_δ-set in Y. Suppose g is not a scatteredly continuous map. Then there is an open set U in Z such that $g^{-1}(U)$ is not G_δ-set in Y.

On the other hand, since $g \circ f$ is a scatteredly continuous map from a perfectly paracompact space X to a regular space Z, we obtain that $(g \circ f)^{-1}(U)$ is a G_δ-set in X (see [5]).

Put $A = (g \circ f)^{-1}(U) \subset X$. Then $f(A) = g^{-1}(U)$. Since A is a G_δ-set in X, we have that $X \backslash A$ is an F_σ-set in X, that is, $X \backslash A = \bigcup\{F_i\colon i \in \mathbb{N}\}$ where each F_i is a closed subset in X. Then $f(X \backslash A) = \bigcup f(F_i)$ is an F_σ-set in Y. But then $Y \backslash f(X \backslash A) = g^{-1}(U)$ is a G_δ-set in Y, which is a contradiction. □

Example 2. Assume that f is a map from a scattered uncountable compact space X to the segment $Y = [0,1]$, and $g\colon [0,1] \to \mathbb{R}$ is the characteristic function of the set $\mathbb{Q}$. The spaces X and Y are both compact. Obviously, the maps $g \circ f\colon X \to \mathbb{R}$ and f are scatteredly continuous. But the characteristic function $g\colon [0,1] \to \mathbb{R}$ is not scatteredly continuous.

Each $f\colon X \to Y$ induces a dual map $f^{\#}\colon \mathbb{R}^Y \to \mathbb{R}^X$ that assigns to each function $\varphi \in \mathbb{R}^Y$, the composition $f^{\#}(\varphi) = \varphi \circ f$.

Proposition 3. *Let X and Y be topological spaces and let f be a mapping of X to Y. Then the following statements are true:*

1. *The map $f^{\#}\colon SC_p(Y) \to f^{\#}(SC_p(Y))$ is continuous.*
2. *If the map f is scatteredly continuous and the space Y is regular, then $f^{\#}(SC_p(Y)) \subset SC_p(X)$.*
3. *Let f be open scatteredly continuous surjective map and let Y be regular. Then $f^{\#}$ is a homeomorphism of the space $SC_p(Y)$ onto the closed subspace $f^{\#}(SC_p(Y))$ of $SC_p(X)$.*
4. *Let f be a closed scatteredly continuous surjective map, let X be perfectly paracompact and let Y be a hereditary Baire Preiss-Simon space. Then $f^{\#}$ is a homeomorphism of the space $SC_p(Y)$ onto the closed subspace $f^{\#}(SC_p(Y))$ of $SC_p(X)$.*
5. *If f is scatteredly continuous then f is injective if and only if $f^{\#}(SC_p(Y))$ is dense in $SC_p(X)$.*

Proof. 1. In [3], in particular, is proved that the map $f^{\#}\colon \mathbb{R}^Y \to \mathbb{R}^X$ is continuous for an arbitrary map $f\colon X \to Y$ and arbitrary sets X and Y. Thus, the map $f^{\#}\colon SC_p(Y) \to f^{\#}(SC_p(Y))$ is continuous.

2. Let f be a scatteredly continuous map. Take an arbitrary map $\varphi \in SC_p(Y)$. Since the maps φ and f are scatteredly continuous and Y is a regular space, the composition $\varphi \circ f$ is scatteredly continuous map according to Proposition 2. Therefore $f^{\#}(SC_p(Y)) \subset SC_p(X)$.

3. Assume that f is an open map and $f(X) = Y$. Since the map $f^{\#}\colon \mathbb{R}^Y \to \mathbb{R}^X$ is a homeomorphism of the space $\mathbb{R}^Y$ onto the closed subspace $f^{\#}(\mathbb{R}^Y)$ of $\mathbb{R}^X$ (see [3]), the map $f^{\#}\colon SC_p(Y) \to SC_p(X)$ is a homeomorphism of the space $SC_p(Y)$ onto the subspace $f^{\#}(SC_p(Y))$ of $SC_p(X)$.

We prove that $f^{\#}(SC_p(Y))$ is a closed subspace of the space $SC_p(X)$. Take any function $\psi \in SC_p(X)$ with $\psi \in \overline{f^{\#}(SC_p(Y))}$ and an arbitrary $y \in Y$. Obviously, each function φ of $f^{\#}(SC_p(Y))$ is constant on $f^{-1}(y)$. Then the function ψ is constant on $f^{-1}(y)$, as well. Therefore we can find a function $g\colon Y \to \mathbb{R}$ such that $\psi = g \circ f$, that is, $\psi = f^{\#}(g)$. Since f is an open map, the map ψ is scatteredly continuous and $\mathbb{R}$ is a regular space, then Lemma 1(1) guarantees that the map g is scatteredly continuous. Hence $\psi \in f^{\#}(SC_p(Y))$, that is, the set $f^{\#}(SC_p(Y))$ is closed in $SC_p(X)$.

4. The proof of this statement is similar to (3), and can be proved using Lemma 1(2).

5. Assume that f is scatteredly continuous and injective, $\psi \in SC_p(X)$ and $W(\psi, x_1, \ldots, x_k, \varepsilon)$ is an arbitrary standard neighborhood of the function ψ in $SC_p(X)$. Put $y_i = f(x_i)$, $i = 1, \ldots, k$. Since the map f is bijective, there is a function $\varphi \in SC_p(Y)$ such that $\varphi(y_i) = \psi(x_i)$, $i = 1, \ldots, k$. Obviously, $f^{\#}(\varphi) \in W(\psi, x_1, \ldots, x_k, \varepsilon)$, that is, the subspace $f^{\#}(SC_p(Y))$ is dense in the space $SC_p(X)$.

Now let map f be scatteredly continuous and let $f^{\#}(SC_p(Y))$ be dense in $SC_p(X)$. We show that f is bijective. Assume that $x_1 \neq x_2$, but $f(x_1) = f(x_2) = y$. Then for all $\varphi \in f^{\#}(SC_p(Y))$ we have $f^{\#}(\varphi)(x_1) = \varphi(f(x_2)) = \varphi(y) = \varphi(f(x_2)) = f^{\#}(\varphi)(x_2)$. Take a function $\psi \in SC_p(X)$ with $\psi(x_1) = 0$ and $\psi(x_2) = 1$. Obviously, $W(\psi, x_1, x_2, \frac{1}{2}) \cap f^{\#}SC_p(Y) = \varnothing$, which contradicts the fact that $f^{\#}(SC_p(Y))$ is dense in $SC_p(X)$. □

4. Extent and normality of the spaces of scatteredly continuous maps. Recall that two subsets A and B of a topological space X are separated, if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

Lemma 2 ([3]). *Let Y be a dense subspace of the product $X = \prod\{X_\alpha\colon \alpha \in A\}$ of separable metrizable spaces X_α and $P \subset Y, Q \subset Y$. Then following conditions are equivalent:*

a) there are open subsets U and V in X such that $P \subset U$, $Q \subset V$ and $U \cap V = \varnothing$;

b) there is a countable set $M \subset A$ such that the sets $\pi_M(Q)$ and $\pi_M(P)$ are separated in $\pi_M(Y) \subset \pi_M(X) = \prod\{X_\alpha : \alpha \in M\}$.

Theorem 2. *Let $SC_p(X)$ be a normal space and $Y \subset X$. Then so is the space $\pi_Y(SC_p(X))$.*

Proof. Assume that P and Q are closed disjoint sets in the space $\pi_Y(SC_p(X))$. The space $\pi_Y(SC_p(X))$ is dense in $\mathbb{R}^Y$. By Lemma 2 it is sufficient to find a countable subset $Z \subset Y$ such that the sets $\pi_Z(P)$ and $\pi_Z(Q)$ are separated in $\pi_Z(SC_p(X))$. Consider the sets $P' = \pi_Y^{-1}(P)$ and $Q' = \pi_Y^{-1}(Q)$. Since the space $SC_p(X)$ is normal, there is a countable set $Z' \subset X$ such that $\pi_{Z'}(P')$ and $\pi_{Z'}(Q')$ are separated in $\pi_{Z'}(SC_p(X))$. We show that, if we replace Z' with the set $Z = Z' \cap Y$, then the sets $\pi_Z(P') = \pi_Z(P)$ and $\pi_Z(Q') = \pi_Z(Q)$ will be separated in $\pi_Z(SC_p(X))$. Suppose this is not true. Assume, for example, that $\pi_Z(P') \cap cl_T(\pi_Z(Q')) \neq \varnothing$ with $T = \pi_Z(SC_p(X))$. Choose $f \in P'$ so that $f|_Z \in cl_T(\pi_Z(Q'))$, and prove that $f|_{Z'} \in cl_{T'}(\pi_{Z'}(Q'))$ with $T' = \pi_{Z'}(SC_p(X))$. Take some finite set $K \subset Z'$ and $\varepsilon > 0$. Put $K_1 = K \cap Y$ and $K_2 = K \cap (X \backslash Y)$. Since $f|_Z \in cl_T(\pi_Z(Q))$, there is a map $g \in Q$, such that $|g(x) - f(x)| < \varepsilon$ for any $x \in K_1$. Then there is a map $g' \in Q'$ such that $\pi_Y(g') = g$. Fix a function h' such that $h'(x) = f(x) - g'(x)$ for any $x \in K_2$ and $h'(x) = 0$ for all $x \in X \backslash K_2$. Obviously, $h' \in SC_p(X)$. Put $h = h' + g'$. Then $h|_Y = g$ (that is $h \in Q'$) and $|h(x) - f(x)| < \varepsilon$ for every $x \in K$. Since K is an arbitrary finite set, $f|_{Z'} \in cl_{T'}(\pi_{Z'}(Q'))$. And we obtain that the sets $\pi_{Z'}(P')$ and $\pi_{Z'}(Q')$ are not separated, which is a contradiction. Hence, $\pi_Z(P)$ and $\pi_Z(Q)$ are separated in $\pi_Z(SC_p(X))$. □

Proposition 4. *Let X be a topological space. If $SC_p(X)$ is a normal space, then every scattered subspace of the space X is countable.*

Proof. Assume that A is a scattered subspace of X. Then $\pi_A(SC_p(X)) = SC_p(A)$. The previous theorem implies the space $\pi_A(SC_p(X))$ is normal, and, therefore, the space $SC_p(A)$ is normal as well. If the subspace A is scattered, then $SC_p(A) = \mathbb{R}^A$. And since the space $\mathbb{R}^A$ is not normal with uncountable A, the set A is countable. □

Recall that the Lindelöf number $l(X)$ of a space X is the smallest cardinal number m such that each open cover of X has a subcover of size $\leqslant m$. Hereditary Lindelöf number $hl(X)$ of a space X is equal to $sup\{l(Y) : Y \subset X\}$.

Corollary 1. *If $SC_p(X)$ is a normal space, then $hl(X) \leq \aleph_0$.*

By the extent $e(X)$ of a topological space X we understand the smallest infinite cardinal number m such that the cardinality of each closed discrete subspace of the space X does not exceed m.

We define a subset $A \subset X$ to be *sc*-embedded into a space X if for any scatteredly continuous map $f : A \to \mathbb{R}$ there is a scatteredly continuous map $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f}|_A = f$. One can show that all undividing subsets of the topological space X are *sc*-embedded into X (see the proof of Proposition 1(3)).

Lemma 3. *If Y is sc-embedded into a space X, then $e(SC_p(Y)) \leqslant e(SC_p(X))$.*

Proof. Let Y be *sc*-embedded into X. Assume that F_Y is a closed discrete subspace of $SC_p(Y)$. Consider the restriction operator $\pi_Y : SC_p(X) \to SC_p(Y)$. Due to Proposition 1(1) this operator is continuous. For each $g \in F_Y$ fix an element $f_g \in SC_p(X)$ such that $f_g \in$

$\pi_Y^{-1}(g)$. Put $F_X = \{f_g : g \in F_Y\}$. Since the restriction $\pi_Y|_{F_X}$ is bijective and continuous, the subspace F_X is discrete in $SC_p(X)$. Let us prove that F_X is a closed subspace of $SC_p(X)$. Suppose this is not true. Take some function $f^* \in \overline{F_X} \backslash F_X$. Since F_Y is a closed subspace of $SC_p(Y)$ and π_Y is a continuous map, $\pi_Y^{-1}(F_Y)$ is a closed subspace of $SC_p(X)$. Thus $f^* \in \pi_Y^{-1}(F_Y)$, that is, $f^* \in \pi_Y^{-1}(g)$ for some $g \in F_Y$. Since $\{g\}$ is an isolated point in the subspace F_Y and π_Y is continuous, $\pi_Y^{-1}(g)$ is a neighborhood of the point f^* in the subspace $\pi_Y^{-1}(F_Y)$. Put $\{f_g\} \in \pi_Y^{-1}(g) \cap F_X$. Then $(\pi_Y^{-1}(g) \backslash \{f_g\}) \cap F_X = \varnothing$. And this contradicts the fact that $f^* \in \overline{F_X}$. □

Proposition 5. *If a space X contains a non-scattered compact, then $e(SC_p(X)) \geqslant 2^{\aleph_0}$.*

Proof. Assume that K is a compact subspace of X. Then there is a continuous map φ from the space K onto a metrizable compact Y. Let ρ be a metric on Y and let $B_i(y) = \{t : \rho(y,t) \leqslant \frac{1}{i}\}$. For each $y \in Y$ we define the map $f_y : K \to \mathbb{R}$ as follows:

$$f_y(x) = \begin{cases} 0, & \text{if } x \in \varphi^{-1}(y); \\ \min\{i : x \notin \varphi^{-1}(B_i(y))\}, & \text{otherwise.} \end{cases}$$

Put $\mathcal{F} = \{f_y : y \in Y\}$. We show that $\mathcal{F}$ is a closed discrete subspace of $SC_p(K)$.

Firstly, we prove that $\mathcal{F} \subset SC_p(K)$, that is, each map f_y is scatteredly continuous. Consider an arbitrary map $f_y \in \mathcal{F}$ and an arbitrary subset $A \subset K$. If $A \subset \varphi^{-1}(y)$, then $f_y(\varphi^{-1}(y)) = 0$ and the function f_y is continuous at every point of A. Let $A \not\subset \varphi^{-1}(y)$. Put $i_0 = \min\{i : f_y(A) = i\}$. Then there is a point $x_0 \in A$ such that $f_y(x_0) = i_0$. And by the definition of the function f_y this means that $x_0 \notin \varphi^{-1}(B_{i_0}(y))$. Then $x_0 \in A \backslash \varphi^{-1}(B_{i_0}(y))$, the subset $A \backslash \varphi^{-1}(B_{i_0}(y))$ is an open subset of A and $f_y(A \backslash \varphi^{-1}(B_{i_0}(y))) = i_0$. Thus, the function $f_y \in \mathcal{F}$ is continuous at the point x_0. And, therefore, f_y is scatteredly continuous for all $y \in Y$.

We prove that $\mathcal{F}$ is a closed subspace of $SC_p(K)$. Assume that we have a function $g \in \overline{\mathcal{F}} \backslash \mathcal{F}$.

Fix a base $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of the function g in the space $SC_p(K)$. Put $P_\alpha = \{x \in K : f_{\varphi(x)} \in U_\alpha\}$.

The family $\{P_\alpha : \alpha \in A\}$ has the finite intersection property, that is, $P_{\alpha_1} \cap P_{\alpha_2} \cap ... \cap P_{\alpha_n} \neq \varnothing$ for every finite system $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. Since K is a compact space, we have that $\bigcap_{\alpha \in A} \overline{P_\alpha} \neq \varnothing$.

Fix a point $z \in \bigcap_{\alpha \in A} \overline{P_\alpha}$. Take a standard neighborhood $W(g, z, \frac{1}{2}) = \{f \in SC_p(K) : |g(z) - f(z)| < \frac{1}{2}\}$ of the function g in a space $SC_p(K)$. If $g(z) = 0$, then $(W(g, z, \frac{1}{2}) \backslash f_{\varphi(z)}) \cap \mathcal{F} = \varnothing$. And this contradicts the fact that $g \in \overline{\mathcal{F}}$. Hence, $g(z) \neq 0$. Since for all $x \in K$ and each $f_y \in \mathcal{F}$ we have that $f_y(x) \in \mathbb{N} \cup \{0\}$, there is $i \in \mathbb{N}$ such that $g(z) = i$. Fix an element of the base $U_{\alpha_0} \in \mathcal{U}$ such that $U_{\alpha_0} \subset W(g, z, \frac{1}{2})$. Since $z \in \bigcap \overline{P_\alpha}$, we obtain that $z \in \overline{P_{\alpha_0}}$. And since $g(z) = i$, for any $x \in K$ such that functions $f_{\varphi(x)}$ lie in the neighborhood U_{α_0}, we have $f_{\varphi(x)}(z) = i$ and $z \notin \varphi^{-1}(B_i(\varphi(x)))$. Then for each $x \in P_{\alpha_0}$ we have $z \notin \varphi^{-1}(B_i(\varphi(x)))$, that is, $\rho(\varphi(x), \varphi(z)) > \frac{1}{i}$. Obviously, point $z \in \operatorname{Int}(\varphi^{-1}(B_i(\varphi(z))))$, but $\operatorname{Int}(\varphi^{-1}(B_i(\varphi(z)))) \cap P_{\alpha_0} = \varnothing$. And this contradicts the fact that $z \in \overline{P_{\alpha_0}}$.

Therefore, the subspace $\mathcal{F}$ is closed in $SC_p(K)$.

We prove that $\mathcal{F}$ is a discrete subspace of $SC_p(K)$. Take any function f_y of $\mathcal{F}$. Fix a point $x \in \varphi^{-1}(y)$ and a standard neighborhood $W(f_y, x, \frac{1}{2})$ of the function f_y in the space $SC_p(K)$. Then $x \notin \varphi^{-1}(y')$ for each $y' \in Y$ such that $y' \neq y$. By the definition of the function $f_{y'}$ one has that $f_{y'}(x) \geqslant 1$. Thus, $W(f_y, x, \frac{1}{2}) \cap \mathcal{F} = \{f_y\}$. Therefore, $\mathcal{F}$ is a discrete subspace of the space $SC_p(K)$.

Since K is a compact subspace of X, one has that K is sc-embedded into X. Hence, by Lemma 3, $e(SC_p(X)) \geqslant e(SC_p(K)) \geqslant 2^{\aleph_0}$. □

Theorem 3 ([3]). *Let X be a normal space with countable Souslin number and $\chi(X) \leqslant 2^{\aleph_0}$, that is, the space X has a base of cardinality $\leqslant 2^{\aleph_0}$ at every point. Then $e(X) < 2^{\aleph_0}$.*

A space X is called k-scattered ([6], [7]) if for an arbitrary non-empty subset $F \subset X$ there is a non-empty open subset U of X such that $U \cap F \neq \varnothing$ and $\overline{U \cap F}$ is compact.

Theorem 4. *If $SC_p(X)$ is a normal space, then every k-scattered subspace of X is countable.*

Proof. Assume that $SC_p(X)$ is a normal space. We show that all compact subspaces of X are scattered. Suppose there is a non-scattered compact K of X. In view of Theorem 2, normality of the space $SC_p(X)$ implies normality of the space $\pi_K(SC_p(X))$. Since K is a closed set, applying the Proposition 1(3), we obtain that $\pi_K(SC_p(X)) = SC_p(K)$. Since K is an uncountable compact, there is a continuous map φ from K onto the segment $I = [0, 1]$. Since $SC_p(K)$ is a normal space, then by Corollary 1, we have that $hl(K) \leqslant \aleph_0$, that is, the compact K is a hereditary Lindelöf space. And, therefore, the space K is perfectly paracompact. Then, according to Proposition 3(4), the map $\varphi^{\#}\colon SC_p(I) \to SC_p(K)$ is a homeomorphism of the space $SC_p(I)$ onto the closed subspace $\varphi^{\#}(SC_p(I))$ of $SC_p(K)$. Since the space $SC_p(K)$ is normal, its closed subspace $\varphi^{\#}(SC_p(I))$ is normal as well. And since the map $\varphi^{\#}\colon SC_p(I) \to \varphi^{\#}(SC_p(I))$ is a homeomorphism, the space $SC_p(I)$ is normal. The space $SC_p(I)$ is a normal space with countable Souslin number and $\chi(SC_p(I)) \leqslant 2^{\aleph_0}$. Then, by Theorem 3, $e(SC_p(I)) < 2^{\aleph_0}$. But, applying Proposition 5, we have that $e(SC_p(I)) \geqslant 2^{\aleph_0}$. This contradiction proves that all compact subspaces of X are scattered.

Let Y be a non-empty k-scattered subspace of X and let A be an arbitrary non-empty subset of Y. Since Y is k-scattered, there is an open subset U of Y such that $U \cap A \neq \varnothing$ and $\overline{U \cap A}$ is a compact subspace of Y. Since every compact subspace of Y is compact in X, we have that $K = \overline{U \cap A}$ is a compact subspace of X. Thus, K is scattered. Then there are a point $x \in K$ and its neighborhood $O(x)$ such that $O(x) \cap K = \{x\}$. And since $x \in \overline{U \cap A}$, we obtain $O(x) \cap U \cap A \neq \varnothing$. The fact $O(x) \cap U \cap A \subset O(x) \cap \overline{U \cap A} = \{x\}$ implies that $(O(x) \cap U) \cap A = \{x\}$. Therefore, the point x is an isolated point in A. Hence, the space Y is scattered. And, by Proposition 4, space Y is countable. □

Corollary 2. *If X is a k-scattered space, then the space $SC_p(X)$ is normal if and only if X is countable.*

Recall that a space X is said to be a k-space if a set $F \subseteq X$ is closed if and only if for each compact subset $K \subseteq X$ the set $F \cap K$ is a compact in K.

Corollary 3. *If X is a k-space and $SC_p(X)$ is a normal space, then X is a sequential space.*

Proof. Let A be a non-closed subset of X. Then there is a compact subspace $K \subset X$ such that $A \cap K$ is a non-closed subset in K. Due to Theorem 4, compact K is countable, and therefore, K is metrizable. Thus, for each point $x \in \overline{A \cap K} \backslash (A \cap K)$ there is a sequence from $A \cap K$ that converges to x. □

A space X is called σ-compact if it is a countable union of its compact subspaces.

Corollary 4. *If X is a σ-compact space, then $SC_p(X)$ is normal if and only if X is countable.*

Question 1. Is there an uncountable space X such that $SC_p(X)$ is normal?

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