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# ON NORMALITY OF SPACES OF SCATTEREDLY CONTINUOUS MAPS

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A map $f\colon X \to Y$ between topological spaces is called scatteredly continuous if for each non-empty subspace $A \subset X$ the restriction $f|_A$ has a point of continuity. By $SC_p(X)$ we denote the space of all scatteredly continuous real-valued functions on $X$ endowed with the topology of pointwise convergence.

In this paper we focus on the normality of the space $SC_p(X)$. Particularly, it is proved that if the function space $SC_p(X)$ is normal, then all compact and all scattered subspaces of $X$ are countable.

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Отображение $f\colon X \to Y$ между топологическими пространствами называют разреженно непрерывным, если для каждого непустого подпространства $A \subset X$ сужение $f|_A$ имеет точку непрерывности. Через $SC_p(X)$ обозначаем пространство всех разреженно непрерывных вещественных функций на пространстве $X$ в топологии поточечной сходимости.

Исследуется нормальность пространства $SC_p(X)$. В частности, доказано, что если пространство $SC_p(X)$ нормально, то все компактные и все разреженные подпространства пространства $X$ счетны.

**1. Introduction.** A map $f\colon X \to Y$ between topological spaces is called *scatteredly continuous* if for each non-empty subspace $A \subset X$ the restriction $f|_A$ has a point of continuity. By $SC_p(X)$ we denote the space of all scatteredly continuous real-valued functions on $X$ endowed with the topology of pointwise convergence. Clearly, that the space of all continuous maps $C_p(X)$ is a subspace of the space $SC_p(X)$, and the function space $SC_p(X)$ is a subspace of the space $\mathbb{R}^X$. It is well known that the space $\mathbb{R}^X$ is normal if and only if $X$ is countable. On the other hand, there are uncountable spaces $X$ such that the function space $C_p(X)$ is normal, in particular if the network weight of $X$ is countable. A natural question arises: under what conditions on a space $X$, is the space of all scatteredly continuous functions $SC_p(X)$ normal? In this paper we prove, in particular, that if the function space $SC_p(X)$ is normal, then all compact and all scattered subspaces of $X$ are countable.

**2. Terminology and notation.** A "space" always means a "topological space". By $\mathbb{R}$ and $\mathbb{Q}$ we denote the usual spaces of real and rational numbers, respectively; $\mathbb{N}$ stands for the set of integer positive numbers.

A standard base of neighborhoods of a function $f\colon X \to \mathbb{R}$ in the space $SC_p(X)$ consists of the sets of the form $W(f, x_1, ..., x_k, \varepsilon) = \{g \in SC(X)\colon |g(x_i) - f(x_i)| < \varepsilon, i = 1, \dots, k\}$ with $k \in \mathbb{N}, x_1, \dots, x_k \in X$ and $\varepsilon > 0$.

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For a subset $A$ of a topological space $X$ by $cl_X(A)$ or $\overline{A}$ we denote the closure of $A$ in $X$ while $\mathrm{Int}(A)$ stands for the interior of $A$ in $X$.

Recall that a space $X$ is called *normal*, if it is a $T_1$-space and for an arbitrary pair of disjoint closed subsets $F_1$, $F_2$ of $X$ there are open subsets $U_1$, $U_2$ of $X$ such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \varnothing$.

All spaces encountered in this paper (unless stated otherwise) are assumed to be Hausdorff. The rest of the notation and terminology is standard and can be found in [1].

**3. The restriction operator and the dual map.** Let $Y$ be a subspace of a space $X$. By $\pi_Y\colon SC_p(X) \to SC_p(Y)$ we denote the restriction operator from $SC_p(X)$ onto $SC_p(Y)$, that is $\pi_Y(f) = f|_Y$ for all $f \in SC_p(X)$. The definition of a scatteredly continuous map implies that $\pi_Y(SC_p(X))$ is a subspace of the space $SC_p(Y)$.

We say that a set $A$ is *dividing* (see [2]), if there is a non-empty set $F$ such that $\overline{A \cap F} = \overline{F \backslash A}$, and $A$ is called *undividing* if $\overline{A \cap F} \neq \overline{F \backslash A}$ for arbitrary non-empty set $F$.

Obviously, all closed, open and scattered subsets of any topological space $X$ are undividing. In [2] it is proved that if $X$ is a hereditary Baire perfectly paracompact space, then a subset $A$ of $X$ is undividing if and only if $A$ is an $F_\sigma$-set and $G_\delta$-set in $X$.

**Theorem 1** ([4])**.** *Let $f\colon X \to Y$ be a scatteredly continuous map from a topological space $X$ to a regular topological space $Y$. Then each non-empty subspace $A \subset X$ contains an open (in $A$) dense subset $U \subset A$ such that the restriction $f|_A : A \to Y$ is continuous at every point of the set $U$.*

**Proposition 1.** *For an arbitrary subspace $Y$ of a topological space $X$ the following statements are true:*

1. *The operator $\pi_Y\colon SC_p(X) \to SC_p(Y)$ is continuous and $\overline{\pi_Y(SC_p(X))} \supset SC_p(Y)$;*
2. *The operator $\pi_Y\colon SC_p(X) \to SC_p(Y)$ is an open map from $SC_p(X)$ onto the subspace $\pi_Y(SC_p(X))$ of $SC_p(Y)$;*
3. *If $Y$ is an undividing set in $X$, then $\pi_Y(SC_p(X)) = SC_p(Y)$;*
4. *If $Y$ is a scattered subspace of a space $X$, then $\pi_Y(SC_p(X)) = \mathbb{R}^Y$;*
5. *The operator $\pi_Y$ is injective if and only if $Y = X$.*

*Proof.* 1. Obviously, $\pi_Y\colon SC_p(X) \to SC_p(Y)$ is continuous. We prove that $\overline{\pi_Y(SC_p(X))} \supset SC_p(Y)$. Take an arbitrary $g \in SC_p(Y)$ and a standard neighborhood $W(g, y_1, ..., y_n, \varepsilon)$ of the point $g \in SC_p(Y)$. We define a function $f\colon X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0, & x \in X \backslash \{y_1, ..., y_n\}, \\ g(y_i), & x \in \{y_1, ..., y_n\}. \end{cases}$$

It is easy to check, that $f \in SC_p(X)$ and $\pi_Y(f) \in W(g, y_1, ..., y_n, \varepsilon)$.

2. Consider an arbitrary standard open set $W(f, x_1, ..., x_k, \varepsilon) \subset SC_p(X)$. Without loss of generality, we may assume that $x_1, ..., x_l \in Y$ and $x_{l+1}, ..., x_k \in X \backslash Y$ with $0 \leq l \leq k$. Obviously, $\pi_Y(W(f, x_1, ..., x_k, \varepsilon)) \subset W(\pi_Y(f), x_1, ..., x_l, \varepsilon) \cap \pi_Y(SC_p(X))$. We show that $\pi_Y(W(f, x_1, ..., x_k, \varepsilon)) = W(\pi_Y(f), x_1, ..., x_l, \varepsilon) \cap \pi_Y(SC_p(X))$, which implies that the set $\pi_Y(W(f, x_1, ..., x_k, \varepsilon))$ is open in the space $\pi_Y(SC_p(X))$. And this means that the operator $\pi_Y\colon SC_p(X) \to SC_p(Y)$ is open.

It remains to show that $\pi_Y(W(f,x_1,...,x_k,\varepsilon)) \supset W(\pi_Y(f),x_1,...,x_l,\varepsilon) \cap \pi_Y(SC_p(X))$. Let $g \in \pi_Y(SC_p(X))$ and $|g(x_i) - \pi(f)(x_i)| < \varepsilon$ , $i = 1,...,l$. Since $g \in \pi_Y(SC_p(X))$, there is a map $g_1 \in SC_p(X)$ such that $g = \pi_Y(g_1)$. We fix a function $\varphi\colon X \to \mathbb{R}$ such that

$$\varphi(x) = \begin{cases} 0, & x \notin \{x_{l+1},...,x_k\} \\ f(x_i) - g_1(x_i), & x \in \{x_{l+1},...,x_k\}. \end{cases}$$

It is easy to check that the function $\varphi$ is scatteredly continuous. Put $h = \varphi + g_1$. Obviously, $h \in W(f,x_1,...,x_k,\varepsilon)$ and $\pi_Y(h) = g$. Therefore, $g \in \pi_Y(W(f,x_1,...,x_k,\varepsilon))$ and the statement (2) is proved.

3. Let $Y$ be some non-empty undividing set in $X$. We show that $\pi_Y(SC_p(X)) = SC_p(Y)$. Consider some map $g \in SC_p(Y)$. Define a function $f\colon X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} g(x), & x \in Y \\ 0, & x \notin Y. \end{cases}$$

It is easy to see that $\pi_Y(f) = g$. Now show that $f \in SC_p(X)$. Let $A$ be an arbitrary non-empty subset of $X$. Put $P = A \cap Y$ and $Q = A \backslash Y$. According to Theorem 1, the space $P$ contains an open (in $P$) dense subspace $U \subset P$ such that the restriction $g\,|_P$ is continuous at every point of the set $U$. Put $B = ((P \backslash \bar{Q}) \cap U) \cup (Q \backslash \bar{P})$. Since $Y$ is an undividing set in $X$ and $U$ is dense in $P$, the set $B \neq \varnothing$. Obviously, the restriction $f|_A$ is continuous at every point of the set $B$.

4. Since every scattered subspace of a topological space is an undividing set, then statement 3 of this proposition implies that $\pi_Y(SC_p(X)) = SC_p(Y)$. Since $Y$ is scattered, $SC_p(Y) = \mathbb{R}^Y$.

5. Assume that $Y \neq X$. Fix an arbitrary point $x_0 \in X \backslash Y$ and maps $f_1\colon X \to \mathbb{R}$ and $f_2\colon X \to \mathbb{R}$ , which are defined as follows:

$$f_1(x) = \begin{cases} 0, & x \in X \backslash \{x_0\} \\ 1, & x = x_0 \end{cases}, \quad f_2(x) = \begin{cases} 0, & x \in X \backslash \{x_0\} \\ 2, & x = x_0 \end{cases}$$

Observe that $f_1, f_2 \in SC_p(X)$ , $f_1 \neq f_2$, but $\pi_Y(f_1) = \pi_Y(f_2)$. Thus, the map $\pi_Y$ is not injective. □

**Proposition 2.** *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be scatteredly continuous maps and let $Y$ be a regular space. Then the composition $g \circ f\colon X \to Z$ is a scatteredly continuous map.*

*Proof.* Let $A$ be an arbitrary subspace of $X$. By Theorem 1, the space $A$ contains an open (in $A$) dense subspace $U \subset A$ such that the restriction $f\,|_A$ is continuous at every point of the set $U$. Using the scattered continuity of $g$, we find a continuity point $y \in f(U)$ of the map $g\,\big|_{f(U)}$. Take an arbitrary point $x \in f^{-1}(y) \cap U$ and notice that the composition $g \circ f$ is continuous at the point $x$. □

The following example shows that the regularity of the space $Y$ in the previous proposition is essential.

**Example 1.** Let $f\colon \mathbb{R} \to \mathbb{R}_{\mathbb{Q}}$ be the identity map from the real line equipped with the standard topology $\tau$ to the real line endowed with the topology generated by the subbase

$\tau \cup \{\mathbb{Q}\}$. Let $\chi_{\mathbb{Q}} \colon \mathbb{R} \to \{0; 1\}$ denote the characteristic function of the set $\mathbb{Q}$. It is easy to show that the maps $f \colon \mathbb{R} \to \mathbb{R}_{\mathbb{Q}}$ and $\chi_{\mathbb{Q}} \colon \mathbb{R} \to \{0; 1\}$ are scatteredly continuous while their composition $\chi_{\mathbb{Q}} \circ f \colon \mathbb{R} \to \{0; 1\}$ is everywhere discontinuous (and hence fails to be scatteredly continuous).

Recall that a space $X$ is called a *Preiss-Simon* space if for an arbitrary non-empty closed subset $A$ of $X$ and each point $x \in A$ there is a sequence $\{U_n \colon n \in \mathbb{N}\}$ of non-empty open subsets of $A$ that converges to $x$ in the sense that each neighborhood of $x$ contains all but finitely many sets $U_n$.

**Lemma 1.** *Let $f$ be a surjective map from a topological space $X$ onto a topological space $Y$ and $g$ be a map from the space $Y$ to some topological space $Z$. Then scattered continuity of the map $g \circ f$ implies the scattered continuity of the map $g$ if one of the following conditions is satisfied:*

1) *the map $f \colon X \to Y$ is open;*
2) *the map $f \colon X \to Y$ is closed, the space $X$ is perfectly paracompact, $Y$ is a hereditary Baire Preiss-Simon space and $Z$ is a regular space.*

*Proof.* 1. Assume that the map $\varphi = g \circ f$ is scatteredly continuous and let $f$ be an open map. To show that $g$ is scatteredly continuous, fix a non-empty subset $B \subset Y$ and let $A = f^{-1}(B)$. It follows that $f|_A \colon A \to B$ is an open map. Since the map $\varphi$ is scatteredly continuous, the restriction $\varphi|_A \colon A \to Z$ has a continuity point $x_0 \in A$. We show that the map $g|_B \colon B \to Z$ is continuous at the point $y_0 = f(x_0)$. Assume that $O(g(y_0))$ is a neighborhood of the point $g(y_0)$ in $Z$. Since $\varphi(x_0) = g(y_0)$, there is a neighborhood $O(x_0)$ of the point $x_0$ in the subspace $A$ such that $\varphi(O(x_0)) \subset O(g(y_0))$. Since the restriction $f|_A \colon A \to B$ is an open map, one has that $f(O(x_0))$ is a neighborhood of the point $y_0$. It is easy to deduce that $g(f(O(x_0))) = \varphi(O(x_0)) \subset O(g(y_0))$.

2. In [5], in particular, is proved that a map $g$ from a hereditary Baire Preiss-Simon space $Y$ to a regular space $Z$ is scatteredly continuous if for any open subset in $Z$ its preimage is a $G_\delta$-set in $Y$. Suppose $g$ is not a scatteredly continuous map. Then there is an open set $U$ in $Z$ such that $g^{-1}(U)$ is not $G_\delta$-set in $Y$.

On the other hand, since $g \circ f$ is a scatteredly continuous map from a perfectly paracompact space $X$ to a regular space $Z$, we obtain that $(g \circ f)^{-1}(U)$ is a $G_\delta$-set in $X$ (see [5]).

Put $A = (g \circ f)^{-1}(U) \subset X$. Then $f(A) = g^{-1}(U)$. Since $A$ is a $G_\delta$-set in $X$, we have that $X \backslash A$ is an $F_\sigma$-set in $X$, that is, $X \backslash A = \bigcup\{F_i \colon i \in \mathbb{N}\}$ where each $F_i$ is a closed subset in $X$. Then $f(X \backslash A) = \bigcup f(F_i)$ is an $F_\sigma$-set in $Y$. But then $Y \backslash f(X \backslash A) = g^{-1}(U)$ is a $G_\delta$-set in $Y$, which is a contradiction. □

**Example 2.** Assume that $f$ is a map from a scattered uncountable compact space $X$ to the segment $Y = [0, 1]$, and $g \colon [0, 1] \to \mathbb{R}$ is the characteristic function of the set $\mathbb{Q}$. The spaces $X$ and $Y$ are both compact. Obviously, the maps $g \circ f \colon X \to \mathbb{R}$ and $f$ are scatteredly continuous. But the characteristic function $g \colon [0, 1] \to \mathbb{R}$ is not scatteredly continuous.

Each $f \colon X \to Y$ induces a dual map $f^{\#} \colon \mathbb{R}^Y \to \mathbb{R}^X$ that assigns to each function $\varphi \in \mathbb{R}^Y$, the composition $f^{\#}(\varphi) = \varphi \circ f$.

**Proposition 3.** *Let $X$ and $Y$ be topological spaces and let $f$ be a mapping of $X$ to $Y$. Then the following statements are true:*

1. *The map* $f^{\#}\colon SC_p(Y) \to f^{\#}(SC_p(Y))$ *is continuous.*
2. *If the map* $f$ *is scatteredly continuous and the space* $Y$ *is regular, then* $f^{\#}(SC_p(Y)) \subset SC_p(X)$.
3. *Let* $f$ *be open scatteredly continuous surjective map and let* $Y$ *be regular. Then* $f^{\#}$ *is a homeomorphism of the space* $SC_p(Y)$ *onto the closed subspace* $f^{\#}(SC_p(Y))$ *of* $SC_p(X)$.
4. *Let* $f$ *be a closed scatteredly continuous surjective map, let* $X$ *be perfectly paracompact and let* $Y$ *be a hereditary Baire Preiss-Simon space. Then* $f^{\#}$ *is a homeomorphism of the space* $SC_p(Y)$ *onto the closed subspace* $f^{\#}(SC_p(Y))$ *of* $SC_p(X)$.
5. *If* $f$ *is scatteredly continuous then* $f$ *is injective if and only if* $f^{\#}(SC_p(Y))$ *is dense in* $SC_p(X)$.

*Proof.* 1. In [3], in particular, is proved that the map $f^{\#}\colon \mathbb{R}^Y \to \mathbb{R}^X$ is continuous for an arbitrary map $f\colon X \to Y$ and arbitrary sets $X$ and $Y$. Thus, the map $f^{\#}\colon SC_p(Y) \to f^{\#}(SC_p(Y))$ is continuous.

2. Let $f$ be a scatteredly continuous map. Take an arbitrary map $\varphi \in SC_p(Y)$. Since the maps $\varphi$ and $f$ are scatteredly continuous and $Y$ is a regular space, the composition $\varphi \circ f$ is scatteredly continuous map according to Proposition 2. Therefore $f^{\#}(SC_p(Y)) \subset SC_p(X)$.

3. Assume that $f$ is an open map and $f(X) = Y$. Since the map $f^{\#}\colon \mathbb{R}^Y \to \mathbb{R}^X$ is a homeomorphism of the space $\mathbb{R}^Y$ onto the closed subspace $f^{\#}(\mathbb{R}^Y)$ of $\mathbb{R}^X$ (see [3]), the map $f^{\#}\colon SC_p(Y) \to SC_p(X)$ is a homeomorphism of the space $SC_p(Y)$ onto the subspace $f^{\#}(SC_p(Y))$ of $SC_p(X)$.

We prove that $f^{\#}(SC_p(Y))$ is a closed subspace of the space $SC_p(X)$. Take any function $\psi \in SC_p(X)$ with $\psi \in \overline{f^{\#}(SC_p(Y))}$ and an arbitrary $y \in Y$. Obviously, each function $\varphi$ of $f^{\#}(SC_p(Y))$ is constant on $f^{-1}(y)$. Then the function $\psi$ is constant on $f^{-1}(y)$, as well. Therefore we can find a function $g\colon Y \to \mathbb{R}$ such that $\psi = g \circ f$, that is, $\psi = f^{\#}(g)$. Since $f$ is an open map, the map $\psi$ is scatteredly continuous and $\mathbb{R}$ is a regular space, then Lemma 1(1) guarantees that the map $g$ is scatteredly continuous. Hence $\psi \in f^{\#}(SC_p(Y))$, that is, the set $f^{\#}(SC_p(Y))$ is closed in $SC_p(X)$.

4. The proof of this statement is similar to (3), and can be proved using Lemma 1(2).

5. Assume that $f$ is scatteredly continuous and injective, $\psi \in SC_p(X)$ and $W(\psi, x_1, ..., x_k, \varepsilon)$ is an arbitrary standard neighborhood of the function $\psi$ in $SC_p(X)$. Put $y_i = f(x_i)$, $i = 1, ..., k$. Since the map $f$ is bijective, there is a function $\varphi \in SC_p(Y)$ such that $\varphi(y_i) = \psi(x_i)$, $i = 1, ..., k$. Obviously, $f^{\#}(\varphi) \in W(\psi, x_1, ..., x_k, \varepsilon)$, that is, the subspace $f^{\#}(SC_p(Y))$ is dense in the space $SC_p(X)$.

Now let map $f$ be scatteredly continuous and let $f^{\#}(SC_p(Y))$ be dense in $SC_p(X)$. We show that $f$ is bijective. Assume that $x_1 \neq x_2$, but $f(x_1) = f(x_2) = y$. Then for all $\varphi \in f^{\#}(SC_p(Y))$ we have $f^{\#}(\varphi)(x_1) = \varphi(f(x_2)) = \varphi(y) = \varphi(f(x_2)) = f^{\#}(\varphi)(x_2)$. Take a function $\psi \in SC_p(X)$ with $\psi(x_1) = 0$ and $\psi(x_2) = 1$. Obviously, $W(\psi, x_1, x_2, \frac{1}{2}) \cap f^{\#}SC_p(Y) = \varnothing$, which contradicts the fact that $f^{\#}(SC_p(Y))$ is dense in $SC_p(X)$. □

**4. Extent and normality of the spaces of scatteredly continuous maps.** Recall that two subsets $A$ and $B$ of a topological space $X$ are separated, if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

**Lemma 2** ([3]). *Let* $Y$ *be a dense subspace of the product* $X = \prod\{X_\alpha \colon \alpha \in A\}$ *of separable metrizable spaces* $X_\alpha$ *and* $P \subset Y, Q \subset Y$. *Then following conditions are equivalent:*

*a) there are open subsets $U$ and $V$ in $X$ such that $P \subset U$, $Q \subset V$ and $U \cap V = \varnothing$;*

*b) there is a countable set $M \subset A$ such that the sets $\pi_M(Q)$ and $\pi_M(P)$ are separated in $\pi_M(Y) \subset \pi_M(X) = \prod\{X_\alpha : \alpha \in M\}$.*

**Theorem 2.** *Let $SC_p(X)$ be a normal space and $Y \subset X$. Then so is the space $\pi_Y(SC_p(X))$.*

*Proof.* Assume that $P$ and $Q$ are closed disjoint sets in the space $\pi_Y(SC_p(X))$. The space $\pi_Y(SC_p(X))$ is dense in $\mathbb{R}^Y$. By Lemma 2 it is sufficient to find a countable subset $Z \subset Y$ such that the sets $\pi_Z(P)$ and $\pi_Z(Q)$ are separated in $\pi_Z(SC_p(X))$. Consider the sets $P' = \pi_Y^{-1}(P)$ and $Q' = \pi_Y^{-1}(Q)$. Since the space $SC_p(X)$ is normal, there is a countable set $Z' \subset X$ such that $\pi_{Z'}(P')$ and $\pi_{Z'}(Q')$ are separated in $\pi_{Z'}(SC_p(X))$. We show that, if we replace $Z'$ with the set $Z = Z' \cap Y$, then the sets $\pi_Z(P') = \pi_Z(P)$ and $\pi_Z(Q') = \pi_Z(Q)$ will be separated in $\pi_Z(SC_p(X))$. Suppose this is not true. Assume, for example, that $\pi_Z(P') \cap cl_T(\pi_Z(Q')) \neq \varnothing$ with $T = \pi_Z(SC_p(X))$. Choose $f \in P'$ so that $f|_Z \in cl_T(\pi_Z(Q'))$, and prove that $f|_{Z'} \in cl_{T'}(\pi_{Z'}(Q'))$ with $T' = \pi_{Z'}(SC_p(X))$. Take some finite set $K \subset Z'$ and $\varepsilon > 0$. Put $K_1 = K \cap Y$ and $K_2 = K \cap (X \backslash Y)$. Since $f|_Z \in cl_T(\pi_Z(Q))$, there is a map $g \in Q$, such that $|g(x) - f(x)| < \varepsilon$ for any $x \in K_1$. Then there is a map $g' \in Q'$ such that $\pi_Y(g') = g$. Fix a function $h'$ such that $h'(x) = f(x) - g'(x)$ for any $x \in K_2$ and $h'(x) = 0$ for all $x \in X \backslash K_2$. Obviously, $h' \in SC_p(X)$. Put $h = h' + g'$. Then $h|_Y = g$ (that is $h \in Q'$) and $|h(x) - f(x)| < \varepsilon$ for every $x \in K$. Since $K$ is an arbitrary finite set, $f|_{Z'} \in cl_{T'}(\pi_{Z'}(Q'))$. And we obtain that the sets $\pi_{Z'}(P')$ and $\pi_{Z'}(Q')$ are not separated, which is a contradiction. Hence, $\pi_Z(P)$ and $\pi_Z(Q)$ are separated in $\pi_Z(SC_p(X))$. □

**Proposition 4.** *Let $X$ be a topological space. If $SC_p(X)$ is a normal space, then every scattered subspace of the space $X$ is countable.*

*Proof.* Assume that $A$ is a scattered subspace of $X$. Then $\pi_A(SC_p(X)) = SC_p(A)$. The previous theorem implies the space $\pi_A(SC_p(X))$ is normal, and, therefore, the space $SC_p(A)$ is normal as well. If the subspace $A$ is scattered, then $SC_p(A) = \mathbb{R}^A$ . And since the space $\mathbb{R}^A$ is not normal with uncountable $A$, the set $A$ is countable. □

Recall that the Lindelöf number $l(X)$ of a space $X$ is the smallest cardinal number $m$ such that each open cover of $X$ has a subcover of size $\leqslant m$. Hereditary Lindelöf number $hl(X)$ of a space X is equal to $sup\{l(Y) : Y \subset X\}$.

**Corollary 1.** *If $SC_p(X)$ is a normal space, then $hl(X) \leq \aleph_0$.*

By the extent $e(X)$ of a topological space $X$ we understand the smallest infinite cardinal number $m$ such that the cardinality of each closed discrete subspace of the space $X$ does not exceed $m$.

We define a subset $A \subset X$ to be *sc*-embedded into a space $X$ if for any scatteredly continuous map $f : A \to \mathbb{R}$ there is a scatteredly continuous map $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f}|_A = f$. One can show that all undividing subsets of the topological space $X$ are *sc*-embedded into $X$ (see the proof of Proposition 1(3)).

**Lemma 3.** *If $Y$ is sc-embedded into a space $X$, then $e(SC_p(Y)) \leqslant e(SC_p(X))$.*

*Proof.* Let $Y$ be *sc*-embedded into $X$. Assume that $F_Y$ is a closed discrete subspace of $SC_p(Y)$. Consider the restriction operator $\pi_Y : SC_p(X) \to SC_p(Y)$. Due to Proposition 1(1) this operator is continuous. For each $g \in F_Y$ fix an element $f_g \in SC_p(X)$ such that $f_g \in$

$\pi_Y^{-1}(g)$. Put $F_X = \{f_g \colon g \in F_Y\}$. Since the restriction $\pi_Y|_{F_X}$ is bijective and continuous, the subspace $F_X$ is discrete in $SC_p(X)$. Let us prove that $F_X$ is a closed subspace of $SC_p(X)$. Suppose this is not true. Take some function $f^* \in \overline{F_X} \backslash F_X$. Since $F_Y$ is a closed subspace of $SC_p(Y)$ and $\pi_Y$ is a continuous map, $\pi_Y^{-1}(F_Y)$ is a closed subspace of $SC_p(X)$. Thus $f^* \in \pi_Y^{-1}(F_Y)$, that is, $f^* \in \pi_Y^{-1}(g)$ for some $g \in F_Y$. Since $\{g\}$ is an isolated point in the subspace $F_Y$ and $\pi_Y$ is continuous, $\pi_Y^{-1}(g)$ is a neighborhood of the point $f^*$ in the subspace $\pi_Y^{-1}(F_Y)$. Put $\{f_g\} \in \pi_Y^{-1}(g) \cap F_X$. Then $(\pi_Y^{-1}(g) \backslash \{f_g\}) \cap F_X = \varnothing$. And this contradicts the fact that $f^* \in \overline{F_X}$. □

**Proposition 5.** *If a space $X$ contains a non-scattered compact, then $e(SC_p(X)) \geqslant 2^{\aleph_0}$.*

*Proof.* Assume that $K$ is a compact subspace of $X$. Then there is a continuous map $\varphi$ from the space $K$ onto a metrizable compact $Y$. Let $\rho$ be a metric on $Y$ and let $B_i(y) = \{t \colon \rho(y,t) \leqslant \frac{1}{i}\}$. For each $y \in Y$ we define the map $f_y \colon K \to \mathbb{R}$ as follows:

$$f_y(x) = \begin{cases} 0, & \text{if } x \in \varphi^{-1}(y); \\ \min\{i \colon x \notin \varphi^{-1}(B_i(y))\}, & \text{otherwise.} \end{cases}$$

Put $\mathcal{F} = \{f_y \colon y \in Y\}$. We show that $\mathcal{F}$ is a closed discrete subspace of $SC_p(K)$.

Firstly, we prove that $\mathcal{F} \subset SC_p(K)$, that is, each map $f_y$ is scatteredly continuous. Consider an arbitrary map $f_y \in \mathcal{F}$ and an arbitrary subset $A \subset K$. If $A \subset \varphi^{-1}(y)$, then $f_y(\varphi^{-1}(y)) = 0$ and the function $f_y$ is continuous at every point of $A$. Let $A \not\subseteq \varphi^{-1}(y)$. Put $i_0 = \min\{i \colon f_y(A) = i\}$. Then there is a point $x_0 \in A$ such that $f_y(x_0) = i_0$. And by the definition of the function $f_y$ this means that $x_0 \notin \varphi^{-1}(B_{i_0}(y))$. Then $x_0 \in A \backslash \varphi^{-1}(B_{i_0}(y))$, the subset $A \backslash \varphi^{-1}(B_{i_0}(y))$ is an open subset of $A$ and $f_y(A \backslash \varphi^{-1}(B_{i_0}(y))) = i_0$. Thus, the function $f_y \in \mathcal{F}$ is continuous at the point $x_0$. And, therefore, $f_y$ is scatteredly continuous for all $y \in Y$.

We prove that $\mathcal{F}$ is a closed subspace of $SC_p(K)$. Assume that we have a function $g \in \overline{\mathcal{F}} \backslash \mathcal{F}$.

Fix a base $\mathcal{U} = \{U_\alpha \colon \alpha \in A\}$ of the function $g$ in the space $SC_p(K)$. Put $P_\alpha = \{x \in K \colon f_{\varphi(x)} \in U_\alpha\}$.

The family $\{P_\alpha \colon \alpha \in A\}$ has the finite intersection property, that is, $P_{\alpha_1} \cap P_{\alpha_2} \cap \ldots \cap P_{\alpha_n} \neq \varnothing$ for every finite system $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Since $K$ is a compact space, we have that $\bigcap_{\alpha \in A} \overline{P_\alpha} \neq \varnothing$.

Fix a point $z \in \bigcap_{\alpha \in A} \overline{P_\alpha}$. Take a standard neighborhood $W(g, z, \frac{1}{2}) = \{f \in SC_p(K) \colon |g(z) - f(z)| < \frac{1}{2}\}$ of the function $g$ in a space $SC_p(K)$. If $g(z) = 0$, then $(W(g, z, \frac{1}{2}) \backslash f_{\varphi(z)}) \cap \mathcal{F} = \varnothing$. And this contradicts the fact that $g \in \overline{\mathcal{F}}$. Hence, $g(z) \neq 0$. Since for all $x \in K$ and each $f_y \in \mathcal{F}$ we have that $f_y(x) \in \mathbb{N} \cup \{0\}$, there is $i \in \mathbb{N}$ such that $g(z) = i$. Fix an element of the base $U_{\alpha_0} \in \mathcal{U}$ such that $U_{\alpha_0} \subset W(g, z, \frac{1}{2})$. Since $z \in \bigcap \overline{P_\alpha}$, we obtain that $z \in \overline{P_{\alpha_0}}$. And since $g(z) = i$, for any $x \in K$ such that functions $f_{\varphi(x)}$ lie in the neighborhood $U_{\alpha_0}$, we have $f_{\varphi(x)}(z) = i$ and $z \notin \varphi^{-1}(B_i(\varphi(x)))$. Then for each $x \in P_{\alpha_0}$ we have $z \notin \varphi^{-1}(B_i(\varphi(x)))$, that is, $\rho(\varphi(x), \varphi(z)) > \frac{1}{i}$. Obviously, point $z \in \operatorname{Int}(\varphi^{-1}(B_i(\varphi(z))))$, but $\operatorname{Int}(\varphi^{-1}(B_i(\varphi(z)))) \cap P_{\alpha_0} = \varnothing$. And this contradicts the fact that $z \in \overline{P_{\alpha_0}}$.

Therefore, the subspace $\mathcal{F}$ is closed in $SC_p(K)$.

We prove that $\mathcal{F}$ is a discrete subspace of $SC_p(K)$. Take any function $f_y$ of $\mathcal{F}$. Fix a point $x \in \varphi^{-1}(y)$ and a standard neighborhood $W(f_y, x, \frac{1}{2})$ of the function $f_y$ in the space $SC_p(K)$. Then $x \notin \varphi^{-1}(y')$ for each $y' \in Y$ such that $y' \neq y$. By the definition of the function $f_{y'}$ one has that $f_{y'}(x) \geqslant 1$. Thus, $W(f_y, x, \frac{1}{2}) \cap \mathcal{F} = \{f_y\}$. Therefore, $\mathcal{F}$ is a discrete subspace of the space $SC_p(K)$.

Since $K$ is a compact subspace of $X$, one has that $K$ is sc-embedded into $X$. Hence, by Lemma 3, $e(SC_p(X)) \geqslant e(SC_p(K)) \geqslant 2^{\aleph_0}$. □

**Theorem 3** ([3])**.** *Let $X$ be a normal space with countable Souslin number and $\chi(X) \leqslant 2^{\aleph_0}$, that is, the space $X$ has a base of cardinality $\leqslant 2^{\aleph_0}$ at every point. Then $e(X) < 2^{\aleph_0}$.*

A space $X$ is called $k$-scattered ([6], [7]) if for an arbitrary non-empty subset $F \subset X$ there is a non-empty open subset $U$ of $X$ such that $U \cap F \neq \varnothing$ and $\overline{U \cap F}$ is compact.

**Theorem 4.** *If $SC_p(X)$ is a normal space, then every k-scattered subspace of $X$ is countable.*

*Proof.* Assume that $SC_p(X)$ is a normal space. We show that all compact subspaces of $X$ are scattered. Suppose there is a non-scattered compact $K$ of $X$. In view of Theorem 2, normality of the space $SC_p(X)$ implies normality of the space $\pi_K(SC_p(X))$. Since $K$ is a closed set, applying the Proposition 1(3), we obtain that $\pi_K(SC_p(X)) = SC_p(K)$. Since $K$ is an uncountable compact, there is a continuous map $\varphi$ from $K$ onto the segment $I = [0, 1]$. Since $SC_p(K)$ is a normal space, then by Corollary 1, we have that $hl(K) \leqslant \aleph_0$, that is, the compact $K$ is a hereditary Lindelöf space. And, therefore, the space $K$ is perfectly paracompact. Then, according to Proposition 3(4), the map $\varphi^{\#}\colon SC_p(I) \to SC_p(K)$ is a homeomorphism of the space $SC_p(I)$ onto the closed subspace $\varphi^{\#}(SC_p(I))$ of $SC_p(K)$. Since the space $SC_p(K)$ is normal, its closed subspace $\varphi^{\#}(SC_p(I))$ is normal as well. And since the map $\varphi^{\#}\colon SC_p(I) \to \varphi^{\#}(SC_p(I))$ is a homeomorphism, the space $SC_p(I)$ is normal. The space $SC_p(I)$ is a normal space with countable Souslin number and $\chi(SC_p(I)) \leqslant 2^{\aleph_0}$. Then, by Theorem 3, $e(SC_p(I)) < 2^{\aleph_0}$. But, applying Proposition 5, we have that $e(SC_p(I)) \geqslant 2^{\aleph_0}$. This contradiction proves that all compact subspaces of $X$ are scattered.

Let $Y$ be a non-empty k-scattered subspace of $X$ and let $A$ be an arbitrary non-empty subset of $Y$. Since $Y$ is k-scattered, there is an open subset $U$ of $Y$ such that $U \cap A \neq \varnothing$ and $\overline{U \cap A}$ is a compact subspace of $Y$. Since every compact subspace of $Y$ is compact in $X$, we have that $K = \overline{U \cap A}$ is a compact subspace of $X$. Thus, $K$ is scattered. Then there are a point $x \in K$ and its neighborhood $O(x)$ such that $O(x) \cap K = \{x\}$. And since $x \in \overline{U \cap A}$, we obtain $O(x) \cap U \cap A \neq \varnothing$. The fact $O(x) \cap U \cap A \subset O(x) \cap \overline{U \cap A} = \{x\}$ implies that $(O(x) \cap U) \cap A = \{x\}$. Therefore, the point $x$ is an isolated point in $A$. Hence, the space $Y$ is scattered. And, by Proposition 4, space $Y$ is countable. □

**Corollary 2.** *If $X$ is a k-scattered space, then the space $SC_p(X)$ is normal if and only if $X$ is countable.*

Recall that a space $X$ is said to be a k-space if a set $F \subseteq X$ is closed if and only if for each compact subset $K \subseteq X$ the set $F \cap K$ is a compact in $K$.

**Corollary 3.** *If $X$ is a k-space and $SC_p(X)$ is a normal space, then $X$ is a sequential space.*

*Proof.* Let $A$ be a non-closed subset of $X$. Then there is a compact subspace $K \subset X$ such that $A \cap K$ is a non-closed subset in $K$. Due to Theorem 4, compact $K$ is countable, and therefore, $K$ is metrizable. Thus, for each point $x \in \overline{A \cap K} \setminus (A \cap K)$ there is a sequence from $A \cap K$ that converges to $x$. □

A space $X$ is called $\sigma$-compact if it is a countable union of its compact subspaces.

**Corollary 4.** *If $X$ is a $\sigma$-compact space, then $SC_p(X)$ is normal if and only if $X$ is countable.*

**Question 1.** Is there an uncountable space $X$ such that $SC_p(X)$ is normal?

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