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ALMOST ANTIPROXIMINAL SETS IN L_1

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We introduce a notion of almost antiproximinality of sets in the space L_1 which is a weakening of the notion of antiproximinality. Also we investigate properties of almost antiproximinal sets and establish a method of construction of almost antiproximinal sets.

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В пространстве L_1 вводится понятие почти антипроксиминальности множеств, которое является ослаблением понятия антипроксиминальности. Исследуются свойства почти антипроксиминальных множеств и устанавливается метод построения почти антипроксиминальных множеств.

1. Introduction. By $d(x, M)$ we denote the distance $\inf\{\|x - y\| \colon y \in M\}$ between an element x of a normed space X and a non-empty set $M \subseteq X$. An element $y \in M$ is called the *nearest point* to x if $\|x - y\| = d(x, M)$. The set of all nearest points to a point x in a set M is denoted by $P_M(x)$.

A set M is called an *antiproximinal (AP) set* if $P_M(x) = \varnothing$ for each $x \in X \setminus M$.

Let X^* be the Banach space conjugate to X. A functional $f \in X^*$ *attains supremum* on $M \subseteq X$ if there exists an element $x \in M$ such that $f(x) = \sup f(M)$. By $\Sigma(M)$ we denote the set of all functionals which attain supremum on M, i.e.

$$\Sigma(M) = \{f \in X^* \colon \exists x \in M \mid f(x) = \sup f(M)\}.$$

Let X be a normed space, $x_0 \in X \setminus \{0\}$. A functional $f_0 \in X^*$ is called a *support functional* in x_0 if $\|f_0\| = 1$ and $f_0(x_0) = \|x_0\|$.

M. Edelstein and A. Thompson in [1] showed that a bounded closed convex subset A of Banach space X is AP if and only if each non-zero support functional of the set A does not attain maximum on the closed unit ball B of X, i.e. $\Sigma(A) \cap \Sigma(B) = \{0\}$.

In 1961 E. Bishop and P. Phelps proved that each Banach space is subreflexive, i.e. the set of functionals which attain their maximum at unit ball is dense in this space [2], and in 1963 they generalized this result to the case of convex sets (see [3]).

Plenty of mathematicians have worked on the problem of the existence of non-empty closed convex bounded AP-sets in Banach spaces. In particular, it is proved in [1]–[7] that such sets exist in the spaces $c_0, c, L_\infty, C(X)$ (with some especial conditions on X).

Besides, V. Klee showed in [4] that a Banach space X contains a non-empty closed convex (not obligatory bounded) AP-set if and only if X is not reflexive.

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In connection with these facts, the following question on the existence of non-empty closed convex bounded AP-sets in L_1 was naturally risen by professor V. Fonf (see [12]).

Question. *Do there exist non-empty closed convex bounded AP-sets in L_1?*

Note that the example of radially bounded (i.e. bounded in each direction) AP-set M in L_1 was constructed in [7], but this set M is not bounded.

In this paper we introduce some weakening of the notion of antiproximinality which we call *almost antiproximinality*, and investigate properties of closed convex bounded almost AP-sets in L_1.

2. Definition of almost AP-set in L_1, examples. By L_1 and L_∞ we denote the spaces $L_1[0,1]$ and $L_\infty[0,1]$ respectively. The space L_∞ we identify with the space L_1^* and denote $y(x) = \langle x, y\rangle = \int_0^1 y(t)x(t)d\mu$. Furthermore, for a set $A \subseteq L_1$, the sets $\{y \in L_\infty \colon \langle x, y\rangle \leq 1\ \forall x \in A\}$ and $\{y \in L_\infty \colon |\langle x, y\rangle| \leq 1\ \forall x \in A\}$ are called the *polar* and the *absolute polar* of the set A respectively. The polar and the absolute polar of a set $B \subseteq L_\infty$ are introduced similarly.

It is easily seen that the set $\Sigma(B)$ of all support functionals at the unit ball B of L_1 consists of all $y \in L_\infty$ such that $\mu(\{t \in [0,1] \colon |y(t)| = \|y\|\}) > 0$. The most obvious examples of non-zero functionals $y \in \Sigma(B)$ are so-called *signs*, i.e. such $y \in L_\infty$ that $|y| = \chi_T$, where χ_T is the characteristic function of a measurable set $T \subseteq [0,1]$ having positive measure.

According to the characterization of AP-sets obtained by M. Edelstein and A. Thompson, the next notion is a natural weakening of the conception of antiproximinality in L_1.

Definition 1. A set $A \subseteq L_1$ is called *almost antiproximinal (almost AP)* if for each set $T \subseteq [0,1]$ of positive measure any functional $y \in L_\infty$ such that $|y| = \chi_T$ does not attain supremum on A.

Suppose that a set $A \subseteq L_1$ has the following property: $x \in A$ if and only if $|x| \in A$. Then almost antiproximinality of A is equivalent to the fact that $\chi_T \notin \Sigma(A)$ for each measurable set $T \subseteq [0,1]$ having positive measure. (Indeed, if a sign attains supremum on a set $T \subseteq [0,1]$ at some point $x \in A$, then χ_T attains supremum at $|x| \in A$).

We denote $A_y = \{x \in L_1 \colon \int_0^1 |xy| d\mu \leq 1\}$ for any function $y \in L_\infty$.

Definition 2. A measurable function $y \colon [0,1] \to \mathbb{R}$ is called *rearrange monotone* if for each $\alpha \in \mathbb{R}$ the set $\{t \in [0,1] \colon y(t) = \alpha\}$ has measure zero.

Proposition 1. *Let $y \in L_\infty$ be rearrange monotone. Then the set $M = A_y$ is an almost AP-set, but is not an AP-set.*

Proof. Let $T \subseteq [0,1]$ and $\mu(T) > 0$. Denote $m = \sup\limits_{\mu(F)=0} \inf\limits_{t \in T\setminus F} |y(t)|$.

We will prove that $\alpha = \sup_{x\in M} \int_0^1 |x|\chi_T d\mu \geq \frac{1}{m}$.

Fix $\varepsilon > 0$. The set $S = \{t \in T \colon m \leq |y(t)| \leq m + \varepsilon\}$ has non-zero measure. Consider the function $x = \frac{1}{(m+\varepsilon)\mu(S)} \cdot \chi_S$. Since

$$\int_0^1 |xy| d\mu \leq \frac{1}{(m+\varepsilon)\mu(S)}(m+\varepsilon)\int_0^1 \chi_S d\mu = 1,$$

one has that $x \in M$. So, $\alpha \geq \int_0^1 x d\mu = \frac{1}{(m+\varepsilon)}$. Tending ε to zero, we obtain that $\alpha \geq \frac{1}{m}$, in particular $\alpha = +\infty$ if $m = 0$.

It remains to show that $\int_0^1 |x|\chi_T d\mu < \frac{1}{m}$ for each $x \in M$ (so it will be proved that $\chi_T \notin \Sigma(M)$).

Fix $x \in M$ and put $T_1 = \{t \in T\colon |x(t)| > 0\}$. If $\mu(T_1) = 0$ then $\int_0^1 |x|\chi_T d\mu = 0 < \frac{1}{m}$. Suppose now $\mu(T_1) > 0$. Since the set $\{t \in [0,1]\colon |y(t)| = m\}$ has zero measure, without loss of generality we can assume that $|y(t)| > m$ for each $t \in T_1$. Then $m|x(t)| < |x(t)y(t)|$ for each $t \in T_1$ and

$$m\int_0^1 |x|\chi_T d\mu = m\int_{T_1} |x|\chi_T d\mu < \int_{T_1} |xy| d\mu \le \int_0^1 |xy| d\mu \le 1.$$

So, $\int_0^1 |x|\chi_T d\mu < \frac{1}{m}$ and M is an almost AP-set.

Now we will prove that M is not an AP-set. Assume $a = \operatorname{essinf}|y|$, $c = \operatorname{esssup}|y|$ and $b = \frac{a+c}{2}$. Since y is rearrange monotone, $a < c$ and the sets $S = \{t \in [0,1]\colon |y(t)| \in (a,b)\}$ and $T = \{t \in [0,1]\colon |y(t)| \in (b,c)\}$ have measure greater than zero.

Consider the functions $y_0\colon [0,1] \to \mathbb{R}$ and $x_0\colon [0,1] \to \mathbb{R}$

$$y_0(t) = \begin{cases} |y(t)|, & t \in S,\\ b, & t \in T,\\ 0, & t \notin T \cup S, \end{cases} \qquad x_0(t) = \begin{cases} \frac{1}{|y(t)|\mu(S)}, & t \in S,\\ 0, & t \notin S. \end{cases}$$

Since $|y_0| \le |y|$, we have that $\int_0^1 xy_0 d\mu \le 1$ for each $x \in A$.

On the other hand, $\int_0^1 |x_0 y| d\mu = \int_S |x_0 y| d\mu = 1 = \int_0^1 |x_0 y_0| d\mu$. So $y_0 \in \Sigma(M) \cap \Sigma(B_1)$ which means that M is not an AP-set. □

Proposition 2. *Intersection of two almost AP-sets in L_1 need not be an AP-set.*

Proof. Let $y_1(t) = 1 + t$, $y_2(t) = 2 - t$,

$$A_1 = \Big\{x \in L_1\colon \int_0^1 |x|y_1 d\mu \le 1\Big\},\ A_2 = \Big\{x \in L_1\colon \int_0^1 |x|y_2 d\mu \le 1\Big\}$$

and $A = A_1 \cap A_2$. By Proposition 1, the sets A_1 and A_2 are almost AP-sets.

Consider the functions $y_0 = \chi_{[0,1]}$ and $x_0 = \frac{2}{3}\chi_{[0,1]} \in A$. Note that $y_0 = \frac{1}{3}(y_1 + y_2)$. Now for each $x \in A$ we have

$$\int_0^1 xy_0 d\mu = \frac{1}{3}\Big(\int_0^1 xy_1 d\mu + \int_0^1 xy_2 d\mu\Big) \le \frac{2}{3} = \int_0^1 x_0 y_0 d\mu.$$

Thus, $y_0 \in \Sigma(A)$. □

A set A of measurable functions on $[0,1]$ is called *solid* if for any measurable functions x_1 and x_2 the condition $|x_1| \le |x_2| \in A$ implies that $x_1 \in A$.

Proposition 3. *Let $M \subseteq L_1$ be a non-empty closed absolute convex bounded solid set and $B = \Sigma(M) \cap M^o \cap L_\infty^+$, where M^o is the absolute polar of the set M. Then $M = \cap_{y \in B} A_y$.*

Proof. Note that the absolute polar M^o and the set $\Sigma(M)$ are solid and $A_{y_1} = A_{y_2}$ if $|y_1| = |y_2|$. So, $\cap_{y \in B} A_y = \cap_{y \in C} A_y$, where $C = \Sigma(M) \cap M^o$.

According to [3], the set $\Sigma(M)$ is norm dense in L_∞. We show that C is dense in M^o. Assume to the contrary that there exist $\varepsilon > 0$, $y \in M^o$, $(y_n)_{n=1}^\infty$, $y_n \in \Sigma(M)$, $(x_n)_{n=1}^\infty$, $x_n \in M$, such that $\lim\limits_{n\to\infty} y_n = y$ and $|y_n(x_n)| > 1 + \varepsilon$ for any $n \in \mathbb{N}$. Using boundedness of M, we choose $n \in \mathbb{N}$ such that $\|(y_n - y)(x_n)\| < \frac{\varepsilon}{2}$. Then $|y(x_n)| > 1 + \frac{\varepsilon}{2}$, which is impossible.

Moreover, C is solid, so $\cap_{y \in B} A_y = \cap_{y \in C} A_y = C^o = (C^{oo})^o = M^{oo} = M$. □

In connection with Propositions 2 and 3 the following question naturally arises.

Question. *For which sets $B \subseteq L_\infty$ the set $M = \cap_{y\in B} A_y$ is an almost AP-set?*

3. Positive polars in L_1 and L_∞, and their properties. Let

$$X = L_1^+ = \{x \in L_1\colon x(t) \geq 0 \text{ almost everywhere (a.e.) on } [0,1]\}$$

and $Y = \{y \in L_\infty^+\colon y(t) \geq 0 \text{ a.e. on } [0,1]\}$. Denote by $\sigma(X,Y)$ the weakest topology on X such that for each $y \in Y$ the function $f_y\colon X \to \mathbb{R}$, $f_y(x) = \int_0^1 xy d\mu$, is continuous. Similarly, by $\sigma(Y,X)$ we denote the weakest topology on Y such that for each $x \in X$ the function $f^x\colon Y \to \mathbb{R}$, $f^x(y) = \int_0^1 xyd\mu$, is continuous. We will shortly denote by σ the topologies $\sigma(X,Y)$ and $\sigma(Y,X)$.

Proposition 4. *The topology σ coincides with the restriction of the weak topology w of L_1 on X.*

Proof. Obviously, the restriction of w on X is stronger than σ. So, it remains to prove that for any $x_0 \in X$ and weak neighborhood U of the point x_0 in L_1 the set $U \cap X$ is a σ-neighborhood of the point x_0 in X. It is sufficient to consider the case $U = \{x \in L_1\colon |\langle x - x_0, y\rangle| \leq 1\}$, where $y \in L_\infty$ is fixed.

We denote $A = \{t \in [0,1]\colon y(t) \geq 0\}$ and $B = \{t \in [0,1]\colon y(t) < 0\}$. Put $y_1 = y\chi_{_A}$, $y_2 = -y\chi_{_B}$, $U_1 = \{x \in X\colon |\langle x - x_0, y_1\rangle| \leq \frac{1}{2}\}$ and $U_2 = \{x \in X\colon |\langle x - x_0, y_2\rangle| \leq \frac{1}{2}\}$. It is clear that $y_1, y_2 \in Y$, U_1, U_2 are σ-neighborhoods of $x_0 \in X$, moreover $U_1 \cap U_2 \subseteq U \cap X$. So, $U \cap X$ is a σ-neighborhood of the point x_0. □

The following proposition can be proved similarly.

Proposition 5. *Topology σ coincides with the restriction of the weak* topology w^* of L_∞ on Y.*

Given non-empty sets $A \subseteq X$ and $B \subseteq Y$, the sets

$$\pi(A) = \{y \in Y\colon \langle x, y\rangle \leq 1 \ \forall x \in A\} \text{ and } \pi(B) = \{x \in X\colon \langle x, y\rangle \leq 1 \ \forall y \in B\}$$

are called the *positive polars* of the sets A and B respectively.

For any $y_1, y_2 \in Y$ we denote $[y_1, y_2] = \{y \in Y\colon y_1 \leq y \leq y_2\}$.

Proposition 6. *Let $A \subseteq X$ be a neighborhood of zero in X. Then $\pi(A)$ is σ-compact in Y, in particular, for each $y \in Y$ the set $[0,y]$ is σ-compact.*

Proof. Note that the set $\tilde{A} = \{x \in L_1\colon |x| \in A\}$ is a neighborhood of zero in L_1. Let $B = \pi(A)$ and $\tilde{B} = \tilde{A}^o$ be the absolute polar of $\tilde{A}$ with respect to the duality $\langle L_1, L_\infty\rangle$.

Now we show that $y \in \tilde{B}$ if and only if $|y| \in B$.

Let $y \in \tilde{B}$ and $x \in A$ be arbitrary elements. Consider the element $x' \in L_1$, which is defined in the following way:

$$x'(t) = \begin{cases} x(t), & y(t) \geq 0, \\ -x(t), & y(t) < 0. \end{cases}$$

Obviously, $x' \in \tilde{A}$ and $x'(t)y(t) = x(t)|y(t)|$ on $[0,1]$. Then $\left|\int_0^1 x|y|d\mu\right| = \left|\int_0^1 x'yd\mu\right| \leq 1$. So, $|y| \in B$.

Conversely, let $y \in L_\infty$ be such that $|y| \in B$ and let $x \in \tilde{A}$ be an arbitrary element. Then $|x| \le x'$ and $\left|\int_0^1 xyd\mu\right| \le \left|\int_0^1 |x||y|d\mu\right| \le 1$. Hence, $y \in \tilde{B}$ if and only if $|y| \in B$, in particular, $B = \tilde{B} \cap Y$. According to the Alaouglu-Burbaki theorem [8, p.117], the set $\tilde{B}$ is w^*-compact. Now using Proposition 3, the equality $B = \tilde{B} \cap Y$ and w^*-closeness of Y in L_∞, we obtain that B is σ-compact in Y. □

Proposition 7. *For any non-empty set $B \subseteq Y$ the positive bipolar $\pi(\pi(B))$ is the σ-closed convex hull of the set $\cup_{b\in B}[0, b]$.*

Proof. Denote $D = \mathrm{co}(\cup_{b\in B}[0, b])$ and $C = \bar{D}^\sigma$. Obviously, $C \subseteq \pi(\pi(B))$. Besides, $B \subseteq D$, therefore $\pi(\pi(B)) \subseteq \pi(\pi(D))$. Now it is sufficient to prove that $\pi(\pi(D)) = C$.

We set $D^o = \{x \in L_1 \colon \langle x, y\rangle \le 1 \quad \forall y \in D\}$. Note that for any $y \in D$ and a measurable set $A \subseteq [0, 1]$ we have $y \cdot \chi_A \in D$.

For any $x \in L_1$ denote $x^+ = x \cdot \chi_A$, where $A = \{t \in [0, 1] \colon x(t) > 0\}$. Now we show that $x \in D^o$ if and only if $x^+ \in D^o$, i.e. $x^+ \in \pi(\pi(D))$. Note that $\int_0^1 xyd\mu \le \int_0^1 x^+yd\mu$ for each $y \in Y$, therefore $x \in D^o$ if $x^+ \in D^o$.

Let $x \in D^o$, $y \in D$ and $A = \{t \in [0, 1] \colon x(t) > 0\}$. Then $y \cdot \chi_A \in D$ and $\int_0^1 x^+yd\mu = \int_0^1 xy\chi_A d\mu \le 1$. Thus, $x^+ \in D^o$.

Consider the set $D^{oo} = \{y \in L_\infty \colon \langle x, y\rangle \le 1 \quad \forall x \in D^o\}$. We prove that $D^{oo} = \pi(\pi(D))$.

First we will show that $D^{oo} \subseteq Y$. Assume $y \in L_\infty$, $A = \{t \in [0, 1] \colon y(t) < 0\}$ and $\mu(A) > 0$. Choose $x \in X$ such that $\{t \in [0, 1] \colon x(t) > 0\} \subseteq A$ and $\int_0^1 xyd\mu < -1$. Then $z = -x \in D^o$ as $z^+ = 0 \in \pi(D)$, and $\int_0^1 zyd\mu = -\int_0^1 xyd\mu > 1$. Thus, $y \notin D^{oo}$.

Since for each $y \in Y$, we have $\int_0^1 xyd\mu \le \int_0^1 x^+yd\mu$,

$$D^{oo} = \left\{y \in Y \colon \int_0^1 xyd\mu \le 1 \ \forall x \in D^o\right\} = \left\{y \in Y \colon \int_0^1 x^+yd\mu \le 1 \ \forall x \in D^o\right\} =$$
$$= \left\{y \in Y \colon \int_0^1 xyd\mu \le 1 \ \forall x \in \pi(D)\right\} = \pi(\pi(D)).$$

Now by the bipolar theorem [9, p.160] and by Proposition 5 we have $\pi(\pi(D)) = D^{oo} = \bar{D}^{w^*} = \bar{D}^\sigma = C$. □

Proposition 8. *Let $B = B_1 \cup B_2 \subseteq Y$, and let B be norm bounded. Then*

$$\overline{\mathrm{co}(\bigcup_{b\in B}[0, b])} = \mathrm{co}\left(\overline{\mathrm{co}(\bigcup_{b\in B_1}[0, b])} \cup \overline{\mathrm{co}(\bigcup_{b\in B_2}[0, b])}\right),$$

where closures are taken in the σ-topology.

Proof. Denote $A_1 = \mathrm{co}(\cup_{b\in B_1}[0, b])$, $A_2 = \mathrm{co}(\cup_{b\in B_2}[0, b])$ and $A = \mathrm{co}(\cup_{b\in B}[0, b])$. Obviously, $\mathrm{co}(\overline{A_1} \cup \overline{A_2}) \subseteq \overline{A}$.

Now we show that $\overline{A} \subseteq \mathrm{co}(\overline{A_1} \cup \overline{A_2})$. Note that $A \subseteq \mathrm{co}(A_1 \cup A_2)$. Therefore, it is sufficient to prove that the set $\mathrm{co}(\overline{A_1} \cup \overline{A_2})$ is closed.

By Proposition 7, we have $\overline{A_1} = \pi(\pi(B_1))$ and $\overline{A_2} = \pi(\pi(B_2))$. Moreover, the norm boundedness of the sets B_1 and B_2 together with Proposition 6 imply that the sets $\overline{A_1}$ and $\overline{A_2}$ are σ-compact.

Consider the following continuous mapping $\varphi\colon [0, 1]^2 \times Y^2 \to Y$, $\varphi(\lambda, \mu, y_1, y_2) = \lambda y_1 + \mu y_2$. The set $S = \{(\lambda, \mu) \in [0, 1]^2 \colon \lambda + \mu = 1\}$ is compact in $[0, 1]^2$. Hence, the set $\mathrm{co}(\overline{A_1} \cup \overline{A_2}) = \{\lambda x_1 + \mu x_2 \colon x_1 \in A_1, x_2 \in A_2\} = \varphi(S \times \overline{A_1} \times \overline{A_2})$ is compact as the continuous image of a compact set. Then $\mathrm{co}(\overline{A_1} \cup \overline{A_2})$ is closed. □

Proposition 9. *Let measurable functions x, y, z be such that $y \leq z$ and $\int_0^1 xzd\mu \leq \int_0^1 xyd\mu$. Then $y = z$ a.e. at $T = \{t \in [0,1] \colon x(t) > 0\}$.*

4. Support functionals at X and Y. For each set $A \subseteq X = L_1^+$ by $\Sigma(A)$ we denote the set of all $y \in Y = L_\infty^+$ such that $\max_{x\in A}\langle x, y\rangle$ exists. By $\Sigma_0(A)$ we denote the set of all $y \in Y = L_\infty^+$ such that $\max_{x\in A}\langle x, y\rangle = 1$, and by $\Sigma_{\max}(A)$ denote the set of all maximal elements of $\Sigma_0(A)$. Given any set $B \subseteq Y$, by $\Sigma_0(B)$ we denote the set of all $x \in X = L_1^+$ such that $\max_{y\in B}\langle x, y\rangle = 1$, and by $\Sigma_{\max}(B)$ denote the set of all maximal elements of $\Sigma_0(B)$.

Proposition 10. *Let $A \subseteq X$ be such that $\Sigma_0(A)$ is norm bounded in L_1. Then for each $y \in \Sigma_0(A)$ there exists $y' \in \Sigma_{\max}(A)$ such that $y \leq y'$.*

Proof. Suppose the contrary. Then by the Levi theorem [10, p.299], we can construct a strictly increasing transfinite sequence $(y_\xi)\colon \xi < \omega_1$ of functions $y_\xi \in \Sigma_0(A)$, where ω_1 is the first uncountable ordinal. Then $\int_0^1 y_\xi d\mu < \int_0^1 y_\eta d\mu$ for $1 \leq \xi < \eta < \omega_1$ and the transfinite sequence of numbers $a_\xi = \int_0^1 y_\xi d\mu$ is strictly increasing, which is impossible. □

Proposition 11. *Let $M \subseteq L_1$ be a closed absolute convex bounded solid neighborhood of zero, $A = M \cap X$ and $D = \Sigma_{\max}(A)$. Then $M = \cap_{y\in D} A_y$.*

Proof. By Proposition 3, we have $M = \cap_{y\in C} A_y$, where $C = \Sigma(A) \cap \pi(A)$. Obviously, $D \subseteq C$. Note that for each $y \in C$ there exists $y' \in \Sigma_0(A)$ such that $y \leq y'$. Since $B = \Sigma_0(A) \subseteq \pi(A)$ and A is neighborhood of zero in X, $\Sigma_0(A)$ is norm bounded in L_∞, and so, it is also norm bounded in L_1. Then by Proposition 10, for each $y' \in \Sigma_0(A)$ there exists $y'' \in \Sigma_{\max}(A)$ such that $y' \leq y''$. Thus, for each $y \in C$ there exists $y'' \in D$ such that $y \leq y''$, in particular, $A_y \supseteq A_{y''}$. Then $M = \cap_{y\in C} A_y = \cap_{y\in D} A_y$. □

Proposition 12. *Let a set $A \subseteq X$ be such that $\Sigma_0(A)$ is a norm bounded non-empty set in L_1 and all functions $y \in \Sigma_{\max}(A)$ are rearrange monotone. Then the set $M = \{x \in L_1 \colon |x| \in A\}$ is an almost AP-set.*

Proof. Suppose the contrary. Then there exists a measurable set $T \subseteq [0,1]$ with $\mu(T) > 0$ such that $\alpha = \max\limits_{x\in M} \int_T xd\mu$ exists. Note that $\alpha \neq 0$. Indeed, if $\alpha = 0$, then $x\chi_T = 0$ for each $x \in A$. Now for any $y_1 \in \Sigma_0(A)$ and $C > 0$ we have $y_1 + C\chi_T \in \Sigma_0(A)$, which contradicts the boundedness of $\Sigma_0(A)$.

Since the set M is balanced, $\alpha > 0$. Consider the function $y_0 = \frac{1}{\alpha}\chi_T$. Then $y_0 \in Y$ and $\max\limits_{x\in M}\langle x, y_0\rangle = \max\limits_{x\in A}\langle x, y_0\rangle = 1$. So, $y_0 \in \Sigma_0(A)$.

Then there exists $x_0 \in A$ such that $\langle x_0, y_0\rangle = 1$. By Proposition 10, there exists $y_1 \in \Sigma_{\max}(A)$ such that $y_0 \leq y_1$. Besides, $\langle x_0, y_0\rangle = \langle x_0, y_1\rangle = 1$. By Proposition 9, $y_0 = y_1$ on the set $S = \{t \in [0,1] \colon x_0(t) > 0\}$. Now since $\langle x_0, y_0\rangle = 1$, we obtain that $\mu(S \cap T) > 0$. Hence, $y_1(t) = \frac{1}{\alpha}$ for each $t \in S \cap T$, which contradicts the conditions of the proposition, because $y_1 \in \Sigma_{\max}(A)$ is rearrange monotone. □

Theorem 1. *Let $B \subseteq Y$, $A = \pi(B)$ and the following conditions hold:*

(i) the set B is norm bounded in L_1;

(ii) for each $x \in \Sigma_0(\pi(A))$ there exists $\varepsilon > 0$ such that the set $B_x = \{y \in B \colon \langle x, y\rangle \geq 1 - \varepsilon\}$ is finite and each $y \in \operatorname{co}(B_x)$ is rearrange monotone.

Then the set $M = \bigcap\limits_{y\in B} A_y$ is almost AP.

Proof. We show that A satisfies the conditions of Proposition 12. Note that by (i), the set $\tilde{B} = \pi(A) = \pi^2(B) = \overline{\mathrm{co}(\cup_{b\in B}[0,b])}$ is norm bounded.

Let $y_0 \in \Sigma_{\max}(A)$ and $x_0 \in A$ be such that $\langle x_0, y_0\rangle = 1$. Obviously, $y_0 \in \tilde{B} = \pi(A) = \{y \in Y : \langle x, y\rangle \le 1 \quad \forall x \in A\}$. Besides, since $x_0 \in A$, one has that $\langle x_0, y\rangle \le 1 = \langle x_0, y_0\rangle$ for any $y \in \tilde{B}$. So $x_0 \in \Sigma_0(\tilde{B})$. Using condition (ii), we choose $\varepsilon > 0$ so that set $B_1 = \{y \in B : \langle x_0, y\rangle \ge 1 - \varepsilon\}$ is bounded and each $y \in \mathrm{co}(B_1)$ is rearrange monotone. Then we put $B_2 = B \setminus B_1$.

Note that by the σ-compactness of $[0, y]$, finiteness of B_1 and Propositions 7 and 8 we have that

$$\tilde{B} = \overline{\mathrm{co}(\bigcup_{b\in B}[0,b])} = \mathrm{co}\Big(\mathrm{co}(\bigcup_{b\in B_1}[0,b]) \cup \overline{\mathrm{co}(\bigcup_{b\in B_2}[0,b])}\Big).$$

Denote $C_1 = \mathrm{co}(\cup_{b\in B_1}[0,b])$ and $C_2 = \overline{\mathrm{co}(\cup_{b\in B_2}[0,b])}$. Choose $y_1 \in C_1, y_2 \in C_2, \alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 + \alpha_2 = 1$ such that $y_0 = \alpha_1 y_1 + \alpha_2 y_2$. Remind that $\langle x_0, y\rangle \le 1$ for any $y \in \tilde{B}$ and $\langle x_0, y\rangle \le 1 - \varepsilon$ for each $y \in C_2$. Now we obtain

$$1 = \langle x_0, y_0\rangle = \alpha_1\langle x_0, y_1\rangle + \alpha_2\langle x_0, y_2\rangle \le \alpha_1 + \alpha_2(1-\varepsilon) = 1 - \alpha_2\varepsilon.$$

So $\alpha_2 = 0$ and $y_0 \in C_1$, i.e. $y_0 = \sum\limits_{b\in B_1} \alpha_b y_b$, where $\alpha_b \ge 0$ and $\sum\limits_{b\in B_1} \alpha_b = 1$, and $y_b \in [0, b]$ for each $b \in B_1$.

We set $y^* = \sum_{b\in B_1} \alpha_b b$ and show that $y_0 = y^*$. Firstly observe that $y_0 \le y^*$ as $y_b \le b$ for any $b \in B_1$. On the other hand, since $\langle x_0, y^*\rangle \ge \langle x_0, y\rangle = 1$ and $y^* \in \pi(A)$, we obtain $\langle x_0, y^*\rangle = \max_{x\in A}\langle x, y^*\rangle = 1$. Hence, $y^* \in \Sigma_0(A)$. Then $y_0 \in \Sigma_{\max}(A)$ yields $y_0 = y^* = \sum_{b\in B_1} \alpha_b b \in \mathrm{co}(B_1)$. Therefore, y_0 is rearrange monotone, according to the choice of B_1.

Thus, A satisfies the conditions of Proposition 12, and M is an almost AP-set. □

Corollary 1. *Let $B \subseteq Y$ be a finite set such that all functions $y \in \mathrm{co}(B)$ are rearrange monotone. Then the set $M = \cap_{y\in B} A_y$ is almost AP.*

Proof. Denote $A = \pi(B)$, $D = \pi(\pi(B))$. The set D satisfies the conditions of Theorem 1 as $\Sigma_{\max}(A) \subseteq \mathrm{co}(B)$. □

Corollary 2. *Let B be a finite collection of polynomials on $[0,1]$ having pairwise distinct degrees ≥ 1. Then the set $M = \cap_{y\in B} A_y$ is almost AP.*

The following example shows the existence of a countable set B such that $\cap_{y\in B} A_y$ is almost AP.

Example. *The set $B = \{2\frac{1}{4} + \frac{3t}{4\pi}, 2 \pm \cos nt, 2 \pm \sin nt, n \in \mathbb{N}, t \in [-\pi,\pi]\}$ satisfies the conditions of Theorem 4.4 for the spaces $X = L_1^+([-\pi,\pi])$ and $Y^+ = L_\infty^+([-\pi,\pi])$, and the set $M = \{x \in L_1[-\pi,\pi] : |\langle x, y\rangle| \le 1 \quad \forall y \in B\}$ is almost AP.*

Indeed, the set B satisfies condition (i) by construction. We show that condition (ii) holds.

Denote $b_0 = 2\frac{1}{4} + \frac{3t}{4\pi}$. Choose $x \in \Sigma_0(\pi(A))$ and find $\varepsilon > 0$ such that the set $B_x = \{b \in B : \langle x, b\rangle > 1 - \varepsilon\}$ is finite and $\mu(\{t \in [-\pi,\pi] : y(t) = \alpha\}) = 0$ for any $y \in \mathrm{co}(B_x)$ and $\alpha \in \mathbb{R}$. First we consider the case when $x = \mathrm{const} = C > 0$ a. e. on $[-\pi,\pi]$. It is easy to show that $\langle x, b\rangle = \int_{-\pi}^{\pi} bx d\mu = 4\pi C$ for any $b \in B, b \ne b_0$ and $\langle x, b_0\rangle = \int_{-\pi}^{\pi} b_0 x d\mu = 4,5\pi C$. Since $x \in \Sigma_0(\pi(A))$, one has that $\langle x, b_0\rangle = 4.5\pi C \le 1$, and hence, $C \le \frac{1}{4.5\pi}$. Then $\langle x, b\rangle \le \frac{8}{9}$ for each $b \in B, b \ne b_0$.

Put $\varepsilon = \frac{1}{9}$. Then $\{b \in B\colon \langle x, b\rangle > 1-\varepsilon\} = \{b \in B\colon \langle x, b\rangle > \frac{8}{9}\} = \{b_0\}$, i.e. the set E is finite. Obviously, $\mathrm{co}(B_x) = \{b_0\}$ and $\mu(\{t \in [-\pi,\pi]\colon b_0(t) = \alpha\}) = 0$ for each $\alpha \in \mathbb{R}$.

Now suppose $x \neq$ const on $[-\pi,\pi]$ (up to sets of zero measure). Then the set $\{n \in \mathbb{N}\colon a_n = \int_{-\pi}^{\pi} x(t)\sin nt d\mu \neq 0$ or $b_n = \int_{-\pi}^{\pi} x(t)\cos nt d\mu \neq 0\}$ is non-empty ([11, p.270]), moreover $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$ ([11, p.260]).

Denote $\alpha = \sup\{|a_n|, |b_n|\colon n \in \mathbb{N}\}$. Obviously, $\alpha > 0$ and the sets $N_1 = \{n \in \mathbb{N}\colon a_n > \frac{\alpha}{2}\}$, $N_2 = \{n \in \mathbb{N}\colon -a_n > \frac{\alpha}{2}\}$, $N_3 = \{n \in \mathbb{N}\colon b_n > \frac{\alpha}{2}\}$ and $N_4 = \{n \in \mathbb{N}\colon -b_n > \frac{\alpha}{2}\}$ are finite because $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$. Now we set $\tilde{B} = \{2+\cos nt\colon n \in N_1\} \cup \{2-\cos nt\colon n \in N_2\} \cup \{2+\sin nt\colon n \in N_3\} \cup \{2-\sin nt\colon n \in N_4\} \cup \{b_0\}$. Note that $\tilde{B}$ is finite. Besides, $1 = \sup\limits_{b\in B}\int_{-\pi}^{\pi} bx d\mu \geq 2\int_{-\pi}^{\pi} x d\mu + \alpha$ and for each $b \in B\setminus\tilde{B}$ we have $\int_{-\pi}^{\pi} bx d\mu \leq 2\int_{-\pi}^{\pi} x d\mu + \frac{\alpha}{2} \leq 1-\frac{\alpha}{2}$. Putting $\varepsilon = \frac{\alpha}{3}$, we obtain that the set $\{b \in B\colon \langle x, b\rangle \geq \int_{-\pi}^{\pi} bx d\mu - \varepsilon\} \subseteq \tilde{B}$ is finite. Since $N_1 \cap N_2 = N_3 \cap N_4 = \varnothing$, each function $y \in \mathrm{co}(\tilde{B})$ is rearrange monotone. Thus, condition (ii) of Theorem 1 holds.

Thus, by Theorem 1 the set $M = \{x \in L_1\colon |x| \in \pi(B)\}$ is almost AP, moreover, $\Sigma_{\max}(A) \subseteq \mathrm{co}(B)$.

Note that for the given examples of almost AP-sets $M = \cap_{y\in B} A_y$ the condition $\Sigma_{\max}(A) \subseteq \mathrm{co}(B)$ holds, where B is, at most, a countable set. The following theorem shows that constructed in such a way sets are not AP-set.

Theorem 2. *Let $A \subseteq X$ be a closed bounded convex set, $B \subseteq \pi(A)$, let $\pi(A)$ be a norm bounded subset of L_1 and let B be, at most, a countable set such that $\Sigma_{\max}(A) \subseteq \mathrm{co}(B)$. Then the set $M = \{x \in L_1\colon |x| \in \pi(B)\}$ is not AP.*

Proof. We set $B_0 = B \cap \Sigma_{\max}(A)$ and show that $\Sigma_{\max}(A) \subseteq \mathrm{co}(B_0)$.

Assume $y_0 \in \Sigma_{\max}(A)$. By the theorem conditions, there exist $n \in \mathbb{N}$, $b_1, ..., b_n \in B$ and $\alpha_1, ..., \alpha_n \in (0,1]$ such that $\alpha_1 + ... + \alpha_n = 1$ and $y_0 = \alpha_1 b_1 + ... + \alpha_n b_n$. Choose $x_0 \in A$ so that $\langle x_0, y_0\rangle = 1$. Since $b_1, ..., b_n \in \pi(A)$, we have that $\langle x_0, b_i\rangle \leq 1$ for $1 \leq i \leq n$. Now we show that $\langle x_0, b_i\rangle = 1$ for $1 \leq i \leq n$. Suppose, on the contrary, that $\langle x_0, b_j\rangle < 1$ for some $1 \leq j \leq n$. Since $\alpha_j > 0$, we obtain that

$$\langle x_0, y_0\rangle = \sum_{i=1}^{n} \alpha_i \langle x_0, b_i\rangle < \sum_{i=1}^{n} \alpha_i = 1,$$

which contradicts the choice of $\alpha_1, ..., \alpha_n$.

Besides, $\langle x, b_i\rangle \leq 1$ for each $x \in A$. Then $b_1, ..., b_n \in \Sigma_0(A)$. Since $\alpha_1, ..., \alpha_n > 0$ and $y_0 = \sum_{i=1}^{n} \alpha_i b_i \in \Sigma_{\max}(A)$, we have that $b_1, ..., b_n \in \Sigma_{\max}(A)$. Thus, $b_1, ..., b_n \in B_0$ and $y_0 \in \mathrm{co}(B_0)$.

Observe that each functional $y \in B_0$ has the maximum value 1 on M. Thus, if $B_0 \cap \Sigma_0 \neq \varnothing$, where Σ_0 is the set of support functionals on unit ball, then M is not AP. It remains to consider the case $B_0 \cap \Sigma_0 = \varnothing$. Let $B_0 = \{y_n\colon n \in \mathbb{N}\}$. Since $b_n \notin \Sigma_0$ for each $n \in \mathbb{N}$, we can choose $\delta_n > 0$ so that $\mu(T_n) = \mu(\{t \in [0,1]\colon y_n(t) > \|y_n\| - \delta_n\}) < \frac{1}{4^n}$ and put $S = [0,1] \setminus \left(\cup_{n=1}^{\infty} T_n\right)$. Obviously, $\mu(S) > 0$ and $C = \{x|_S\colon x \in M\}$ is a closed bounded convex set in the Banach space $L_1(S)$. By the Bishop-Phelps theorem [3], there exist functions $u_0 \in C$ and $v_0 \in L_\infty(S)$ such that $1 = \int_S u_0 v_0 d\mu = \max\limits_{u\in C} \int_S u v_0 d\mu$.

Consider the functions $y_0 \in Y$ and $x_0 \in X$

$$y_0(t) = \begin{cases} |v_0(t)|, & t \in S, \\ 0, & t \notin S, \end{cases} \qquad x_0(t) = \begin{cases} |u_0(t)|, & t \in S, \\ 0, & t \notin S. \end{cases}$$

Note that $x_0 \in A$ and $y_0 \in \Sigma_0(A)$. Since $\pi(A)$ is norm bounded in L_1, so is $\Sigma_0(A)$, and by Proposition 10, there exist $\tilde{y} \in \Sigma_{\max}(A)$ such that $y_0 \leq \tilde{y}$. Thus, there exist $\tilde{y}_1, ..., \tilde{y}_n \in B_0$ and $\alpha_1, ..., \alpha_n > 0$ such that $\sum\limits_{i=1}^{n} \alpha_i = 1$ and $\tilde{y} = \sum\limits_{i=1}^{n} \alpha_i \tilde{y}_i$.

Let $\tilde{y}_1 = y_k$. Note that $\langle x_0, y_k \rangle = 1$. Consider the function $y^* \in L_\infty$,

$$y^*(t) = \begin{cases} y_k(t), & t \in [0,1] \setminus T_k, \\ \|y_k\| - \delta_k, & t \in T_k, \end{cases}$$

which obviously belongs to Σ_0. We have

$$\int_0^1 y^* x_0 d\mu = \int_S y^* x_0 d\mu = \int_S y_k x_0 d\mu = 1.$$

On the other hand, for each $x \in M$ we have $|\langle x, y^* \rangle| \leq \langle |x|, y^* \rangle \leq \langle |x|, y_n \rangle \leq 1$. Thus, $y^* \in \Sigma(M) \cap \Sigma_0$ and M is not AP-set. □

REFERENCES

1. Edelstein M., Thompson A.C. *Some results on nearest points and support properties of convex sets in* c_0// Pacific J. Math. – 1972. – V.40. – P. 553–560.
2. Bishop E., Phelps R.R. *A proof that every Banach space is subreflexive*// Bull. Amer. Math. Soc. – 1961. – V.67. – P. 97–98.
3. Bishop E., Phelps R.R. *Support functionals of convex sets*// Proc.Simposia in Pure Math. (Convexity) Amer. Math. Soc. – 1963. – V.7. – P. 27–35.
4. Klee V. *Remarks on nearest points in normed linear spaces*// Proc. Colloquium on Convexity, Copenhagen. – 1965. – P. 168–176.
5. Cobzaş S. *Antiproximinal sets in the spaces* c_0 *and* c// Math. Notes. – 1975. – V.17. – P. 449–457.
6. Fonf V.P. *On antiproximinal sets in spaces of continuous functions on compacta*// Mat. Zametki. – 1983. – V.33, №4. – P. 549–558. (in Russian)
7. Balaganskii V.S. *Antiproximinal sets in the space of continuous functions*// Math. Notes. – 1996. – V.60, №5. – P. 485–494.
8. Kantorovich L.V., Akilov G.P. Functional analysis. – Moscow: Nauka, 1984. – 752p. (in Russian)
9. Schaefer H. Topological vector spaces. – Moscow: Mir, 1971. – 359p. (in Russian)
10. Kolmogorov A.N., Fomin S.V. Elements of the theory of functions and functional analysis. – Moscow: Nauka, 1976. – 544p. (in Russian)
11. Natanson I.P. Theory of functions of real variable. – Moscow: Nauka, 1974. – 480p. (in Russian)
12. Martínez-Abejón A., Odell E., Popov M.M. *Some open problems on the classical function space* L_1// Mat. Stud. – 2005. – V.24, №2. – P. 173–191.

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