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CONVERGENCE OF A FORMAL POWER SERIES AND GELFOND-LEONT’EV DERIVATIVES


Given a formal power series, we establish conditions on the Gelfond-Leont’ev derivatives under which the series represents a function analytic in the disk \( \{ z : |z| < R \} \), \( R \in (0, +\infty) \). We also give a survey of well-know results for the case \( R = +\infty \).


Получены условия на производные Гельфонда-Леонтьева формального степенного ряда, обеспечивающие аналитичность в круге \( \{ z : |z| < R \} \), \( R \in (0, +\infty) \) его суммы. Приведен обзор ранее известных результатов для случая \( R = +\infty \).

1. For \( R \in [0, +\infty] \) we denote by \( A(R) \) the class of power series

\[
f(z) = \sum_{k=0}^{\infty} f_k z^k,
\]

having a radius of convergence \( \geq R \), and we say that \( f \in A^+(R) \) if \( f \in A(R) \) and \( f_k > 0 \) for all \( k \geq 0 \). For \( f \in A(0) \) and \( l(z) = \sum_{k=0}^{\infty} l_k z^k \in A^+(0) \) the formal power series

\[
D^n_l f(z) = \sum_{k=0}^{+\infty} \frac{l_k}{l_{k+n}} f_{k+n} z^k
\]

is called [1–2] the Gelfond-Leont’ev derivative of order \( n \). If \( l(z) = e^z \), that is \( l_k = 1/k! \), then \( D^n_l f(z) = f^{(n)}(z) \) is a usual derivative of order \( n \). We may assume that \( l_0 = 1 \).

As in [2], let \( \Lambda \) be the class of all positive sequences \( \lambda = (\lambda_k) \) with \( \lambda_1 \geq 1 \), and let \( \Lambda^* = \{ \lambda \in \Lambda : \ln \lambda_k \leq ak \) for every \( k \in \mathbb{N} \) and some \( a \in [0, +\infty) \} \). We say that \( f \in A_{\lambda}(0) \) if \( f \in A(0) \) and \( |f_k| \leq \lambda_k |f_1| \) for all \( k \geq 1 \). Finally, let \( N \) be a class of increasing sequences \( (n_p) \) of nonnegative integers, \( n_0 = 0 \).

In the next two subsections we give some known results which preceded the main results.

2. Investigation of conditions on Gelfond-Leont’ev derivatives, under which series (1) represents an entire function, started in [2]. In particular, the following theorem is proved.

**Theorem 1** ([2]). Let \( l \in A^+(0) \). Then for every \( f \in A(0) \) and \( \lambda \in \Lambda \) the condition \( (\forall n \in \mathbb{Z}_+) \{ D^n_l f \in A_{\lambda}(0) \} \) implies the inclusion \( f \in A(+\infty) \) if and only if \( l \in A^+(+\infty) \), i. e.

\[
\lim_{k \to +\infty} \sqrt[k]{l_k} = 0.
\]

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Denote by $N$ the class of a sequence $(n_p)$ such that $n_p \in \mathbb{N}, n_p < n_{p+1}$ ($p \geq 1$).

Assuming that $l \in A^+(+\infty)$ the second author investigated ([2]) conditions on $(n_p) \in N$ that provide the implication

$$\forall p \in \mathbb{Z}_+ \{D^p_l f \in A_\lambda(0)\} \Rightarrow f \in A(+\infty).$$

**Theorem 2** ([2]). Let $(n_p) \in N$. Then for every $\lambda \in \Lambda$, $f \in A(0)$ and $l \in A^+(+\infty)$ condition (4) holds if and only if

$$\lim_{p \to +\infty} (n_{p+1} - n_p) < \infty. \quad (5)$$

We put $x_k = l_kl_{k+1}/l_k^2$ and we say that $l \in A^+(R)$ if $l \in A^+(R)$ and the sequence $(x_k)$ is nondecreasing. In the following theorem the sequence $(n_{p+1} - n_p)$ can be unbounded, but we require that $\lambda \in \Lambda^*$ and $l \in A^*_+(+\infty)$.

**Theorem 3** ([2]). Let $(n_p) \in N$ and $l \in A^*_+(+\infty)$. Then for every $\lambda \in \Lambda^*$ and $f \in A(0)$ condition (4) holds if and only if

$$\lim_{p \to +\infty} \frac{1}{n_p+1} \left\{ \ln \frac{1}{l_{n_p+1}} - \sum_{j=1}^{p} \ln \frac{1}{l_{n_j-n_{j-1}+1}} \right\} = +\infty. \quad (6)$$

The condition $l \in A^*_+(+\infty)$ (i. e. the nondecrease of the sequence $(x_k)$) in Theorem 3 cannot be removed in general. Actually, in [2] it is shown, that there exist sequences $(n_p)$, $\lambda \in \Lambda^*$ and functions $l \in A^+(+\infty)$ and $f \not\in A(+\infty)$ such that the sequence $(x_k)$ is oscillating, condition (6) does not hold and $(\forall p \in \mathbb{Z}_+) \{D^p_l f \in A_\lambda(0)\}$.

The following problem is examined in [2]: for what functions $l \in A^*_+(+\infty)$ one has that, if $(\forall p \in \mathbb{Z}_+) \{D^p_l f \in A_\lambda(0)\}$ holds for every $(n_p)$ and $\lambda \in \Lambda^*$ then $f \in A(+\infty)$? If we denote $\omega_k = \frac{1}{k+1} \ln \frac{l_k}{l_{k+1}} - \frac{1}{k} \ln \frac{1}{l_k}$ then [2] the nondecrease of the sequence $(x_k)$ implies the positivity and the nonincrease of the sequence $(\omega_k)$, that is, $\lim_{k \to \infty} \omega_k = \omega$ exists.

**Theorem 4** ([2]). Let $l \in A^*_+(+\infty)$. Then for every $(n_p) \in N$, $\lambda \in \Lambda^*$ and $f \in A(0)$ condition (4) holds if and only if $\omega > 0$.

The following theorem asserts that, under some conditions on $l \in A^+(+\infty)$, one value $n \in \mathbb{N}$ suffices in order that the condition $D^n_l f \in A_\lambda(0)$ implies the inclusion $f \in A(+\infty)$.

**Theorem 5** ([2]). Let $l \in A^+(+\infty)$. Then for every $n \in \mathbb{N}$, $\lambda \in \Lambda^*$ and $f \in A(0)$ the condition $D^n_l f \in A_\lambda(0)$ implies the inclusion $f \in A(+\infty)$, if and only if

$$\lim_{k \to \infty} \sqrt[k]{l_k/l_{k+1}} = +\infty. \quad (7)$$

We note that condition (7) holds provided $\omega_k \to +\infty (k \to \infty)$. For the sequence $l_k = \exp\{-\omega k^2\}$ with $\omega > 0$ we have $\omega_k \sim \omega(k \to \infty)$, and condition (7) holds if $l_k = \exp\{-\beta k^2\}$, where $0 < \beta(x) \sim +\infty(k \to \infty)$.

It is shown in [3] that if $l \in A^+(+\infty)$ and the sequence $(x_k)$ is nonincreasing then $1 > x_k \geq 0$ ($k_0 \leq k \to +\infty$) and $\omega_k \geq \omega(k \to +\infty)$. Therefore, if $1 > x_k \geq 0$ ($k_0 \leq k \to +\infty$) then by Theorem 5, for every $n \in \mathbb{N}$, $\lambda \in \Lambda^*$ and $f \in A(0)$ the condition $D^n_l f \in A_\lambda(0)$ implies the inclusion $f \in A(+\infty)$. However, the following result is more general.
Theorem 6 ([3]). Let \( l \in A^{+}(+\infty) \) and \( 1 > \aleph_{k} \searrow \aleph \geq 0 \ (k_{0} \leq k \to +\infty) \). Then for every \((n_{p}) \in N, \lambda \in \Lambda^{*} \) and \( f \in A(0) \) condition (4) holds.

In Theorems 3–6 \( \lambda \in \Lambda^{*} \), i.e. the sequence \( \lambda \) can increase not faster than the exponential function. In [4] an analogue of Theorem 3 is obtained for the case if \( \lambda \) is allowed to increase considerable fast. We assume that a positive sequence \( \psi = (\psi_{k}) \) satisfies the condition \( \psi_{k}^{2} \leq \psi_{k-1} \psi_{k+1}, k \geq 2 \), and let \( \Lambda_{\psi} = \{ \lambda: \ln \lambda_{k} \leq \ln \psi_{k} + ak(k \in N), a \equiv \text{const.} \}

Theorem 7 ([4]). Let \((n_{p}) \in N, l \in A^{+}(+\infty)\) and \( \psi_{k}^{2} \leq \psi_{k-1} \psi_{k+1}, k \geq 2 \). Then for every \( \lambda \in \Lambda_{\psi} \) and \( f \in A(0) \) condition (4) holds if and only if

\[
\lim_{p \to +\infty} \frac{1}{n_{p} + 1} \left\{ \ln \frac{1}{l_{n_{p} + 1}} - \sum_{j=1}^{p} \ln \frac{\psi_{n_{j} - n_{j-1} + 1}}{\lambda_{n_{j} - n_{j-1} + 1}} \right\} = +\infty. \tag{8}
\]

We remark that the condition \( \psi_{k}^{2} \leq \psi_{k-1} \psi_{k+1}, k \geq 2 \) holds if, for example, \( \psi_{k} = k! \) or \( \psi_{k} = \exp\{ak^n\} \ (a > 0, n \in N) \), and Theorem 7 has the following consequence.

Corollary 1 ([4]). Let \( l_{k} = \exp\{-\omega_{1}k^{2}\} \) and \( \psi_{k} = \exp\{-\omega_{2}k^{2}\}, 0 < \omega_{1}, \omega_{2} < \infty \). Then for every \((n_{p}) \in N, \lambda \in \Lambda_{\psi} \) and \( f \in A(0) \) condition (4) holds.

3. In all mentioned results conditions on the Gelfond-Leont’ev derivatives of formal power series (1) implies that the convergence radius \( R[f] = +\infty \). The following question naturally arises: find conditions on the Gelfond-Leont’ev derivatives, under which series (1) is convergent in some neighborhoods of the origin, i.e. \( R[f] > 0 \). Such results are obtained in the papers [5–7]. In particular, in [5] it is proved the following proposition, which is new also for the case \( R[f] = +\infty \).

Proposition 1 ([5]). Let \( R \in (0, +\infty) \). Then \( f \in A(R) \) if and only if there exists a sequence \( \lambda \in \Lambda \) such that \( f \in A_{\lambda}(0) \) and \( \lim_{k \to \infty} \sqrt[k]{\lambda_{k}} \leq 1/R \).

The following result generalizes Theorem 1.

Theorem 8 ([5]). Let \( l \in A^{+}(0) \). Then for every \( f \in A(0) \) and \( \lambda \in \Lambda \) the condition \( (\forall n \in \mathbb{Z}_{+}) \{D^{n}f \in A_{\lambda}(0)\} \) implies the inclusion \( f \in A(R) \) if and only if

\[
\lim_{k \to +\infty} \sqrt[k]{l_{k}} \leq \frac{l_{2}}{l_{1} \lambda_{2} R}. \tag{9}
\]

Obviously, for \( R = +\infty \) conditions (3) and (9) are equivalent. We remark also that Theorem 8 implies that if \( l \in A^{+}(0) \) then in order that for every \( f \in A(0) \) and \( \lambda \in \Lambda \) the condition \( (\forall n \in \mathbb{Z}_{+}) \{D^{n}f \in A_{\lambda}(0)\} \) imply the analyticity of \( f \) in some neighborhood of the origin, it is necessary and sufficient that \( \sqrt[l]{l_{k}} = O(1), k \to \infty \).

The following two results are analogues of Theorem 2.

Theorem 9 ([6]). Let \((n_{p}) \in N \). In order that for every \( \lambda \in \Lambda, f \in A(0) \) and \( l \in A^{+}(R) \) \((0 < R \leq +\infty)\) the condition \( (\forall p \in \mathbb{Z}_{+}) \{D^{p}f \in A_{\lambda}(0)\} \) imply the analyticity of \( f \) in some neighborhood of the origin is necessary and sufficient that condition (5) hold.

Theorem 10 ([6]). Let \((n_{p}) \in N \). For condition (5) to hold is necessary and sufficient that for every \( \lambda \in \Lambda, f \in A(0) \) and \( l \in A^{+}(R) \) \((0 < R \leq +\infty)\) the condition \( (\forall p \in \mathbb{Z}_{+}) \{D^{p}f \in A_{\lambda}(0)\} \) imply the estimate \( R[f] \geq PR[l] \), where \( R[f] \) and \( R[l] \) are the convergence radii of the functions \( f \) and \( l \), and \( P \) is a positive constant.
We note that \( P = \frac{1}{\ln \max \{\lambda_k/l_k : 2 \leq k \leq m+1\} \} \), where \( m = \max \{n_{p+1} - n_p : p \geq 0\} \), and the estimate \( R[f] \geq PR[l] \) in Theorem 10 is sharp. Also if \( \lim_{p \to +\infty} (n_{p+1} - n_p) = \infty \) then for every \( R \in (0, +\infty) \) there exist \( \lambda \in \Lambda, f \in A(0) \) and \( l \in A^+(0) \) such that \( (\forall p \in \mathbb{Z}_+) \{D_l^{n_p} f \in A_{\lambda}(0)\} \), but \( R[f] = 0 \) and \( R[l] = R \).

We denote \( \Lambda_* = \{\lambda : \lambda_{k-1}\lambda_{k+1} \geq \lambda_k^2(k \geq 2)\} \). Then Theorem 3 has the following analogue.

**Theorem 11** ([7]). Let \( (n_p) \in \mathbb{N} \). Then for every \( \lambda \in \Lambda_* \), \( l \in A^*(0) \) and \( f \in A(0) \) the condition \( (\forall p \in \mathbb{Z}_+) \{D_l^{n_p} f \in A_{\lambda}(0)\} \) implies the inclusion \( f \in A(R) \) if and only if

\[
\lim_{p \to +\infty} \frac{1}{n_p + 1} \left\{ \ln \frac{1}{l_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \right\} \geq \ln R. \tag{10}
\]

Each of the conditions \( \lambda \in \Lambda_* \), \( l \in A^*(0) \) in Theorem 11 cannot be relaxed. Theorem 11 is a consequence of the following result.

**Theorem 12** ([7]). Let \( (n_p) \in \mathbb{N} \), and let a sequence \( \lambda \in \Lambda \) and a function \( l \in A^+(0) \) be such that for all \( p \in \mathbb{Z}_+ \) and \( k = 2, \ldots, n_{p+1} - n_p \)

\[
\ln \frac{l_{n_{p+k-1}l_{n_{p+k+1}}}}{l_{n_p+k}^2} - \ln \frac{l_{k-1}l_{k+1}}{l_k^2} + \ln \frac{\lambda_{k-1}\lambda_{k+1}}{\lambda_k^2} \geq 0.
\]

If \( (\forall p \in \mathbb{Z}_+) \{D_l^{n_p} f \in A_{\lambda}(0)\} \) then the estimate

\[
\ln R[f] \geq \lim_{p \to +\infty} \frac{1}{n_p + 1} \left\{ \ln \frac{1}{l_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \right\}
\]

is true and sharp.

We remark that one can obtain an analogue of Theorem 12 for the case when the sequence \( \lambda \in \Lambda \) satisfies a condition similar to \( \lambda \in \Lambda_* \).

**Theorem 13** ([7]). Let \( (n_p) \in \mathbb{N}, l \in A^+(0) \) and let a sequence \( \lambda \in \Lambda \) be such that \( \ln \lambda_k \leq a(k-1) \) for all \( k \geq 1 \) and some \( a > 0 \). If \( (\forall p \in \mathbb{Z}_+) \{D_l^{n_p} f \in A_{\lambda}(0)\} \) then the estimate

\[
\ln R[f] \geq \lim_{p \to +\infty} \frac{1}{n_p + 1} \left\{ \ln \frac{1}{l_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{1}{l_{n_j-n_{j-1}+1}} \right\} - a \tag{11}
\]

is true and sharp.

In Theorem 13 one cannot replace the condition \( \ln \lambda_k \leq a(k-1) \) with the condition \( \lambda \in \Lambda_* \) and, moreover, with the condition \( \lim_{k \to \infty} (\ln \lambda_k)/k = a \). However, the following theorem is true.

**Theorem 14** ([7]). Let \( (n_p) \in \mathbb{N}, \ln \lambda_k = o(k)(k \to \infty) \) and let a function \( l \in A^+(0) \) be such that the sequence \( (\mu_{k-1}\mu_{k+1}/\mu_k^2) \) is nondecreasing, where \( \mu_k = l_k/\lambda_k \). If \( (\forall p \in \mathbb{Z}_+) \{D_l^{n_p} f \in A_{\lambda}(0)\} \) then the estimate (11) with \( a = 0 \) is true and sharp.

4. Continuing investigations from [5–7], we show, at first, that the full analogue of Theorem 5 is valid.
**Theorem 15.** Let $l \in A^+(0)$. Then for every $n \in \mathbb{N}$, $\lambda \in \Lambda^*$ and $f \in A(0)$ the condition $D^nf \in A_\lambda(0)$ implies the inequality $R[f] > 0$ if and only if

$$\lim_{k \to \infty} \sqrt[1] {l_k/l_{k+1}} > 0.$$  \hspace{1cm} (12)

**Proof.** We first assume that $(l_k)$ satisfies condition (12) and $D^nf \in A_\lambda(0)$ for some fixed number $n \in \mathbb{N}$. Then \( \frac{l_{n+k}}{l_{n+k}} \leq \lambda_k \frac{l_{n+k}}{l_{n+k}} (k \geq 1) \), that is,

$$\frac{1}{n+k} \ln \frac{1}{l_{n+k}} \geq \frac{1}{n+k} \ln \frac{l_k}{l_{n+k}} - \frac{\ln \lambda_k}{n+k} - \frac{1}{n+k} \ln \left( \frac{l_{n+k}}{l_{n+k}} \right)$$

and, thus,

$$\ln R[f] = \lim_{k \to \infty} \frac{1}{n+k} \ln \frac{1}{l_{n+k}} \geq \lim_{k \to \infty} \frac{1}{n+k} \ln \frac{l_k}{l_{n+k}} - a.$$  \hspace{1cm} (13)

But by condition (12), there exists $c > 0$ such that $\sqrt[1] {l_k/l_{k+1}} \geq c$ for all $k \geq 1$. Therefore, we obtain from (13)

$$\ln R[f] \geq \lim_{k \to \infty} \frac{1}{n+k} \sum_{j=k}^{n+k-1} \ln \frac{l_j}{l_{j+1}} - a \geq \lim_{k \to \infty} \frac{1}{n+k} \sum_{j=k}^{n+k-1} j \ln c - a =$$

$$= \lim_{k \to \infty} \frac{\ln c}{n+k} \frac{2nk + n(n-1)}{2} - a = n \ln c - a > -\infty,$$

that is, $R[f] > 0$ and the sufficiency of (12) is proved.

Now we assume that condition (12) does not hold, i.e. there exists an increasing sequence $(k_j)$ such that $\sqrt[1] {l_{k_j}/l_{k_j+1}} = \alpha_j \to 0$ $(j \to \infty)$. We put $k_0 = 0$, $f_{k_j+1} = \frac{l_{k_j+1}}{l_{j+1}} (j \geq 0)$ and $f_k = 0$ for $k \neq k_j + 1$. For series (1) with such coefficients we have $\sqrt[1] {l_{k_j+1}/l_{k_j}} = (1/\alpha_j)^{k_j/(k_j+1)} \to +\infty (j \to \infty)$, that is, $R[f] = 0$.

On the other hand, for the series we have

$$D^1f(z) = \sum_{j=0}^{\infty} l_{k_j+1} z^{k_j+1} = z + \sum_{j=1}^{\infty} z^{k_j+1},$$

that is, $D^1f \in A_\lambda(0)$, if we choose $\lambda_k \equiv 1$. Thus, if condition (12) does not hold then there exist $n = 1$, $\lambda \in \Lambda^*$ and a formal power series (1) such that $D^nf \in A_\lambda(0)$, but $R[f] = 0$. Theorem 15 is proved. 

If we denote $c = \lim_{k \to \infty} \sqrt[1] {l_k/l_{k+1}}$ then one can show using arguments analogous to that in the proof of Theorem 15 that the estimate $\ln R[f] \geq n \ln c - a$ is valid. On the other hand, if we choose $l_k = c^{k(k+1)} (k \geq 0)$, $\lambda_k = e^{a(k-1)} (k \geq 1)$, $f_0 = \cdots = f_n = 0, f_{n+1} = 1$ and $f_{n+k} = f_{n+1} + \lambda_k l_{n+k}/l_k (k \geq 1)$ then for series (1) with such coefficients we have $D^nf(z) = \sum_{k=0}^{\infty} \lambda_k z^k$ (i.e. $D^nf \in A_\lambda(0)$) and $\ln R[f] = \lim_{k \to \infty} \frac{1}{n+k} \left( \ln \frac{l_{n+k}}{l_{n+k}} - a(k-1) \right) = n \ln c - a$. Thus, the following proposition is true.

**Proposition 2.** Let $n \in \mathbb{N}$, $\lambda \in \Lambda^*$, $l \in A^+(0)$ and $\lim_{k \to \infty} \sqrt[1] {l_k/l_{k+1}} = c$. If $D^nf \in A_\lambda(0)$ then the estimate $R[f] \geq c^a e^{-a}$ is valid and sharp.
As above, we denote \( \kappa_k = \frac{l_{k+1} - l_k}{l_k} \) and \( \omega_k = \frac{1}{k+1} \ln \frac{1}{l_{k+1}} - \frac{1}{k} \ln \frac{1}{l_k} \). It is easy to show [2–3] that \( \omega_k = \frac{1}{k(k+1)} \sum_{j=1}^{k} j \ln \frac{1}{\kappa_j} \). Hence, if \( \lim_{k \to \infty} \kappa_k = \kappa \) then \( \lim_{k \to \infty} \omega_k = \omega = \frac{1}{2} \ln \frac{1}{\kappa} \). On the other hand,

\[
\ln c = \lim_{k \to \infty} \left( \frac{1}{k} \ln \frac{1}{l_{k+1}} - \frac{1}{k} \ln \frac{1}{l_k} \right) = \lim_{k \to \infty} \left( \omega_k + \frac{1}{k(k+1)} \ln \frac{1}{l_{k+1}} \right) = \lim_{k \to \infty} \left( \omega_k + \frac{1}{k} \sum_{j=1}^{k} \omega_j \right) = 2\omega = \ln \frac{1}{\kappa}.
\]

Therefore, Proposition 2 and arguments of its proof yield the following result.

**Theorem 16.** Let \( n \in \mathbb{N} \), \( \lambda \in \Lambda^* \), \( l \in A^+(0) \) and \( \lim_{k \to \infty} \kappa_k = \kappa \). If \( D^n_l f \in A_\lambda(0) \) then the estimate \( R[f] \geq e^{2\omega} e^{-a} = \kappa^{-n} e^{-a} \) is valid and sharp.

Theorem 16 has the following corollary.

**Corollary 2.** Let \( (n_p) \in \mathbb{N} \), \( \lambda \in \Lambda^* \), \( l \in A^+(0) \) and \( \lim_{k \to \infty} \kappa_k = \kappa < 1 \) (i.e. \( \omega > 0 \)). If \( D^n_l f \in A_\lambda(0) \) for all \( p \in \mathbb{Z}_+ \) then \( R[f] = +\infty \).

Since \( \kappa < 1 \) implies \( l \in A^+(+\infty) \), from Corollary 2 and, thus, from Theorem 16 we obtain Theorem 6 and the sufficiency of the condition \( \omega > 0 \) in Theorem 4.

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