УДК 517.53

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CONVERGENCE OF A FORMAL POWER SERIES AND GELFOND-LEONT'EV DERIVATIVES

S. I. Fedynyak, M. M. Sheremeta. *Convergence of a formal power series and Gelfond-Leont'ev derivatives*, Mat. Stud. **35** (2011), 149–154.

Given a formal power series, we establish conditions on the Gelfond-Leont'ev derivatives under which the series represents a function analytic in the disk $\{z\colon |z|<R\}, R\in(0,+\infty]$. We also give a survey of well-know results for the case $R=+\infty$.

С. И. Федыняк, М. Н. Шеремета. *Сходимость формального степенного ряда и производные Гельфонда-Леонтьева* // Мат. Студiї. – 2011. – Т.35, №2. – C.149–154.

Получены условия на производные Гельфонда-Леонтьева формального степенного ряда, обеспечивающие аналитичность в круге $\{z\colon |z|<R\}, R\in(0,+\infty]$ его суммы. Приведен обзор ранее известных результатов для случая $R=+\infty$.

1. For $R\in[0,+\infty]$ we denote by $A(R)$ the class of power series

$$f(z)=\sum_{k=0}^{\infty} f_k z^k, \tag{1}$$

having a radius of convergence $\geq R$, and we say that $f\in A^+(R)$ if $f\in A(R)$ and $f_k>0$ for all $k\geq 0$. For $f\in A(0)$ and $l(z)=\sum_{k=0}^{\infty} l_k z^k\in A^+(0)$ the formal power series

$$D_l^n f(z)=\sum_{k=0}^{+\infty}\frac{l_k}{l_{k+n}} f_{k+n} z^k \tag{2}$$

is called [1–2] the Gelfond-Leont'ev derivative of order n. If $l(z)=e^z$, that is $l_k=1/k!$, then $D_l^n f(z)=f^{(n)}(z)$ is a usual derivative of order n. We may assume that $l_0=1$.

As in [2], let Λ be the class of all positive sequences $\lambda=(\lambda_k)$ with $\lambda_1\geq 1$, and let $\Lambda^*=\{\lambda\in\Lambda\colon \ln\lambda_k\leq ak$ for every $k\in\mathbb{N}$ and some $a\in[0,+\infty)\}$. We say that $f\in A_\lambda(0)$ if $f\in A(0)$ and $|f_k|\leq\lambda_k|f_1|$ for all $k\geq 1$. Finally, let N be a class of increasing sequences (n_p) of nonnegative integers, $n_0=0$.

In the next two subsections we give some known results which preceded the main results.

2. Investigation of conditions on Gelfond-Leont'ev derivatives, under which series (1) represents an entire function, started in [2]. In particular, the following theorem is proved.

Theorem 1 ([2]). *Let $l\in A^+(0)$. Then for every $f\in A(0)$ and $\lambda\in\Lambda$ the condition $(\forall n\in\mathbb{Z}_+)\{D_l^n f\in A_\lambda(0)\}$ implies the inclusion $f\in A(+\infty)$ if and only if $l\in A^+(+\infty)$, i. e.*

$$\lim_{k\to+\infty}\sqrt[k]{l_k}=0. \tag{3}$$

2010 *Mathematics Subject Classification:* 30D50, 30D99.

Denote by N the class of a sequence (n_p) such that $n_p \in \mathbb{N}, n_p < n_{p+1}$ $(p \geq 1)$.

Assuming that $l \in A^+(+\infty)$ the second author investigated ([2]) conditions on $(n_p) \in N$ that provide the implication

$$(\forall p \in \mathbb{Z}_+)\{D_l^p f \in A_\lambda(0)\} \Rightarrow f \in A(+\infty). \tag{4}$$

Theorem 2 ([2]). *Let $(n_p) \in N$. Then for every $\lambda \in \Lambda$, $f \in A(0)$ and $l \in A^+(+\infty)$ condition (4) holds if and only if*

$$\overline{\lim_{p\to+\infty}}(n_{p+1} - n_p) < \infty. \tag{5}$$

We put $\varkappa_k = l_k l_{k+1}/l_k^2$ and we say that $l \in A_*^+(R)$ if $l \in A^+(R)$ and the sequence $(\varkappa_k)$ is nondecreasing. In the following theorem the sequence $(n_{p+1} - n_p)$ can be unbounded, but we require that $\lambda \in \Lambda^*$ and $l \in A_*^+(+\infty)$.

Theorem 3 ([2]). *Let $(n_p) \in N$ and $l \in A_*^+(+\infty)$. Then for every $\lambda \in \Lambda^*$ and $f \in A(0)$ condition (4) holds if and only if*

$$\lim_{p\to+\infty} \frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}} - \sum_{j=1}^{p} \ln\frac{1}{l_{n_j-n_{j-1}+1}}\right\} = +\infty. \tag{6}$$

The condition $l \in A_*^+(+\infty)$ (i. e. the nondecrease of the sequence $(\varkappa_k)$) in Theorem 3 cannot be removed in general. Actually, in [2] it is shown, that there exist sequences (n_p), $\lambda \in \Lambda^*$ and functions $l \in A^+(+\infty)$ and $f \notin A(+\infty)$ such that the sequence $(\varkappa_k)$ is oscillating, condition (6) does not hold and $(\forall p \in \mathbb{Z}_+)$ $\{D_l^{n_p} f \in A_\lambda(0)\}$.

The following problem is examined in [2]: for what functions $l \in A_*^+(+\infty)$ one has that, if $(\forall p \in \mathbb{Z}_+)$ $\{D_l^{n_p} f \in A_\lambda(0)\}$ holds for every (n_p) and $\lambda \in \Lambda^*$ then $f \in A(+\infty)$? If we denote $\omega_k = \frac{1}{k+1}\ln\frac{1}{l_{k+1}} - \frac{1}{k}\ln\frac{1}{l_k}$ then [2] the nondecrease of the sequence $(\varkappa_k)$ implies the positivity and the nonincrease of the sequence (ω_k), that is, $\lim\limits_{k\to\infty} \omega_k = \omega$ exists.

Theorem 4 ([2]). *Let $l \in A_*^+(+\infty)$. Then for every $(n_p) \in N$, $\lambda \in \Lambda^*$ and $f \in A(0)$ condition (4) holds if and only if $\omega > 0$.*

The following theorem asserts that, under some conditions on $l \in A^+(+\infty)$, one value $n \in \mathbb{N}$ suffices in order that the condition $D_l^{n_p} f \in A_\lambda(0)$ implies the inclusion $f \in A(+\infty)$.

Theorem 5 ([2]). *Let $l \in A^+(+\infty)$. Then for every $n \in \mathbb{N}$, $\lambda \in \Lambda^*$ and $f \in A(0)$ the condition $D_l^n f \in A_\lambda(0)$ implies the inclusion $f \in A(+\infty)$, if and only if*

$$\lim_{k\to\infty} \sqrt[k]{l_k/l_{k+1}} = +\infty. \tag{7}$$

We note that condition (7) holds provided $\omega_k \to +\infty$ $(k \to \infty)$. For the sequence $l_k = \exp\{-\omega k^2\}$ with $\omega > 0$ we have $\omega_k \searrow \omega (k \to \infty)$, and condition (7) holds if $l_k = \exp\{-\beta(k)k^2\}$, where $0 < \beta(x) \nearrow +\infty (k \to \infty)$.

It is shown in [3] that if $l \in A^+(+\infty)$ and the sequence $(\varkappa_k)$ is nonincreasing then $1 > \varkappa_k \searrow \varkappa \geq 0$ $(k_0 \leq k \to +\infty)$ and $\omega_k \nearrow \omega = \frac{-\ln\varkappa}{2} \leq +\infty$ $(k \to +\infty)$. Therefore, if $1 > \varkappa_k \searrow \varkappa \geq 0$ $(k_0 \leq k \to +\infty)$ then by Theorem 5, for every $n \in \mathbb{N}$, $\lambda \in \Lambda^*$ and $f \in A(0)$ the condition $D_l^n f \in A_\lambda(0)$ implies the inclusion $f \in A(+\infty)$. However, the following result is more general.

Theorem 6 ([3])**.** *Let $l \in A^+(+\infty)$ and $1 > \varkappa_k \searrow \varkappa \geq 0$ $(k_0 \leq k \to +\infty)$. Then for every $(n_p) \in N$, $\lambda \in \Lambda^*$ and $f \in A(0)$ condition (4) holds.*

In Theorems 3–6 $\lambda \in \Lambda^*$, i. e. the sequence λ can increase not faster than the exponential function. In [4] an analogue of Theorem 3 is obtained for the case if λ is allowed to increase considerable fast. We assume that a positive sequence $\psi = (\psi_k)$ satisfies the condition $\psi_k^2 \leq \psi_{k-1}\psi_{k+1}, k \geq 2$, and let $\Lambda_\psi = \{\lambda\colon \ln \lambda_k \leq \ln \psi_k + ak (k \in \mathbb{N}\}, a \equiv \text{const}$.

Theorem 7 ([4])**.** *Let $(n_p) \in N$, $l \in A_*^+(+\infty)$ and $\psi_k^2 \leq \psi_{k-1}\psi_{k+1}, k \geq 2$. Then for every $\lambda \in \Lambda_\psi$ and $f \in A(0)$ condition (4) holds if and only if*

$$\lim_{p\to+\infty} \frac{1}{n_p+1}\left\{\ln \frac{1}{l_{n_p+1}} - \sum_{j=1}^{p} \ln \frac{\psi_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}}\right\} = +\infty. \tag{8}$$

We remak that the condition $\psi_k^2 \leq \psi_{k-1}\psi_{k+1}, k \geq 2$ holds if, for example, $\psi_k = k!$ or $\psi_k = \exp\{\alpha k^n\}$ $(\alpha > 0, n \in \mathbb{N})$, and Theorem 7 has the following consequence.

Corollary 1 ([4])**.** *Let $l_k = \exp\{-\omega_1 k^2\}$ and $\psi_k = \exp\{-\omega_2 k^2\}$, $0 < \omega_1, \omega_2 < \infty$. Then for every $(n_p) \in N$, $\lambda \in \Lambda_\psi$ and $f \in A(0)$ condition (4) holds.*

3. In all mentioned results conditions on the Gelfond-Leont'ev derivatives of formal power series (1) implies that the convergence radius $R[f] = +\infty$. The following question naturally arises: find conditions on the Gelfond-Leont'ev derivatives, under which series (1) is convergent in some neighborhoods of the origin, i.e. $R[f] > 0$. Such results are obtained in the papers [5–7]. In particular, in [5] it is proved the following proposition, which is new also for the case $R[f] = +\infty$.

Proposition 1 ([5])**.** *Let $R \in (0, +\infty]$. Then $f \in A(R)$ if and only if there exists a sequence $\lambda \in \Lambda$ such that $f \in A_\lambda(0)$ and $\varlimsup\limits_{k\to\infty} \sqrt[k]{\lambda_k} \leq 1/R$.*

The following result generalizes Theorem 1.

Theorem 8 ([5])**.** *Let $l \in A^+(0)$. Then for every $f \in A(0)$ and $\lambda \in \Lambda$ the condition $(\forall n \in \mathbb{Z}_+)\{D_l^n f \in A_\lambda(0)\}$ implies the inclusion $f \in A(R)$ if and only if*

$$\varlimsup_{k\to+\infty} \sqrt[k]{l_k} \leq \frac{l_2}{l_1\lambda_2 R}. \tag{9}$$

Obviously, for $R = +\infty$ conditions (3) and (9) are equivalent. We remark also that Theorem 8 implies that if $l \in A^+(0)$ then in order that for every $f \in A(0)$ and $\lambda \in \Lambda$ the condition $(\forall n \in \mathbb{Z}_+)\{D_l^n f \in A_\lambda(0)\}$ imply the analyticity of f in some neighborhood of the origin, it is necessary and sufficient that $\sqrt[k]{l_k} = O(1), k \to \infty$.

The following two results are analogues of Theorem 2.

Theorem 9 ([6])**.** *Let $(n_p) \in N$. In order that for every $\lambda \in \Lambda$, $f \in A(0)$ and $l \in A^+(R)$ $(0 < R \leq +\infty)$ the condition $(\forall p \in \mathbb{Z}_+)$ $\{D_l^{n_p} f \in A_\lambda(0)\}$ imply the analyticity of f in some neighborhood of the origin is necessary and sufficient that condition (5) hold.*

Theorem 10 ([6])**.** *Let $(n_p) \in N$. For condition (5) to hold is necessary and sufficient that for every $\lambda \in \Lambda$, $f \in A(0)$ and $l \in A^+(R)$ $(0 < R \leq +\infty)$ the condition $(\forall p \in \mathbb{Z}_+)$ $\{D_l^{n_p} f \in A_\lambda(0)\}$ imply the estimate $R[f] \geq PR[l]$, where $R[f]$ and $R[l]$ are the convergence radii of the functions f and l, and P is a positive constant.*

We note that $P = \frac{1}{l_1 \max\{\lambda_k/l_k\colon 2\le k\le m+1\}}$, where $m = \max\{n_{p+1}-n_p\colon p\ge 0\}$, and the estimate $R[f]\ge PR[l]$ in Theorem 10 is sharp. Also if $\varlimsup\limits_{p\to+\infty}(n_{p+1}-n_p)=\infty$ then for every $R\in(0,+\infty]$ there exist $\lambda\in\Lambda$, $f\in A(0)$ and $l\in A^+(0)$ such that $(\forall p\in\mathbb{Z}_+)$ $\{D_l^{n_p}f\in A_\lambda(0)\}$, but $R[f]=0$ and $R[l]=R$.

We denote $\Lambda_* = \{\lambda\in\Lambda\colon \lambda_{k-1}\lambda_{k+1}\ge\lambda_k^2 (k\ge 2)\}$. Then Theorem 3 has the following analogue.

Theorem 11 ([7])**.** *Let $(n_p)\in N$. Then for every $\lambda\in\Lambda_*$, $l\in A_*^+(0)$ and $f\in A(0)$ the condition $(\forall p\in\mathbb{Z}_+)$ $\{D_l^{n_p}f\in A_\lambda(0)\}$ implies the inclusion $f\in A(R)$ if and only if*

$$\varliminf_{p\to+\infty}\frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p}\ln\frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}}\right\}\ge\ln R. \tag{10}$$

Each of the conditions $\lambda\in\Lambda_*$, $l\in A_*^+(0)$ in Theorem 11 cannot be relaxed. Theorem 11 is a consequence of the following result.

Theorem 12 ([7])**.** *Let $(n_p)\in N$, and let a sequence $\lambda\in\Lambda$ and a function $l\in A^+(0)$ be such that for all $p\in\mathbb{Z}_+$ and $k=2,\dots,n_{p+1}-n_p$*

$$\ln\frac{l_{n_p+k-1}l_{n_p+k+1}}{l_{n_p+k}^2} - \ln\frac{l_{k-1}l_{k+1}}{l_k^2} + \ln\frac{\lambda_{k-1}\lambda_{k+1}}{\lambda_k^2}\ge 0.$$

If $(\forall p\in\mathbb{Z}_+)$ $\{D_l^{n_p}f\in A_\lambda(0)\}$ then the estimate

$$\ln R[f]\ge\varliminf_{p\to+\infty}\frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p}\ln\frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}}\right\}$$

is true and sharp.

We remark that one can obtain an analogue of Theorem 12 for the case when the sequence $\lambda\in\Lambda$ satisfies a condition similar to $\lambda\in\Lambda^*$.

Theorem 13 ([7])**.** *Let $(n_p)\in N$, $l\in A_*^+(0)$ and let a sequence $\lambda\in\Lambda$ be such that $\ln\lambda_k\le a(k-1)$ for all $k\ge 1$ and some $a>0$. If $(\forall p\in\mathbb{Z}_+)$ $\{D_l^{n_p}f\in A_\lambda(0)\}$ then the estimate*

$$\ln R[f]\ge\varliminf_{p\to+\infty}\frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p}\ln\frac{1}{l_{n_j-n_{j-1}+1}}\right\} - a \tag{11}$$

is true and sharp.

In Theorem 13 one cannot replace the condition $\ln\lambda_k\le a(k-1)$ with the condition $\lambda\in\Lambda^*$ and, moreover, with the condition $\varlimsup\limits_{k\to\infty}(\ln\lambda_k)/k=a$. However, the following theorem is true.

Theorem 14 ([7])**.** *Let $(n_p)\in N$, $\ln\lambda_k=o(k)(k\to\infty)$ and let a function $l\in A^+(0)$ be such that the sequence $(\mu_{k-1}\mu_{k+1}/\mu_k^2)$ is nondecreasing, where $\mu_k=l_k/\lambda_k$. If $(\forall p\in\mathbb{Z}_+)$ $\{D_l^{n_p}f\in A_\lambda(0)\}$ then the estimate (11) with $a=0$ is true and sharp.*

4. Continuing investigations from [5–7], we show, at first, that the full analogue of Theorem 5 is valid.

Theorem 15. *Let $l \in A^+(0)$. Then for every $n \in \mathbb{N}$, $\lambda \in \Lambda^*$ and $f \in A(0)$ the condition $D_l^n f \in A_\lambda(0)$ implies the inequality $R[f] > 0$ if and only if*

$$\varliminf_{k\to\infty} \sqrt[k]{l_k/l_{k+1}} > 0. \tag{12}$$

Proof. We first assume that (l_k) satisfies condition (12) and $D_f^n \in A_\lambda(0)$ for some fixed number $n \in \mathbb{N}$. Then $\frac{l_k|f_{n+k}|}{l_{n+k}} \le \lambda_k \frac{l_1|f_{n+1}|}{l_{n+1}}$ $(k \ge 1)$, that is,

$$\frac{1}{n+k} \ln \frac{1}{|f_{n+k}|} \ge \frac{1}{n+k} \ln \frac{l_k}{l_{n+k}} - \frac{\ln \lambda_k}{n+k} - \frac{1}{n+k} \ln \left(\frac{l_1|f_{n+1}|}{f_{n+1}} \right)$$

and, thus,

$$\ln R[f] = \varliminf_{k\to\infty} \frac{1}{n+k} \ln \frac{1}{|f_{n+k}|} \ge \varliminf_{k\to\infty} \frac{1}{n+k} \ln \frac{l_k}{l_{n+k}} - a. \tag{13}$$

But by condition (12), there exists $c > 0$ such that $\sqrt[k]{l_k/l_{k+1}} \ge c$ for all $k \ge 1$. Therefore, we obtain from (13)

$$\ln R[f] \ge \varliminf_{k\to\infty} \frac{1}{n+k} \sum_{j=k}^{n+k-1} \ln \frac{l_j}{l_{j+1}} - a \ge \varliminf_{k\to\infty} \frac{1}{n+k} \sum_{j=k}^{n+k-1} j \ln c - a =$$
$$= \varliminf_{k\to\infty} \frac{\ln c}{n+k} \frac{2nk + n(n-1)}{2} - a = n \ln c - a > -\infty,$$

that is, $R[f] > 0$ and the sufficiency of (12) is proved.

Now we assume that condition (12) does not hold, i.e. there exists an increasing sequence (k_j) such that $\sqrt[k_j]{l_{k_j}/l_{k_j+1}} = \alpha_j \to 0$ $(j \to \infty)$. We put $k_0 = 0$, $f_{k_j+1} = \frac{l_{k_j+1}}{f_{k_j}}$ $(j \ge 0)$ and $f_k = 0$ for $k \ne k_j + 1$. For series (1) with such coefficients we have $\sqrt[k_j]{f_{k_j+1}} = \sqrt[k_j]{l_{k_j+1}/l_{k_j}} = (1/\alpha_j)^{k_j/(k_j+1)} \to +\infty$ $(j \to \infty)$, that is, $R[f] = 0$.

On the other hand, for the series we have

$$D_f^1(z) = \sum_{j=0}^{\infty} \frac{l_{k_j}}{l_{k_j+1}} l_{k_j+1} z^{k_j+1} = z + \sum_{j=1}^{\infty} z^{k_j+1},$$

that is, $D_f^1 \in A_\lambda(0)$, if we choose $\lambda_k \equiv 1$. Thus, if condition (12) does not hold then there exist $n = 1$, $\lambda \in \Lambda^*$ and a formal power series (1) such that $D_f^n \in A_\lambda(0)$, but $R[f] = 0$. Theorem 15 is proved. □

If we denote $c = \varliminf_{k\to\infty} \sqrt[k]{l_k/l_{k+1}}$ then one can show using arguments analogous to that in the proof of Theorem 15 that the estimate $\ln R[f] \ge n \ln c - a$ is valid. On the other hand, if we choose $l_k = c^{k(k+1)} (k \ge 0)$, $\lambda_k = e^{a(k-1)} (k \ge 1)$, $f_0 = \cdots = f_n = 0, f_{n+1} = 1$ and $f_{n+k} = f_{n+1}\lambda_k l_{n+k}/l_k (k \ge 1)$ then for series (1) with such coefficients we have $D_l^n f(z) = \sum_{k=0}^{\infty} \lambda_k z^k$ (i.e. $D_l^n f \in A_\lambda(0)$) and $\ln R[f] = \varliminf_{k\to\infty} \frac{1}{n+k} \left(\ln \frac{l_k}{l_{n+k}} - a(k-1) \right) = n \ln c - a$. Thus, the following proposition is true.

Proposition 2. *Let $n \in \mathbb{N}$, $\lambda \in \Lambda^*$, $l \in A^+(0)$ and $\varliminf_{k\to\infty} \sqrt[k]{l_k/l_{k+1}} = c$. If $D_l^n f \in A_\lambda(0)$ then the estimate $R[f] \ge c^n e^{-a}$ is valid and sharp.*

As above, we denote $\varkappa_k = \frac{l_{k-1}l_{k+1}}{l_k^2}$ and $\omega_k = \frac{1}{k+1}\ln\frac{1}{l_{k+1}} - \frac{1}{k}\ln\frac{1}{l_k}$. It is easy to show [2–3] that $\omega_k = \frac{1}{k(k+1)}\sum_{j=1}^{k} j\ln\frac{1}{\varkappa_j}$. Hence, if $\lim_{k\to\infty}\varkappa_k = \varkappa$ then $\lim_{k\to\infty}\omega_k = \omega = \frac{1}{2}\ln\frac{1}{\varkappa}$. On the other hand,

$$\ln c = \varliminf_{k\to\infty}\Big(\frac{1}{k}\ln\frac{1}{l_{k+1}} - \frac{1}{k}\ln\frac{1}{l_k}\Big) = \varliminf_{k\to\infty}\Big(\omega_k + \frac{1}{k(k+1)}\ln\frac{1}{l_{k+1}}\Big) =$$
$$= \varliminf_{k\to\infty}\Big(\omega_k + \frac{1}{k}\sum_{j=1}^{k}\omega_j\Big) = 2\omega = \ln\frac{1}{\varkappa}.$$

Therefore, Proposition 2 and arguments of its proof yield the following result.

Theorem 16. *Let* $n\in\mathbb{N}$, $\lambda\in\Lambda^*$, $l\in A^+(0)$ *and* $\lim_{k\to\infty}\varkappa_k = \varkappa$. *If* $D_l^n f\in A_\lambda(0)$ *then the estimate* $R[f]\ge e^{2n\omega}e^{-a} = \varkappa^{-n}e^{-a}$ *is valid and sharp.*

Theorem 16 has the following corollary.

Corollary 2. *Let* $(n_p)\in N$, $\lambda\in\Lambda^*$, $l\in A^+(0)$ *and* $\lim_{k\to\infty}\varkappa_k = \varkappa < 1$ *(i. e.* $\omega > 0$*). If* $D_l^{n_p} f\in A_\lambda(0)$ *for all* $p\in\mathbb{Z}_+$ *then* $R[f] = +\infty$.

Since $\varkappa < 1$ implies $l\in A^+(+\infty)$, from Corollary 2 and, thus, from Theorem 16 we obtain Theorem 6 and the sufficiency of the condition $\omega > 0$ in Theorem 4.

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Received 7.12.2008