THE INDICATOR OF ENTIRE FUNCTIONS WITH RAPIDLY OSCILLATING COEFFICIENTS


Let \( f(z) = \sum_{n=0}^{\infty} c_n z^n \) be an entire function of order \( \rho_f \in (0, +\infty) \), let \( \sigma_f \) and \( h_f(\theta) \) be the type and the indicator of the function \( f \), respectively, let \( (p_n) \) be a sequence of positive integers with Hadamard gaps and \( f_t(z) = \sum_{n=0}^{\infty} e^{ip_n t} c_n z^n \), where \( t \in \mathbb{R} \). Then, for almost every \( t \in \mathbb{R} \), the equality \( h_f(t)(\theta) = \sigma_f \) holds for every \( \theta \in \mathbb{R} \).

1. Introduction. For any transcendental entire function

\[ f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1} \]

let \( M_f(r) = \max\{|f(z)| : |z| = r\} \) be the maximum modulus, and \( S_f(r) = \left( \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \right)^{\frac{1}{2}} \) be the mean. As usual, we define the order \( \rho_f \) of the function \( f \), and, in case when \( 0 < \rho_f < +\infty \), its type \( \sigma_f \) and indicator \( h_f(\theta) \) in accordance with equalities

\[ \rho_f = \lim_{r \to +\infty} \frac{\ln \ln M_f(r)}{\ln r}, \quad \sigma_f = \lim_{r \to +\infty} \frac{\ln M_f(r)}{r^{\rho_f}}, \quad h_f(\theta) = \lim_{r \to +\infty} \frac{\ln |f(re^{i\theta})|}{r^{\rho_f}}. \]

We recall that, by the classical Hadamard formulas,

\[ \rho_f = \frac{n}{n \to -\infty} \frac{n \ln n}{\ln |c_n|}, \quad \sigma_f = \frac{n}{n \to -\infty} \frac{n|c_n|^\rho_f/n}{e^{\rho_f}}. \]

For a set \( A \subset \mathbb{R} \) by \( A' = \mathbb{R} \setminus A \).

Let \( (p_n) \) be a sequence of positive integers with Hadamard gaps, i.e. there exists a number \( q > 1 \) such that

\[ \frac{p_{n+1}}{p_n} > q \quad (n \in \mathbb{Z}_+). \tag{2} \]

2010 Mathematics Subject Classification: 30D20, 30B20.
Along with the entire function (1) we consider the function

\[ f_t(z) = \sum_{n=0}^{\infty} e^{ip_n t} c_n z^n, \quad t \in \mathbb{R}. \quad (3) \]

Using Hadamard's formulas, we obtain \( \rho_{f_t} = \rho_f, \sigma_{f_t} = \sigma_f \) for all \( t \in \mathbb{R} \). Furthermore, \( S_{f_t}(r) = S_f(r) \) and, by the Parseval equality,

\[
\int_0^{2\pi} |f_t(re^{i\theta})|^2 d\theta = 2\pi S_f^2(r) \quad (t \in \mathbb{R}); \quad \int_0^{2\pi} |f_t(z)|^2 dt = 2\pi S_f^2(r) \quad (z \in \mathbb{C}).
\]

On the international conference on complex analysis in memory of A. A. Gol’dberg (Lviv, 2010) in a conversation with the author A. Eremenko formulated the following question: \textit{is it true that for an entire function of the form (3) almost everywhere by \( t \) the equality \( h_{f_t}(\theta) = \sigma_f \) holds for all \( \theta \in \mathbb{R} \) (i. e., \( h_{f_t}(\theta) \equiv \sigma_f \))?}

The answer to this question is positive.

**Theorem 1.** Let \( f \) be an entire function of the form (1) such that \( \rho_f \in (0, +\infty) \), let \((p_n)\) be a sequence of positive integers with Hadamard gaps, and let \( f_t \) be the entire function of the form (3), and \( A = \{ t \in \mathbb{R} : h_{f_t}(\theta) \equiv \sigma_f \} \). Then \( A' \) is a set of Lebesgue measure zero and first Baire category in \( \mathbb{R} \).

It is easy to show that Theorem 1 follows from

**Theorem 2.** Let \( f \) be a transcendental entire function of the form (1), \((p_n)\) be a sequence of positive integers with Hadamard gaps, \( f_t \) be the entire function of the form (3), \( \theta \in \mathbb{R} \) be a fixed number, \((r_k)\) be a sequence increasing to \(+\infty\), and

\[
B = \left\{ t \in \mathbb{R} : \lim_{k \to \infty} \frac{|f_t(r_k e^{i\theta})|}{S_f(r_k)} \geq 1 \right\}.
\]

Then \( B' \) is a set of Lebesgue measure zero and first Baire category in \( \mathbb{R} \).

Using Theorem 1 and the Pólya theorem on the connection between the conjugate diagram and the indicator diagram of an entire function of exponential type (see for example [1, p. 114]), we prove the following statement.

**Theorem 3.** Let \( f \) be an analytic function in the disk \( \{ z \in \mathbb{C} : |z| < 1 \} \), represented by power series (1) with the radius of convergence \( R_f = 1 \), let \((p_n)\) be a sequence of positive integers with Hadamard gaps, and let \( C \) be the set of all \( t \in \mathbb{R} \) such that the circle \( \{ z \in \mathbb{C} : |z| = 1 \} \) is the natural boundary for the function (3). Then \( C' \) is a set of Lebesgue measure zero and first Baire category in \( \mathbb{R} \).

Note that the entire functions of the form (3) was introduced by J. M. Steele [2] and called by entire functions with rapidly oscillating coefficients. Properties of such functions were studied also in [3]–[5]. In [6] entire functions were considered with rapidly oscillating coefficients of two variables.

If a sequence \((p_n)\) of positive integers satisfies condition (2) with \( q = 2 \), then the sequences \((\cos 2\pi p_n t)\) and \((\sin 2\pi p_n t)\) are multiplicative systems (see for example [7]). Properties of analytic functions, represented by power series of the form \( f_t(z) = \sum_{n=0}^{\infty} (X_n(t) + iY_n(t)) c_n z^n \)
or their bivariate analogies, in the case when \((X_n(t))\) and \((Y_n(t))\) are multiplicative systems were investigated in [8]–[13].

Some applications the Baire categories to the theory of analytic functions are given in [14]–[18], [5].

2. Auxiliary results. In the proof of Theorem 2 the main tool is Lemma 2 given below. We obtain Lemma 2 with the help of the following lemma of A. Zygmund [19, p. 326].

Lemma 1. Let \(E \subset [0, 2\pi]\) be a set of positive measure \(\delta\), and let \(q > 1\) be some number. Then for each \(\lambda > 1\) there exists a positive integer \(h_0 = h_0(\lambda)\) such that for every trigonometric series \(P(t) = \sum_{n=0}^{\infty}(a_n \cos(p_n t) + b_n \sin(p_n t))\) with \(a_n, b_n \in \mathbb{R}, p_n \in \mathbb{N}, \frac{p_{n+1}}{p_n} > q, p_0 \geq h_0\) and \(\sum_{n=0}^{\infty}(a_n^2 + b_n^2) < +\infty\) we have

\[
\frac{\delta}{2\lambda} \sum_{n=0}^{\infty}(a_n^2 + b_n^2) \leq \int_E P^2(t) dt \leq \frac{\delta \lambda}{2} \sum_{n=0}^{\infty}(a_n^2 + b_n^2).
\]

Lemma 2. Let \(E \subset [0, 2\pi]\) be a set of positive measure \(\delta\), \(f\) a transcendental entire function of the form (1), \((p_n)\) a sequence of positive integers with Hadamard gaps, and \(\theta \in \mathbb{R}\) a fixed number. Then for the function given by (3) we have

\[
\lim_{r \to +\infty} \frac{\int_E |f_t(r e^{i\theta})|^2 dt}{S_f^2(r)} = \delta.
\] (4)

Proof. Let \(E \subset [0, 2\pi]\) be a set of positive measure \(\delta\), \((p_n)\) a sequence of positive integers with Hadamard gaps, and \(q > 1\) some number such that (2) holds. We fix an arbitrary number \(\lambda > 1\) and let \(h_0 = h_0(\lambda)\) be a positive integer whose existence follows from Lemma 1. Put \(n_0 = \max\{n_1, h_0\}\).

We consider a transcendental entire function \(f\) of the form (1) and set

\[
p(z) = \sum_{n=0}^{n_0-1} c_n z^n, \quad g(z) = \sum_{n=n_0}^{\infty} c_n z^n, \quad p_t(z) = \sum_{n=0}^{n_0-1} e^{ip_n t} c_n z^n, \quad g_t(z) = \sum_{n=n_0}^{\infty} e^{ip_n t} c_n z^n.
\]

Then we put \(\gamma_n = \arg c_n\), and let \(P_t^1(z)\) and \(P_t^2(z)\) be the real and imaginary parts of the function \(g_t(z)\), respectively. Then, as easily verified,

\[
P_t^1(re^{i\theta}) = \sum_{n=n_0}^{\infty} (|c_n|r^n \cos(\theta + \gamma_n) \cos p_n t - |c_n|r^n \sin(\theta + \gamma_n) \sin p_n t),
\]

\[
P_t^2(re^{i\theta}) = \sum_{n=n_0}^{\infty} (|c_n|r^n \sin(\theta + \gamma_n) \cos p_n t + |c_n|r^n \cos(\theta + \gamma_n) \sin p_n t).
\]

By Lemma 1, for \(j = 1, 2\) we obtain

\[
\frac{\delta}{2\lambda} S_{g}^2(r) \leq \int_E (P_t^j(re^{i\theta}))^2 dt \leq \frac{\delta \lambda}{2} S_{g}^2(r),
\]

whence it follows that

\[
\frac{\delta}{\lambda} S_{g}^2(r) \leq \int_E |g_t(re^{i\theta})|^2 dt \leq \delta \lambda S_{g}^2(r).
\] (5)
Since the entire function $f$ is transcendental, we obtain \( \ln r = o(\ln S_f(r)) (r \to +\infty) \). Hence,
\[
(\forall \varepsilon > 0)(\exists r_0(\varepsilon))(\forall r \geq r_0(\varepsilon)) : S_p(r) \leq S_f^e(r),
\]
from which, in particular, we have
\[
S_f(r) \sim S_g(r) \quad (r \to +\infty).
\]

Next, note that
\[
\int_E |f_t(re^{i\theta})|^2 dt = \int_E |g_t(re^{i\theta})|^2 dt + \int_E |p_t(re^{i\theta})|^2 dt + \int_E (p_t(re^{i\theta})g_t(re^{i\theta}) + \overline{p_t(re^{i\theta})}g_t(re^{i\theta})) dt.
\]

Because
\[
\int_E |p_t(re^{i\theta})|^2 dt \leq \int_0^{2\pi} |p_t(re^{i\theta})|^2 dt = 2\pi S_g^2(r)
\]
and, by the Schwarz inequality,
\[
\left| \int_E (p_t(re^{i\theta})g_t(re^{i\theta}) + \overline{p_t(re^{i\theta})}g_t(re^{i\theta})) dt \right| \leq 2 \int_E |p_t(re^{i\theta})||g_t(re^{i\theta})| dt \leq 2 \int_0^{2\pi} |p_t(re^{i\theta})||g_t(re^{i\theta})| dt \leq 2 \left( \int_0^{2\pi} |p_t(re^{i\theta})|^2 dt \right)^{\frac{1}{2}} \left( \int_0^{2\pi} |g_t(re^{i\theta})|^2 dt \right)^{\frac{1}{2}} = 4\pi S_p(r)S_g(r),
\]

we have, using (6),
\[
\int_E |f_t(re^{i\theta})|^2 dt = \int_E |g_t(re^{i\theta})|^2 dt + o(S_f^2(r)) \quad (r \to +\infty).
\]

From (5), (7) and (8) it follows that
\[
\frac{\delta}{\lambda} \leq \lim_{r \to +\infty} \frac{\int_E |f_t(re^{i\theta})|^2 dt}{S_f^2(r)} \leq \lim_{r \to +\infty} \frac{\int_E |f_t(re^{i\theta})|^2 dt}{S_g^2(r)} \leq \delta \lambda,
\]
from which, due to the arbitrariness of \( \lambda > 1 \), we have (4). This completes the proof of Lemma 2.

**3. Proof of Theorem 2.** Suppose that the conditions of Theorem 2 are satisfied. For arbitrary positive integers \( k, n, m \) we introduce the set
\[
D_{k,n} = \left\{ t \in \mathbb{R} : |f_t(re^{i\theta})| \leq \left( 1 - \frac{1}{n} \right) S_f(r) \right\}, \quad E_{m,n} = \bigcap_{k=m}^{\infty} D_{k,n}.
\]

Since
\[
t \in B' \iff (\exists m)(\exists n)(\forall k \geq m) : t \in D_{k,n} \iff (\exists m)(\exists n) : t \in E_{m,n} \iff t \in \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E_{m,n},
\]
we have that \( B' = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E_{m,n} \).

First, we prove that for any positive integers \( m \) and \( n \) the set \( E_{m,n} \) is of Lebesgue measure zero. Suppose the contrary, i.e. there exist some fixed positive integers \( m \) and \( n \) such that
the measure of the set $E_{m,n}$ is not zero. Then, from the periodicity of the function $f_t(z)$ as a function of $t$ it follows that the set $E = E_{m,n} \cap [0,2\pi]$ has positive measure $\delta$. Applying Lemma 2, we get

$$\lim_{k \to \infty} \frac{\int_E |f_t(r_ke^{i\theta})|^2 dt}{S_f^2(r_k)} = \delta. \quad (9)$$

On the other hand, if $t \in E$, then $t \in D_{k,n}$ for all $k \geq m$. Therefore,

$$\int_E |f_t(r_ke^{i\theta})|^2 dt \leq \delta \left( 1 - \frac{1}{n} \right)^2 S_f^2(r_k) \quad (k \geq m),$$

which contradicts the relation (9). Thus, $E_{m,n}$ is a set of measure zero for all positive integers $m$ and $n$.

Next, we prove that the set $E_{m,n}$ is nowhere dense. Let $(a,b)$ be an arbitrary interval of the real line. Since the measure of the set $E_{m,n}$ is zero, this interval contains a point $t_0 \notin E_{m,n}$. Then $t_0 \notin D_{k,n}$ for some $k \geq m$, i. e.

$$|f_{t_0}(r_ke^{i\theta})| > \left( 1 - \frac{1}{n} \right) S_f(r_k).$$

From the continuity of the function $f_t(z)$ as a function of $t$ it follows that for all $t$ in some neighborhood $(c,d) \subset (a,b)$ of the point $t_0$ the inequality

$$|f_t(r_ke^{i\theta})| > \left( 1 - \frac{1}{n} \right) S_f(r_k)$$

holds, i. e. $(c,d) \subset E'_{m,n}$. This means that the set $E_{m,n}$ is nowhere dense.

Since the set $B'$ is a countable union of nowhere dense sets of measure zero, this set is of first Baire category and Lebesgue measure zero. Theorem 2 is proved.

4. Proof of Theorem 1. It is well known that for any transcendental entire function $f$ of the form (1) in the definition of its order $\rho_f$ and, if $0 < \rho_f < +\infty$, in the definition of its type $\sigma_f$ we can replace $M_f(r)$ with $S_f(r)$. It is easy to see that this fact follows from the inequalities $S_f(r) \leq M_f(r)$ and

$$M_f(r) \leq \sum_{n=0}^{\infty} |c_n|r^n = \sum_{n=0}^{\infty} |c_n|(qr)^n \frac{1}{q^n} \leq \left( \sum_{n=0}^{\infty} |c_n|^2(qr)^{2n} \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} \frac{1}{q^{2n}} \right)^{\frac{1}{2}} = S_f(qr) \left( \frac{q^2}{q^2 - 1} \right)^{\frac{1}{2}} \quad (q > 1, \ r \geq 0).$$

Let $f$ be an arbitrary entire function of the form (1) and of the order $\rho_f \in (0, +\infty)$. From what has been said it follows that there exists a positive sequence $(r_k)$ increasing to $+\infty$ such that

$$\lim_{k \to \infty} \frac{\ln S_f(r_k)}{r_k^{\rho_f}} = \sigma_f. \quad (10)$$

Consider any sequence $(p_n)$ of positive integers with Hadamard gaps and let $\Theta$ be a countable and everywhere dense set in $\mathbb{R}$ (for example $\Theta = \mathbb{Q}$). For the function given by (3) and every $\theta \in \Theta$ we put

$$B_{\theta} = \left\{ t \in \mathbb{R}: \lim_{k \to \infty} \frac{|f_t(r_ke^{i\theta})|}{S_f(r_k)} \geq 1 \right\}.$$
By Theorem 2, each of the sets $B'_t$ is of measure zero and first Baire category. It is clear that the set $F = \bigcup_{\theta \in \Theta} B'_t$ is also of measure zero and first Baire category. Therefore, to complete the proof of Theorem 1 it suffices to show that $A' \subset F$.

We fix some $t \notin F$. For every $\theta \in \Theta$ we have $t \notin B'_t$, i.e. $t \in B_t$. By the definition of the set $B_t$ and the relation (10) we obtain $h_{f_t}(\theta) = \sigma_f(\theta \in \Theta)$. Then, since the set $\Theta$ is everywhere dense and the indicator $h_{f_t}(\theta)$ is a continuous function, $h_{f_t}(\theta) \equiv \sigma_f$, i.e. $t \in A$, and therefore $t \notin A'$. Consequently, from $t \notin F$ it follows that $t \notin A'$.

This implies that $A' \subset F$. Theorem 1 is proved.

5. Proof of Theorem 3. Let $f$ be an analytic function in the disk $\{z \in \mathbb{C} : |z| < 1\}$, represented by power series (1) with the radius of convergence $R_f = 1$. Set

$$
\varphi(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}}, \quad g(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n.
$$

It is easy to see that the function $\varphi$ is analytic in the domain $\{z \in \mathbb{C} : |z| > 1\}$, and also some point $e^{i\gamma}$ on the unit circle is a singular point of the function $\varphi$ if and only if $e^{-i\gamma}$ is a singular point of $f$. Moreover, by Stirling’s formula and Hadamard’s formulas given above, $g$ is an entire function of order $\rho_g = 1$ and type $\sigma_g = 1$.

Let $I \subset \{z \in \mathbb{C} : |z| \leq 1\}$ be the conjugate diagram of the function $g$, i.e. the smallest convex compact set containing all singularities of the function $\varphi$, and let $k_g(\theta)$ be the supporting function of the set $I$. By the Polya theorem on the connection between the conjugate diagram and the indicator diagram of an entire function of exponential type, we have $k_g(\theta) \equiv h_g(\theta)$. From this and from the continuity of the indicator it follows immediately the equivalence of the following assertions:

(i) there exists a point $\theta \in \mathbb{R}$ such that $h_g(\theta) < 1$;
(ii) $h_g(\theta) < 1$ in some interval;
(iii) $k_g(\theta) < 1$ in some interval;
(iv) the function $\varphi$ can be analytically continued through some arc of the unit circle in some domain $G_1$ such that $G_1 \cap \{z \in \mathbb{C} : |z| < 1\} \neq \emptyset$;
(v) the function $f$ can be analytically continued through some arc of the unit circle in some domain $G_2$ such that $G_2 \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset$.

Therefore, we can conclude that the unit circle is the natural boundary for the function $f$ if and only if $h_g(\theta) \equiv 1$.

Let $(p_n)$ be a sequence of positive integers with Hadamard gaps, and let $C'$ be the set of all $t \in \mathbb{R}$ such that the circle $\{z \in \mathbb{C} : |z| = 1\}$ is the natural boundary for the function (3). Set

$$
g_t(z) = \sum_{n=0}^{\infty} e^{ip_n t} \frac{c_n}{n!} z^n, \quad A = \{t \in \mathbb{R} : h_{g_t}(\theta) \equiv 1\}.
$$

From what has been said above it follows that $C' = A'$. Then by Theorem 1 the set $C'$ is of the Lebesgue measure zero and first Baire category. Theorem 3 is proved.

REFERENCES