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THE INDICATOR OF ENTIRE FUNCTIONS WITH RAPIDLY OSCILLATING COEFFICIENTS

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Let $f(z)=\sum_{n=0}^{\infty} c_nz^n$ be an entire function of order $\rho_f\in(0,+\infty)$, let σ_f and $h_f(\theta)$ be the type and the indicator of the function f, respectively, let (p_n) be a sequence of positive integers with Hadamard gaps and $f_t(z)=\sum_{n=0}^{\infty} e^{ip_nt}c_nz^n$, where $t\in\mathbb{R}$. Then, for almost every $t\in\mathbb{R}$, the equality $h_{f_t}(\theta)=\sigma_f$ holds for every $\theta\in\mathbb{R}$.

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Пусть $f(z)=\sum_{n=0}^{\infty} c_nz^n$ — целая функция порядка $\rho_f\in(0,+\infty)$, σ_f и $h_f(\theta)$ — соответственно тип и индикатор функции f, (p_n) — лакунарная по Адамару последовательность натуральных чисел и $f_t(z)=\sum_{n=0}^{\infty} e^{ip_nt}c_nz^n$, где $t\in\mathbb{R}$. Тогда почти наверное по $t\in\mathbb{R}$ равенство $h_{f_t}(\theta)=\sigma_f$ выполняется для всех $\theta\in\mathbb{R}$.

1. Introduction. For any transcendental entire function

$$f(z)=\sum_{n=0}^{\infty} c_nz^n \tag{1}$$

let $M_f(r)=\max\{|f(z)|\colon |z|=r\}$ be the maximum modulus, and $S_f(r)=(\sum_{n=0}^{\infty}|c_n|^2r^{2n})^{\frac12}$ be the mean. As usual, we define the order ρ_f of the function f, and, in case when $0<\rho_f<+\infty$, its type σ_f and indicator $h_f(\theta)$ in accordance with equalities

$$\rho_f=\varlimsup_{r\to+\infty}\frac{\ln\ln M_f(r)}{\ln r},\quad \sigma_f=\varlimsup_{r\to+\infty}\frac{\ln M_f(r)}{r^{\rho_f}},\quad h_f(\theta)=\varlimsup_{r\to+\infty}\frac{\ln|f(re^{i\theta})|}{r^{\rho_f}}.$$

We recall that, by the classical Hadamard formulas,

$$\rho_f=\varlimsup_{n\to\infty}\frac{n\ln n}{-\ln|c_n|},\quad \sigma_f=\varlimsup_{n\to\infty}\frac{n|c_n|^{\rho_f/n}}{e\rho_f}.$$

For a set $A\subset\mathbb{R}$ by A' we denote its complement in $\mathbb{R}$: $A'=\mathbb{R}\backslash A$.

Let (p_n) be a sequence of positive integers with Hadamard gaps, i.e. there exists a number $q>1$ such that

$$\frac{p_{n+1}}{p_n}>q\quad (n\in\mathbb{Z}_+). \tag{2}$$

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Along with the entire function (1) we consider the function

$$f_t(z) = \sum_{n=0}^{\infty} e^{ip_n t} c_n z^n, \quad t \in \mathbb{R}. \tag{3}$$

Using Hadamard's formulas, we obtain $\rho_{f_t} = \rho_f$, $\sigma_{f_t} = \sigma_f$ for all $t \in \mathbb{R}$. Furthermore, $S_{f_t}(r) = S_f(r)$ and, by the Parseval equality,

$$\int_0^{2\pi} |f_t(re^{i\theta})|^2 d\theta = 2\pi S_f^2(r) \quad (t \in \mathbb{R}); \qquad \int_0^{2\pi} |f_t(z)|^2 dt = 2\pi S_f^2(r) \quad (z \in \mathbb{C}).$$

On the international conference on complex analysis in memory of A. A. Gol'dberg (Lviv, 2010) in a conversation with the author A. Eremenko formulated the following question: *is it true that for an entire function of the form (3) almost everywhere by t the equality $h_{f_t}(\theta) = \sigma_f$ holds for all $\theta \in \mathbb{R}$ (i. e., $h_{f_t}(\theta) \equiv \sigma_f$)?*

The answer to this question is positive.

Theorem 1. *Let f be an entire function of the form (1) such that $\rho_f \in (0, +\infty)$, let (p_n) be a sequence of positive integers with Hadamard gaps, and let f_t be the entire function of the form (3), and $A = \{t \in \mathbb{R} \colon h_{f_t}(\theta) \equiv \sigma_f\}$. Then A' is a set of Lebesgue measure zero and first Baire category in $\mathbb{R}$.*

It is easy to show that Theorem 1 follows from

Theorem 2. *Let f be a transcendental entire function of the form (1), (p_n) be a sequence of positive integers with Hadamard gaps, f_t be the entire function of the form (3), $\theta \in \mathbb{R}$ be a fixed number, (r_k) be a sequence increasing to $+\infty$, and*

$$B = \left\{ t \in \mathbb{R} \colon \overline{\lim_{k\to\infty}} \frac{|f_t(r_k e^{i\theta})|}{S_f(r_k)} \geq 1 \right\}.$$

Then B' is a set of Lebesgue measure zero and first Baire category in $\mathbb{R}$.

Using Theorem 1 and the Pólya theorem on the connection between the conjugate diagram and the indicator diagram of an entire function of exponential type (see for example [1, p. 114]), we prove the following statement.

Theorem 3. *Let f be an analytic function in the disk $\{z \in \mathbb{C} \colon |z| < 1\}$, represented by power series (1) with the radius of convergence $R_f = 1$, let (p_n) be a sequence of positive integers with Hadamard gaps, and let C be the set of all $t \in \mathbb{R}$ such that the circle $\{z \in \mathbb{C} \colon |z| = 1\}$ is the natural boundary for the function (3). Then C' is a set of Lebesgue measure zero and first Baire category in $\mathbb{R}$.*

Note that the entire functions of the form (3) was introduced by J. M. Steeele [2] and called by entire functions with rapidly oscillating coefficients. Properties of such functions were studied also in [3]–[5]. In [6] entire functions were considered with rapidly oscillating coefficients of two variables.

If a sequence (p_n) of positive integers satisfies condition (2) with $q = 2$, then the sequences $(\cos 2\pi p_n t)$ and $(\sin 2\pi p_n t)$ are multiplicative systems (see for example [7]). Properties of analytic functions, represented by power series of the form $f_t(z) = \sum_{n=0}^{\infty}(X_n(t) + iY_n(t))c_n z^n$

or their bivariate analogies, in the case when $(X_n(t))$ and $(Y_n(t))$ are multiplicative systems were investigated in [8]–[13].

Some applications the Baire categories to the theory of analytic functions are given in [14]–[18], [5].

2. Auxiliary results. In the proof of Theorem 2 the main tool is Lemma 2 given below. We obtain Lemma 2 with the help of the following lemma of A. Zygmund [19, p. 326].

Lemma 1. *Let $E \subset [0, 2\pi]$ be a set of positive measure δ, and let $q > 1$ be some number. Then for each $\lambda > 1$ there exists a positive integer $h_0 = h_0(\lambda)$ such that for every trigonometric series $P(t) = \sum_{n=0}^{\infty}(a_n \cos p_n t + b_n \sin p_n t)$ with $a_n, b_n \in \mathbb{R}$, $p_n \in \mathbb{N}$, $\frac{p_{n+1}}{p_n} > q$, $p_0 \ge h_0$ and $\sum_{n=0}^{\infty}(a_n^2 + b_n^2) < +\infty$ we have*

$$\frac{\delta}{2\lambda}\sum_{n=0}^{\infty}(a_n^2 + b_n^2) \le \int_E P^2(t)dt \le \frac{\delta\lambda}{2}\sum_{n=0}^{\infty}(a_n^2 + b_n^2).$$

Lemma 2. *Let $E \subset [0, 2\pi]$ be a set of positive measure δ, f a transcendental entire function of the form (1), (p_n) a sequence of positive integers with Hadamard gaps, and $\theta \in \mathbb{R}$ a fixed number. Then for the function given by (3) we have*

$$\lim_{r\to+\infty} \frac{\int_E |f_t(re^{i\theta})|^2 dt}{S_f^2(r)} = \delta. \tag{4}$$

Proof. Let $E \subset [0, 2\pi]$ be a set of positive measure δ, (p_n) a sequence of positive integers with Hadamard gaps, and $q > 1$ some number such that (2) holds. We fix an arbitrary number $\lambda > 1$ and let $h_0 = h_0(\lambda)$ be a positive integer whose existence follows from Lemma 1. Put $n_0 = \max\{n_1, h_0\}$.

We consider a transcendental entire function f of the form (1) and set

$$p(z) = \sum_{n=0}^{n_0-1} c_n z^n, \quad g(z) = \sum_{n=n_0}^{\infty} c_n z^n, \quad p_t(z) = \sum_{n=0}^{n_0-1} e^{ip_n t} c_n z^n, \quad g_t(z) = \sum_{n=n_0}^{\infty} e^{ip_n t} c_n z^n.$$

Then we put $\gamma_n = \arg c_n$, and let $P_t^1(z)$ and $P_t^2(z)$ be the real and imaginary parts of the function $g_t(z)$, respectively. Then, as easily verified,

$$P_t^1(re^{i\theta}) = \sum_{n=n_0}^{\infty}(|c_n|r^n \cos(\theta + \gamma_n)\cos p_n t - |c_n|r^n \sin(\theta + \gamma_n)\sin p_n t),$$

$$P_t^2(re^{i\theta}) = \sum_{n=n_0}^{\infty}(|c_n|r^n \sin(\theta + \gamma_n)\cos p_n t + |c_n|r^n \cos(\theta + \gamma_n)\sin p_n t).$$

By Lemma 1, for $j = 1, 2$ we obtain

$$\frac{\delta}{2\lambda}S_g^2(r) \le \int_E (P_t^j(re^{i\theta}))^2 dt \le \frac{\delta\lambda}{2}S_g^2(r),$$

whence it follows that

$$\frac{\delta}{\lambda}S_g^2(r) \le \int_E |g_t(re^{i\theta})|^2 dt \le \delta\lambda S_g^2(r). \tag{5}$$

Since the entire function f is transcendental, we obtain $\ln r = o(\ln S_f(r))$ $(r \to +\infty)$. Hence,

$$(\forall \varepsilon > 0)(\exists r_0(\varepsilon))(\forall r \geq r_0(\varepsilon))\colon S_p(r) \leq S_f^{\varepsilon}(r), \tag{6}$$

from which, in particular, we have

$$S_f(r) \sim S_g(r) \quad (r \to +\infty). \tag{7}$$

Next, note that

$$\int_E |f_t(re^{i\theta})|^2 dt = \int_E |g_t(re^{i\theta})|^2 dt + \int_E |p_t(re^{i\theta})|^2 dt + \int_E \Big(p_t(re^{i\theta})\overline{g_t(re^{i\theta})} + \overline{p_t(re^{i\theta})} g_t(re^{i\theta})\Big) dt.$$

Because

$$\int_E |p_t(re^{i\theta})|^2 dt \leq \int_0^{2\pi} |p_t(re^{i\theta})|^2 dt = 2\pi S_g^2(r)$$

and, by the Schwarz inequality,

$$\left|\int_E \Big(p_t(re^{i\theta})\overline{g_t(re^{i\theta})} + \overline{p_t(re^{i\theta})} g_t(re^{i\theta})\Big) dt\right| \leq 2\int_E |p_t(re^{i\theta}||g_t(re^{i\theta})| dt \leq$$

$$\leq 2\int_0^{2\pi} |p_t(re^{i\theta}||g_t(re^{i\theta})| dt \leq 2\left(\int_0^{2\pi} |p_t(re^{i\theta}|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{2\pi} |g_t(re^{i\theta}|^2 dt\right)^{\frac{1}{2}} = 4\pi S_p(r) S_g(r),$$

we have, using (6),

$$\int_E |f_t(re^{i\theta})|^2 dt = \int_E |g_t(re^{i\theta})|^2 dt + o(S_f^2(r)) \quad (r \to +\infty). \tag{8}$$

From (5), (7) and (8) it follows that

$$\frac{\delta}{\lambda} \leq \varliminf_{r\to+\infty} \frac{\int_E |f_t(re^{i\theta})|^2 dt}{S_f^2(r)} \leq \varlimsup_{r\to+\infty} \frac{\int_E |f_t(re^{i\theta})|^2 dt}{S_f^2(r)} \leq \delta\lambda,$$

from which, due to the arbitrariness of $\lambda > 1$, we have (4). This completes the proof of Lemma 2. □

3. Proof of Theorem 2. Suppose that the conditions of Theorem 2 are satisfied. For arbitrary positive integers k, n, m we introduce the set

$$D_{k,n} = \Big\{t \in \mathbb{R}\colon |f_t(r_k e^{i\theta})| \leq \Big(1 - \frac{1}{n}\Big) S_f(r_k)\Big\},\ E_{m,n} = \bigcap_{k=m}^{\infty} D_{k,n}.$$

Since

$$t \in B' \ \Leftrightarrow\ (\exists m)(\exists n)(\forall k \geq m)\colon t \in D_{k,n} \ \Leftrightarrow\ (\exists m)(\exists n)\colon t \in E_{m,n} \ \Leftrightarrow\ t \in \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} E_{m,n},$$

we have that $B' = \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} E_{m,n}$.

First, we prove that for any positive integers m and n the set $E_{m,n}$ is of Lebesgue measure zero. Suppose the contrary, i. e. there exist some fixed positive integers m and n such that the measure of the set $E_{m,n}$ is not zero. Then, from the periodicity of the function $f_t(z)$ as a function of t it follows that the set $E = E_{m,n} \cap [0, 2\pi]$ has positive measure δ. Applying Lemma 2, we get

$$\lim_{k\to\infty} \frac{\int_E |f_t(r_k e^{i\theta})|^2 dt}{S_f^2(r_k)} = \delta. \tag{9}$$

On the other hand, if $t \in E$, then $t \in D_{k,n}$ for all $k \geq m$. Therefore,

$$\int_E |f_t(r_k e^{i\theta})|^2 dt \leq \delta\Big(1 - \frac{1}{n}\Big)^2 S_f^2(r_k) \quad (k \geq m),$$

which contradicts the relation (9). Thus, $E_{m,n}$ is a set of measure zero for all positive integers m and n.

Next, we prove that the set $E_{m,n}$ is nowhere dense. Let (a, b) be an arbitrary interval of the real line. Since the measure of the set $E_{m,n}$ is zero, this interval contains a point $t_0 \notin E_{m,n}$. Then $t_0 \notin D_{k,n}$ for some $k \geq m$, i. e.

$$\big|f_{t_0}(r_k e^{i\theta})\big| > \Big(1 - \frac{1}{n}\Big) S_f(r_k).$$

From the continuity of the function $f_t(z)$ as a function of t it follows that for all t in some neighborhood $(c, d) \subset (a, b)$ of the point t_0 the inequality

$$\big|f_t(r_k e^{i\theta})\big| > \Big(1 - \frac{1}{n}\Big) S_f(r_k)$$

holds, i. e. $(c, d) \subset E'_{m,n}$. This means that the set $E_{m,n}$ is nowhere dense.

Since the set B' is a countable union of nowhere dense sets of measure zero, this set is of first Baire category and Lebesgue measure zero. Theorem 2 is proved.

4. Proof of Theorem 1. It is well known that for any transcendental entire function f of the form (1) in the definition of its order ρ_f and, if $0 < \rho_f < +\infty$, in the definition of its type σ_f we can replace $M_f(r)$ with $S_f(r)$. It is easy to see that this fact follows from the inequalities $S_f(r) \leq M_f(r)$ and

$$M_f(r) \leq \sum_{n=0}^{\infty} |c_n| r^n = \sum_{n=0}^{\infty} |c_n| (qr)^n \frac{1}{q^n} \leq \left(\sum_{n=0}^{\infty} |c_n|^2 (qr)^{2n}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{1}{q^{2n}}\right)^{\frac{1}{2}} =$$
$$= S_f(qr) \left(\frac{q^2}{q^2 - 1}\right)^{\frac{1}{2}} \quad (q > 1,\ r \geq 0).$$

Let f be an arbitrary entire function of the form (1) and of the order $\rho_f \in (0, +\infty)$. From what has been said it follows that there exists a positive sequence (r_k) increasing to $+\infty$ such that

$$\lim_{k\to\infty} \frac{\ln S_f(r_k)}{r_k^{\rho_f}} = \sigma_f. \tag{10}$$

Consider any sequence (p_n) of positive integers with Hadamard gaps and let Θ be a countable and everywhere dense set in $\mathbb{R}$ (for example $\Theta = \mathbb{Q}$). For the function given by (3) and every $\theta \in \Theta$ we put

$$B_\theta = \left\{ t \in \mathbb{R} \colon \varlimsup_{k\to\infty} \frac{|f_t(r_k e^{i\theta})|}{S_f(r_k)} \geq 1 \right\}.$$

By Theorem 2, each of the sets B'_θ is of measure zero and first Baire category. It is clear that the set $F = \bigcup_{\theta\in\Theta} B'_\theta$ is also of measure zero and first Baire category. Therefore, to complete the proof of Theorem 1 it suffices to show that $A' \subset F$.

We fix some $t \notin F$. For every $\theta \in \Theta$ we have $t \notin B'_\theta$, i. e. $t \in B_\theta$. By the definition of the set B_θ and the relation (10) we obtain $h_{f_t}(\theta) = \sigma_f$ $(\theta \in \Theta)$. Then, since the set Θ is everywhere dense and the indicator $h_{f_t}(\theta)$ is a continuous function, $h_{f_t}(\theta) \equiv \sigma_f$, i. e. $t \in A$, and therefore $t \notin A'$. Consequently, from $t \notin F$ it follows that $t \notin A'$. This implies that $A' \subset F$. Theorem 1 is proved.

5. Proof of Theorem 3. Let f be an analytic function in the disk $\{z \in \mathbb{C}\colon |z| < 1\}$, represented by power series (1) with the radius of convergence $R_f = 1$. Set

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}}, \quad g(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n.$$

It is easy to see that the function φ is analytic in the domain $\{z \in \mathbb{C}\colon |z| > 1\}$, and also some point $e^{i\gamma}$ on the unit circle is a singular point of the function φ if and only if $e^{-i\gamma}$ is a singular point of f. Moreover, by Stirling's formula and Hadamard's formulas given above, g is an entire function of order $\rho_g = 1$ and type $\sigma_g = 1$.

Let $I \subset \{z \in \mathbb{C}\colon |z| \le 1\}$ be the conjugate diagram of the function g, i. e. the smallest convex compact set containing all singularities of the function φ, and let $k_g(\theta)$ be the supporting function of the set I. By the Pólya theorem on the connection between the conjugate diagram and the indicator diagram of an entire function of exponential type, we have $k_g(-\theta) \equiv h_g(\theta)$. From this and from the continuity of the indicator it follows immediately the equivalence of the following assertions:

(i) there exists a point $\theta \in \mathbb{R}$ such that $h_g(\theta) < 1$;

(ii) $h_g(\theta) < 1$ in some interval;

(iii) $k_g(\theta) < 1$ in some interval;

(iv) the function φ can be analytically continuated through some arc of the unit circle in some domain G_1 such that $G_1 \cap \{z \in \mathbb{C}\colon |z| < 1\} \ne \varnothing$;

(v) the function f can be analytically continuated through some arc of the unit circle in some domain G_2 such that $G_2 \cap \{z \in \mathbb{C}\colon |z| > 1\} \ne \varnothing$.

Therefore, we can conclude that the unit circle is the natural boundary for the function f if and only if $h_g(\theta) \equiv 1$.

Let (p_n) be a sequence of positive integers with Hadamard gaps, and let C be the set of all $t \in \mathbb{R}$ such that the circle $\{z \in \mathbb{C}\colon |z| = 1\}$ is the natural boundary for the function (3). Set

$$g_t(z) = \sum_{n=0}^{\infty} e^{ip_n t} \frac{c_n}{n!} z^n, \quad A = \{t \in \mathbb{R}\colon h_{g_t}(\theta) \equiv 1\}.$$

From what has been said above it follows that $C' = A'$. Then by Theorem 1 the set C' is of the Lebesgue measure zero and first Baire category. Theorem 3 is proved.

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