УДК 517.5

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ON THE BEST POSSIBLE DESCRIPTION OF AN EXCEPTIONAL SET IN ASYMPTOTIC ESTIMATES FOR LAPLACE–STIELTJES INTEGRALS

O. B. Skaskiv, D. Yu. Zikrach. *On the best possible description of an exceptional set in asymptotic estimates for Laplace–Stieltjes integrals*, Mat. Stud. **35** (2011), 131–141.

For the Laplace–Stieltjes integrals new description of the exceptional set in asymptotic upper estimates in terms of the maximum of the integrand function is obtained.

О. Б. Скаскив, Д. Ю. Зикрач. *О неулучшаемом описании исключительного множества в асимптотических оценках интегралов Лапласа–Стильтьеса* // Мат. Студiї. – 2011. – Т.35, №2. – С.131–141.

Для интегралов Лапласа-Стильтьеса получено новое описание исключительного множества в асимптотических оценках сверху через максимум подинтегральной функции.

Let $\mathbb{R}_+ = (0, +\infty)$. For $x, y \in \mathbb{R}^p_+$, we denote

$$\langle x, y\rangle = \sum_{i=1}^{p} x_i y_i,\ |x| = \Big(\sum_{i=1}^{p} x_i^2\Big)^{\frac{1}{2}},\ \|x\| = \sum_{i=1}^{p} x_i.$$

Let ν be a countably additive nonnegative measure on $\mathbb{R}^p_+$ with unbounded support $\operatorname{supp}\nu$, $f(x)$ an arbitrary nonnegative ν-measurable function on $\mathbb{R}^p_+$. By $\mathcal{I}^p(\nu)$ we denote the class of function $F\colon \mathbb{R}^p \to [0, +\infty)$ of the form

$$F(\sigma) = \int_{\mathbb{R}^p_+} f(x) e^{\langle\sigma, x\rangle} \nu(dx),\ \sigma \in \mathbb{R}^p. \tag{1}$$

By $\nu(E)$ we denote the ν-measure of a ν-measurable set $E \subset \mathbb{R}^p$.

The class of nonnegative continuous functions $\psi(t)$ on $[0, +\infty)$ such that $\psi(t) \to +\infty$ as $t \to +\infty$ is denoted by L, the subclass of functions $\psi \in L$ such that $\psi(t) \nearrow +\infty$ as $t \to +\infty$ is denoted by L^+; L_1 is the subclass of L consisting of functions $\psi \in L$ such that $\int^{+\infty} \frac{dt}{\psi(t)} < +\infty$; $L_1^+ = L_1 \cap L^+$; L_2 is the class differentiable concave functions $\omega \in L^+$ such that

$$\frac{1}{t} = O(\omega'(t))\ (t \to +\infty);$$

2010 *Mathematics Subject Classification:* 30B50.

L_2^0 is the subclass of L which contains functions $\omega \in L_2$ such that for all function $\varepsilon(t) \to +0$ $(t \to +\infty)$

$$\omega'(t) \searrow 0 \ (t \to +\infty) \text{ i } \varlimsup_{t\to+\infty} \omega'((1-\varepsilon(t))t)/\omega'(t) = 1;$$

L_3 is the class of differentiable functions $\omega \in L$ such that $\omega'(t) \ln t = o(1)$ $(t \to +\infty)$ and for all function $\varepsilon(t) \to +0$ $(t \to +\infty)$

$$\frac{1}{\omega'(t\varepsilon(t))} = o\Big(\frac{1}{\omega'(t)}\Big) \ (t \to +\infty).$$

Example. For

$$\omega(t) = t^\alpha, \ 0 < \alpha < 1, \quad \omega(t) = (\ln t)^{1+\beta}, \ \beta \geq 0,$$

we have $\omega \in \bigcap_{j=2}^4 L_j$.

1. Asymptotic relations with restrictions only to the measure: a new description of exceptional sets. In papers [1, 2] the problem of obtaining asymptotic upper estimates of functions $F \in \mathcal{I}^p(\nu)$ (for $p = 1$ in [1]) via

$$\mu_*(\sigma, F) = \sup\{f(x)e^{\langle\sigma,x\rangle} \colon x \in \operatorname{supp}\nu\}$$

with restrictions only to the measure ν was considered.

Theorem 6 ([1]) implies that for each function $F \in \mathcal{I}^1(\nu)$ as soon as the function $\omega \in L_2$ condition

$$(\exists \psi_1 \in L_1^+, \psi_2 \in L_1) \colon \varlimsup_{t\to+\infty} \omega'(\psi_1^{-1}(t)) \ln \nu_1(t - \sqrt{\psi_2(t)}; t + \sqrt{\psi_2(t)}] \leq d, \tag{2}$$

holds for $\nu_1(a, b] = \nu(\{t \in \mathbb{R} \colon a < t \leq b\})$, then there exists a set $E \subset [0; +\infty)$ finite Lebesgue measure such that

$$\varlimsup_{\substack{\sigma\to+\infty,\\ \sigma\notin E}} (\omega(\ln F(\sigma)) - \omega(\ln \mu_*(\sigma, F))) \leq d. \tag{3}$$

In the general case, according to [9, 10] the finiteness of the measure of an exceptional set E in relation

$$\omega(\ln F(\sigma)) - \omega(\ln \mu_*(\sigma, F)) \leq d + o(1) \tag{4}$$

with $d = 0$ is the best possible description. It is proved in [9, 10] for entire Dirichlet series and $\omega(x) = \ln x$, i.e. for integrals of the form (1) with an atomic measure $\nu = \sum \delta_{\lambda_n}$, where δ_{λ_n} is a unit measure concentrated at the point λ_n (Dirac's measure). The such proposition is contained in Theorem 1 ([2]) in the case of the class $\mathcal{I}^p(\nu)$ $(p \geq 2)$.

Theorem A. (Theorem 1([2])) *Let $\nu_0(0, t] = \nu(\{x \in \mathbb{R}^p \colon \|x\| < t\}), t > 0$. Then for each function $F \in \mathcal{I}^p(\nu)$ satisfying (2) with $\nu_1(a, b] = \nu_0(a, b] = \nu(\{x \in \mathbb{R}^p \colon a < \|x\| \leq b\})$, there exists a measurable set $E \subset \mathbb{R}^p$ such that*

$$\operatorname{meas_p}(E \cap \mathcal{C}(R)) = O(R^{p-1}) \ (R \to +\infty) \tag{5}$$

and the relation (4) *holds as* $|\sigma|\to+\infty$ ($\sigma\in K\backslash E$)*, where* $K\subset\mathbb{R}^p$ *is an arbitrary real cone with the vertex at the point* $O=(0,\dots,0)$ *such that*

$$\overline{K}\backslash\{O\}\subset\gamma(F)=\{\sigma\in\mathbb{R}^p\colon\ \varliminf_{t\to\infty}\frac{1}{t}\ln F(t\sigma)=+\infty\},$$

and $\mathcal{C}(R)$ *is a direct unbounded cylinder with the axis* $\{\sigma\in\mathbb{R}^p\colon\sigma_1=\sigma_2=\ldots=\sigma_p\}$ *and guide surface be a* $(p-1)$*-dimensional ball of radius* $R>0$ *centered at the point* O*.*

Note ([1]) that condition (2) with $d=0$ and $\omega\in L_3$ is equivalent to the condition

$$\int_{t_0}^{+\infty}\frac{k(\ln\nu_1(0,t])}{t^2}dt<+\infty,\quad t_0>0,\tag{6}$$

where $k(t)$ is the inverse function to $\frac{1}{\omega'(t)}$.

Therefore, choosing the measure ν such that for each bounded set $G\subset\mathbb{R}^p$

$$\nu(G)=\sum_{\|n\|=0}^{+\infty}\delta_{\lambda_n}(G),\tag{7}$$

where $\delta_\lambda(G)$ is a unit Dirac's measure concentrated at point λ, then Theorem A yields Theorem 3 [3] for entire multiple Dirichlet series

$$F(z)=\sum_{\|n\|=0}^{+\infty}a_ne^{\langle z,\lambda_n\rangle},\tag{8}$$

where $\Lambda_p=(\lambda_n)_{\|n\|=1}^{+\infty}$ is a fixed sequence such that $\lambda_n=(\lambda_{n_1}^{(1)},\dots,\lambda_{n_p}^{(p)})$ for $n=(n_1,\dots,n_p)\in\mathbb{Z}_+^p$ and $0\le\lambda_k^{(j)}\uparrow+\infty$ ($k\to+\infty$) for all $1\le j\le p$.

By $H(\Lambda_p)$ we denote the class of entire multiple Dirichlet series with a fixed sequence of the exponents $\Lambda_p=(\lambda_n)$. For $F\in H(\Lambda_p)$ and $\sigma\in\mathbb{R}_+^p$ we denote

$$M(\sigma,F)=\sup\{|F(\sigma+iy)|\colon y\in\mathbb{R}^p\},\quad \mu(\sigma,F)=\max\{|a_n|e^{\langle\sigma,\lambda_n\rangle}\colon n\in\mathbb{Z}_+^p\}.$$

For each measurable set $E\in\mathbb{R}^p$ and $\alpha>1$ we define

$$\tau_\alpha(E)=\int\limits_E\frac{d\sigma_1\dots d\sigma_p}{|\sigma|^{\alpha-1}}.$$

If we choose $\omega(x)=\ln x$, then condition (6) can be rewritten as

$$\int_0^{+\infty}\frac{d\ln\nu_1(0,t]}{t}<+\infty\quad \nu_1(a,b]=\nu\{x\in\mathbb{R}^p\colon a<\|x\|\le b\}.$$

In [4, 5, 6] it is proved that if the last condition is satisfied for $\nu_1(0,t]=n_\lambda(t)=\sum_{\|\lambda_n\|\le t}1$, then for each entire multiple Dirichlet series $F\in H(\Lambda_p)$ and for each cone K with the vertex at the origin $O=(0,\dots,0)$ such that $\overline{K}\backslash\{O\}\subset\mathbb{R}_+^p$, Borel's relation

$$\ln M(\sigma,F)=(1+o(1))\ln\mu(\sigma,F)\tag{9}$$

is valid as $|\sigma|\to+\infty, \sigma\in K\backslash E$, where the set $E\subset\mathbb{R}^p_+$ such that

$$\tau_p(E)<+\infty. \tag{10}$$

And this description of exceptional set in Borel's relation is the best possible in a certain sense. For entire Dirichlet series $H(\Lambda_1)$ (that is the class $H(\Lambda_p)$ for $p=1$) similar results was obtained in [8, 9, 10]. Note that this, in particular, implies that in the case of the class

$$\mathcal{I}^p=\bigcup_{\nu}\mathcal{I}^p(\nu)$$

the description of an exceptional set in relation

$$\ln F(\sigma)\le(1+o(1))\ln\mu_*(\sigma,F) \tag{11}$$

can not be improved considerable.

The aim of this paper is to prove that in the class $\mathcal{I}^p$ condition (6) implies relation (4) with $d=0$ outside an exceptional set satisfying condition (10). The following theorem is true.

Theorem 1. *Let $F\in\mathcal{I}^p(\nu)$. If the condition*

$$\int_0^{+\infty}\frac{d\ln\nu_0(0,t]}{t}<+\infty, \tag{12}$$

holds, then the relation

$$\ln F(\sigma)\le(1+o(1))\ln\mu_*(\sigma,F) \tag{13}$$

holds as $|\sigma|\to+\infty, \sigma\in K\backslash E$, where K is an arbitrary real cone in $\mathbb{R}^p_+$ with the vertex at the point O such that $\overline{K}\backslash\{O\}\subset\mathbb{R}^p_+$ and the set E satisfies (10).

Proof. For $\sigma_0\in\mathbb{R}^p_+,\ |\sigma_0|=1$, we define

$$\nu_{\sigma_0}(0,t]=\nu(\{x\in\mathbb{R}^p_+\colon\langle\sigma_0,x\rangle\le t\}).$$

Let $F\in\mathcal{I}^p(\nu)$. Without loss of generality, we suppose that $F(0)=1$.

For fixed $\sigma_0\in\mathbb{R}^p_+, |\sigma_0|=1$, we consider the function $g(t)=\ln F(t\sigma_0), t\in\mathbb{R}_+$. It is proved in [2] (Proposition 5) that $g(t)$ is a convex function for $t>0$. Let us consider the probabilistic space $\Omega=\mathbb{R}^p_+$ with the probabilistic measure

$$P(dx)=f(x)e^{t\langle\sigma_0,x\rangle}\frac{\nu(dx)}{F(t\sigma_0)}.$$

and the random variable $\xi=\langle\sigma_0,x\rangle$. Similar to [2] we can prove that $\mathbf{M}\xi=g'(t)$.

It is proved in [2] (Proposition 5′) that for every K real cone with the vertex at the point O such that $\overline{K}\setminus\{O\}\subset\gamma(F)$

$$\lim_{|\sigma|\to+\infty,\ \sigma\in K}\frac{\ln F(\sigma)}{|\sigma|}=+\infty.$$

Since, $\mathbb{R}^p_+\subset\gamma(F)$, we obtain

$$g'(t)\ge\frac{g(t)-g(0)}{t}=\frac{\ln F(t\sigma_0)}{t}\ge\inf\left\{\frac{\ln F(t\sigma)}{t}\colon|\sigma|=1,\sigma\in K\right\}\to+\infty\ (t\to+\infty).$$

So, $g'(t) \to +\infty$ $(t \to +\infty)$.

By Markov's inequality $P\{\xi > a\} \le \frac{\mathbf{M}\xi}{a}$ $(a > 0)$ for $a = 2\mathbf{M}\xi = 2g'(t)$ and $x = t\sigma_0$, we have $P\{\xi > 2g'(t)\} \le \frac{1}{2}$. Thus,

$$F(t\sigma_0) = \int\limits_{\{x\in\mathbb{R}^p_+ \colon \langle\sigma_0,x\rangle\le 2g'(t)\}} f(x)e^{t\langle\sigma_0,x\rangle}d\nu(x) + \int\limits_{\{x\in\mathbb{R}^p_+ \colon \langle\sigma_0,x\rangle> 2g'(t)\}} f(x)e^{t\langle\sigma_0,x\rangle}d\nu(x) \le$$
$$\le \mu(t\sigma_0, F)\nu(\{x \in \mathbb{R}^p_+ \colon \langle\sigma_0, x\rangle \le 2g'(t)\}) + F(t\sigma_0)P(\{x \in \mathbb{R}^p_+ \colon \langle\sigma_0, x\rangle > 2g'(t)\}) \le$$
$$\le \mu(t\sigma_0, F)\nu_{\sigma_0}(0, 2g'(t)] + \frac{1}{2}F(t\sigma_0).$$

Hence,
$$F(t\sigma_0) \le 2\mu(t\sigma_0, F)\nu_{\sigma_0}(0, 2g'(t)]. \tag{14}$$

Let ([5]) $y^* := \inf\{\inf\{y_j \colon y = (y_1, \dots, y_j, \dots, y_p), |y| = 1, y \in \overline{K}\} \colon 1 \le j \le p\}$. Since $\overline{K}\backslash\{O\} \subset \mathbb{R}^p_+$, we have $y^* > 0$, and for $y \in \overline{K}, |y| = 1$, $t \in \mathbb{R}_+$, we obtain

$$\nu_y(0, t] = \nu(\{x \in \mathbb{R}^p_+ \colon \langle y, x\rangle \le t\}) \le \nu(\{x \in \mathbb{R}^p_+ \colon y^*\|x\| \le t\}) = \nu_0\Big(0, \frac{t}{y^*}\Big].$$

Applying the previous inequality to (14), we have

$$F(t\sigma_0) \le 2\mu(t\sigma_0, F)\sup\{\nu_y(0, 2g'(t)] \colon y \in \overline{K}, |y| = 1\} \le 2\mu(t\sigma_0, F)\nu_0\Big(0, \frac{2g'(t)}{y^*}\Big]. \tag{15}$$

We prove that $\exists\psi \in L^+_1 \colon \ln\nu_0(0, t] = o(\psi^{-1}(t))(t \to +\infty)$ holds. We denote

$$l(t) = \int\limits_t^{+\infty} \frac{\ln\nu_0(0, t]}{t^2}dt, \qquad C(t) = (l(t))^{-\frac{1}{2}} \quad (t > 0).$$

Since

$$\int\limits_0^{+\infty} \frac{d\ln\nu_0(0, t]}{t} = \frac{\ln\nu_0(0, t]}{t} + \int\limits_0^{+\infty} \frac{\ln\nu_0(0, t]}{t^2}dt < +\infty,$$

we have $C(t) \nearrow +\infty(t \to +\infty)$. Now, we choose a positive function ψ increasing to $+\infty$ as $t \to +\infty$ such that the inverse function has the form

$$\psi^{-1}(t) = \begin{cases} C(t)\ln\nu_0(0, t], & \text{if } t \ge t_0, \\ \frac{1}{2}C(t_0)\ln\nu_0(0, t_0](1 + \frac{t}{t_0}), & \text{if } t \in [0; t_0], \end{cases}$$

where $t_0 > 0$ such that $C(t_0)\ln\nu_0(0, t_0] > 0$. Therefore,

$$\int\limits_{t_0}^{+\infty} \frac{\psi^{-1}(t)}{t^2}dt = \int\limits_{t_0}^{+\infty} \frac{C(t)\ln\nu_0(0, t]}{t^2}dt = -\int\limits_{t_0}^{+\infty} \frac{dl(t)}{\sqrt{l(t)}} =$$
$$= 2(l(t_0))^{1/2} = 2\Bigg(\int\limits_{t_0}^{+\infty} \frac{\ln\nu_0(0, t]}{t^2}dt\Bigg)^{1/2} < +\infty.$$

It is clear that ψ is nondecreasing, hence $\int\limits_A^{+\infty} \frac{\psi^{-1}(t)}{t^2}dt \geq \frac{\psi^{-1}(A)}{A}$. Thus, by Cauchy's criterion we have $t = o(\psi(t))$ $(t \to +\infty)$. Therefore, since

$$\int\limits_0^{+\infty} \frac{dt}{\psi(t)} = \int\limits_0^{t_0} \frac{dt}{\psi(t)} + \int\limits_{t_0}^{+\infty} \frac{dt}{\psi(t)},$$

$$\int\limits_{t_0}^{+\infty} \frac{dt}{\psi(t)} = \int\limits_{\psi(t_0)}^{+\infty} \frac{d\psi^{-1}(t)}{t} = \frac{\psi^{-1}(t)}{t}\Big|_{\psi(t_0)}^{+\infty} + \int\limits_{\psi(t_0)}^{+\infty} \frac{\psi^{-1}(t)}{t^2}dt < +\infty,$$

we obtain $\psi \in L_1^+$ and $\ln \nu_0(0,t] = o(\psi^{-1}(t))$ $(t \to +\infty)$.

We denote $E(\sigma_0) = \{\sigma = t\sigma_0 \colon t > 0, \frac{2}{y^*}g'(t) > \psi(g(t))\}$ for fixed $\sigma_0 \in K$, and

$$E = \bigcup_{|\sigma_0|=1, \sigma_0 \in \mathbb{R}_+^p} E(\sigma_0).$$

Then for $\sigma = t\sigma_0, \sigma \in K \backslash E$ we have

$$\ln F(\sigma) \leq \ln 2 + \ln \mu_*(\sigma, F) + \ln \nu_0\Big(0, \frac{2g'(t)}{y^*}\Big] = \ln \mu_*(\sigma, F) + o\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big) \leq$$
$$\leq \ln \mu_*(\sigma, F) + o(\psi^{-1}(\psi(\ln F(\sigma)))) = \ln \mu_*(\sigma, F) + o(\ln F(\sigma)) \quad (|\sigma| \to +\infty).$$

Hence, the relation (13) holds as $|\sigma| \to +\infty$ $(\sigma \in K \setminus E)$.

Let $S_1 = \{\sigma \in K \colon |\sigma| = 1\}$. Finally, we obtain the following estimate for the exceptional set E

$$\tau_p(E \cap \mathbb{R}_+^p) = \iint\limits_E \frac{d\sigma}{|\sigma|^{p-1}} = \int\limits_{S_1} \Big(\int\limits_{E(\sigma_0)} dt\Big) ds \leq \frac{2}{y^*} \int\limits_{S_1} \Big(\int\limits_{E(\sigma_0)} \frac{g'(t)}{\psi(g(t))} dt\Big) ds \leq$$
$$\leq \frac{2}{y^*} \int\limits_{S_1} \Big(\int\limits_{g(0)}^{g(+\infty)} \frac{du}{\psi(u)}\Big) ds \leq C \int\limits_0^{+\infty} \frac{du}{\psi(u)} < +\infty.$$

Theorem 1 is completely proved. □

Necessity of condition (12) in Theorem 1 for $p = 1$ is proved in [11]. It follows from Theorem 3 ([11]) that if a measure ν is a countably additive measure on $\mathbb{R}_+$ such that

$$\int\limits_0^{+\infty} \frac{d \ln \nu(0,t]}{t} = +\infty, \quad \ln \nu(0,t] = O(t) \ (t \to +\infty),$$

where $\nu(0,t] = \nu(\{x \in \mathbb{R}_+ \colon x \leq t\})$, then there exists a nonnegative function $F \in \mathcal{I}^1(\nu)$, a constant $d > 0$, and a fixed point $\sigma_0 > 0$ such that for all $\sigma \geq \sigma_0$

$$\ln F(\sigma) \geq (1 + d) \ln \mu(\sigma, F). \tag{16}$$

If $p \geq 2$ and a measure ν on $\mathbb{R}_+^p$ is a direct product of countably additive measures ν_j on $\mathbb{R}_+$ then the necessity of condition (12) in Theorem 1 follows from the following theorem.

Theorem 2. *Let ν be a direct product of countably-additive measures ν_j on $\mathbb{R}_+$, $\nu = \nu_1 \times \ldots \times \nu_p$. If condition* (12) *does not hold and $\ln \nu_0(0,t] = O(t)$ $(t \to +\infty)$, then there exist a function $F \in \mathcal{I}^p(\nu)$, a constant $d > 0$, and a measurable set E such that for all $\sigma \in E$ inequality* (16) *holds and $\tau_p(E) = +\infty$.*

Proof. If condition (12) does not hold and $\ln \nu_0(0,t] = O(t)$ $(t \to +\infty)$, then there exists $j \in \{1, \ldots, p\}$ such that

$$\int\limits_0^{+\infty} \frac{d \ln \nu_j(0,t]}{t} = +\infty, \quad \ln \nu_j(0,t] = O(t) \ (t \to +\infty), \tag{17}$$

where $\nu_j(0,t] = \nu\{x \in \mathbb{R}_+ \colon x \le t\}$.

Without loss of generality we may suppose that condition (17) holds for $j = 1$. Then by Theorem 3 ([11]) there exists a function

$$F_1(\sigma_1) = \int\limits_0^{+\infty} f_1(x) e^{\sigma_1 x} d\nu_1(x),$$

such that for $\sigma_1 \ge \sigma_0$ the inequality $\ln F_1(\sigma_1) \ge (1+d) \ln \mu_*(\sigma_1, F_1)$ holds.

Convexity of $\ln \mu(t, F_1)$ implies that $l(t) = \frac{1}{t} \ln \mu(t, F_1) \nearrow +\infty$ $(t \to +\infty)$. We choose $l_1(t) \equiv \ln l(t)$ and $l_2(t) = t l(t)/l_1(t)$. It is easy to see that $\dfrac{1}{t} l_2(t) \uparrow +\infty$ $(t_0 \le t \uparrow +\infty)$. Therefore, there exist positive functions $f_j(y), j \in \{2, \ldots, p\}$ such that for each $s \ge t_0$ and $j \in \{2, \ldots, p\}$

$$\sup\{\ln f_2(y) + ys \colon t_0 \le y < +\infty\} \le \frac{1}{p} l_2(s).$$

For each $\sigma \in \mathbb{R}_+^p$ we define the functions

$$F_j(s) = \int\limits_{t_0}^{+\infty} f_j(y) e^{sy} d\nu_j(y), \quad F(\sigma) = \int\limits_{\mathbb{R}_+^p} f_1(y_1) f_2(y_2) \cdots f_p(y_p) e^{\langle \sigma, y \rangle} d\nu(y).$$

Since for each $s \in \mathbb{R}$ and $j \in \{2, \ldots, p\}$ $F_j(s) < +\infty$, we have $F \in \mathcal{I}^p(\nu)$.

Let $t \ge t_0$. Then

$$\sum_{j=2}^p \ln \mu(s, F_j) = \sum_{j=2}^p \sup\{\ln f_j(y) + ys \colon y \in \operatorname{supp} \nu_j \cap [t_0; +\infty)\} \le$$

$$\le \sum_{j=2}^p \sup\{\ln f_j(y) + ys \colon t_0 \le y < +\infty\} \le \sum_{j=2}^p \frac{1}{p} l_2(s) = l_2(s) = o(\ln \mu(s, F_1)) \ (s \to +\infty),$$

that is,

$$\left(1 + \frac{d}{2}\right) \sum_{j=2}^p \ln \mu(s, F_j) \le \frac{d}{2} \ln \mu(s, F_1)$$

for all sufficiently large s.

Since

$$\nu(0;t] = \nu\{y \in \mathbb{R}^p_+ \colon \|y\| \le t\} \le \prod_{j=1}^p \nu_j(0,t]$$

and $F_j(s) \ge 1$ for all $2 \le j \le p$, we obtain the following inequality

$$\ln F(\sigma) \ge \sum_{j=1}^p \ln F_j(\sigma_j) \ge \ln F_1(\sigma_1) \ge (1+d)\ln\mu(\sigma_1, F_1) \ge$$
$$\ge \left(1+\frac{d}{2}\right)\ln\mu(\sigma_1,F_1) + \frac{d}{2}\ln\mu(\sigma_1,F_1) \ge \left(1+\frac{d}{2}\right)\ln\mu(\sigma_1,F_1)+$$
$$+\left(1+\frac{d}{2}\right)\sum_{j=2}^p \ln\mu(\sigma_1,F_j) \ge \left(1+\frac{d}{2}\right)\sum_{j=1}^p \ln\mu(\sigma_j,F_j).$$

for $\sigma \in E = \{\sigma \in \mathbb{R}^p_+ \colon \sigma_1 \ge t_0, t_0 \le \sigma_j \le \sigma_1\ j \in \{2,\ldots,p\}\}$

It remains to note that for all $\sigma \in E$

$$\ln\mu(\sigma,F) = \sum_{j=1}^p \ln\mu(\sigma_j,F_j).$$

We show that $\tau_p(E) = +\infty$

$$\tau_p(E) = \int\limits_E \frac{d\sigma_1\ldots d\sigma_p}{|\sigma|^{p-1}} = \int_{t_0}^{+\infty} d\sigma_1 \int_{t_0}^{\sigma_1} d\sigma_2 \ldots \int_{t_0}^{\sigma_1} \frac{d\sigma_p}{|\sigma|^{p-1}} \ge$$
$$\ge \int_{t_0}^{+\infty} d\sigma_1 \int_{t_0}^{\sigma_1} d\sigma_2 \ldots \int_{t_0}^{\sigma_1} \frac{d\sigma_p}{(\sigma_1\sqrt{p})^{p-1}} = \int_{t_0}^{+\infty} \frac{(\sigma_1-t_0)^{p-1}}{(\sigma_1\sqrt{p})^{p-1}} d\sigma_1 = +\infty.$$

Theorem 2 is completely proved. □

Conjecture. Condition (12) in Theorem 1 is necessary in the case when the measure ν on $\mathbb{R}^p_+$ is arbitrary. Is it true in general case?

For the class $H(\Lambda_p)$ the description (10) of a exceptional set in Borel's relation (9) can not improved in the following sense.

Theorem 3 ([6]). *Let $h \in L^+$. There there exists a sequence $\Lambda_p = (\lambda_n)_{n\in\mathbb{Z}^p_+}$ satisfying the condition*

$$\int\limits_0^{+\infty} \frac{d\ln n_0(t)}{t} dt < +\infty, \tag{18}$$

a function $F \in H_+(\Lambda_p)$, a constant $d > 0$ and a measurable set $E \subset \mathbb{R}^p_+$ such that:

1. $(\forall\ x \in E)\colon\ \ln M(x,F) \ge (1+d)\ln\mu(x,F)$;
2. $\displaystyle\int\limits_E \frac{h(|x|)dx_1\ldots dx_p}{|x|^{p-1}} = +\infty.$

Corollary 1. *For each function $h \in L^+$ there exist a countably additive measure ν on $\mathbb{R}^p_+$, satisfying condition* (12)*, a function $F \in \mathcal{I}^p(\nu)$, a constant $d > 0$ and a measurable set $E \subset \mathbb{R}^p_+$ such that:*

1. $(\forall\, \sigma \in E)\colon\ \ln F(\sigma) \geq (1+d) \ln \mu_*(\sigma, F)$;
2. $\displaystyle\int\limits_E \frac{h(|x|) d\sigma_1 \dots d\sigma_p}{|\sigma|^{p-1}} = +\infty.$

Proof. We choose the measure ν of the form (7). Then condition (18) is equivalent to condition (12), and it remains to apply the previous theorem.

This completes the proof of Corollary 1. □

Theorem 4. *Let $F \in \mathcal{I}^p(\nu), \omega \in L_2, k(t)$ be the inverse function to $\frac{1}{\omega'(t)}$. If condition* (6) *holds for $\nu_1(0,t] = \nu_0(0,t]$, then relation* (4) *with $d = 0$ holds as $|\sigma| \to +\infty, \sigma \in K\backslash E$, where K is an arbitrary real cone in $\mathbb{R}^p_+$ with the vertex at the point O such that $\overline{K}\backslash\{O\} \subset \mathbb{R}^p_+$, and for the measurable set E* (10) *holds.*

Proof. Without loss of generality we may suppose that $F(0) = 1$. Repeating arguments similar to that in proof of Theorem 1 in the part of obtaining inequalities (15), and saving notation, we obtain

$$F(t\sigma_0) \leq 2\mu(t\sigma_0, F)\nu_0\Big(0, \frac{2g'(t)}{y^*}\Big]. \tag{19}$$

We prove that $\exists \psi \in L_1^+\colon \ln \nu_0(0,t] = o(\psi^{-1}(t)) (t \to +\infty)$. As above we define the function

$$l(t) = \int\limits_t^{+\infty} \frac{k(\ln \nu_0(0,t])}{t^2} dt, \quad C(t) = (l(t))^{-\frac{1}{2}} \quad (t > 0).$$

As in the proof of Theorem 1 we have $\exists \psi \in L_1^+\colon k(\ln \nu_0(0,t]) = o(\psi^{-1}(t))\ (t \to +\infty)$. Since $k(t)$ is the inverse function to $\frac{1}{\omega'(t)}$ and $\omega \in L_2$ we obtain

$$\ln \nu_0(0,t] = o(k^{-1}(\psi^{-1}(t))) = o\Big(\frac{1}{\omega'(\psi^{-1}(t))}\Big) = o(\psi^{-1}(t))\ (t \to +\infty).$$

It now follows from the proof of Theorem 1 that inequality (13) holds as $\sigma = t\sigma_0, \sigma \in K\backslash E_1$, where

$$E_1 = \bigcup_{|\sigma_0|=1,\ \sigma_0 \in \mathbb{R}^p_+} E_1(\sigma_0), E_1(\sigma_0) = \Big\{\sigma = t\sigma_0\colon t > 0, \frac{2}{y^*} g'(t) > \psi(g(t))\Big\}.$$

Moreover $\tau_p(E \cap \mathbb{R}^p_+) < +\infty$.

Hence

$$\ln \mu_*(\sigma, F) \geq \frac{1}{2} \ln F(\sigma)\ (|\sigma| \to +\infty) \tag{20}$$

as $\sigma = t\sigma_0, \sigma \in K\backslash E_1$.

Since $\omega \in L_2$, ω' is a decreasing function. Then from (15) and by the mean value Lagrange's theorem of finite increments we obtain

$$\omega(\ln F(\sigma)) - \omega(\ln \mu_*(\sigma, F)) \le \omega'(\ln \mu_*(\sigma, F))(\ln F(\sigma) - \ln \mu_*(\sigma, F)) \le$$
$$\le \omega'(\ln \mu_*(\sigma, F))\Big(\ln 2 + \ln \nu_0\Big(0, \frac{2g'(t)}{y^*}\Big]\Big)$$

as $\sigma = t\sigma_0, \sigma \in K\backslash E_1$.

Let

$$E_2 = \bigcup_{|\sigma_0|=1,\ \sigma_0 \in \mathbb{R}^p_+} E_2(\sigma_0),\ \ E_2(\sigma_0) = \Big\{\sigma = t\sigma_0 \colon t > 0, \frac{2}{y^*} g'(t) > \psi\Big(\frac{g(t)}{2}\Big)\Big\}.$$

Then for $\sigma = t\sigma_0$, $\sigma \in K\backslash(E_1 \cup E_2)$,

$$\omega'(\ln \mu_*(\sigma, F)) \le \omega'\Big(\frac{1}{2} \ln F(\sigma)\Big) = \omega'\Big(\frac{1}{2} g(t)\Big) \le \omega'\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big)$$

as $|\sigma| \to +\infty$.

Since

$$\ln \nu_0(0, t] = o(k^{-1}(\psi^{-1}(t))) = o(1/\omega'(\psi^{-1}(t)))\ (t \to +\infty),$$

we have

$$\omega(\ln F(\sigma)) - \omega(\ln \mu_*(\sigma, F)) \le \omega'(\ln \mu_*(\sigma, F))\Big(\ln 2 + \ln \nu_0\Big(0, \frac{2g'(t)}{y^*}\Big]\Big) \le$$
$$\le \omega'\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big)\Big(\ln 2 + o\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big) = o(1) + \omega'\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big) \ln 2. \tag{21}$$

Therefore ψ^{-1} is nondecreasing and $\omega \in L_2$, we obtain (4).

Finally, we obtain the following estimate for the exceptional set $E = E_1 \cup E_2$

$$\tau_p(E_2 \cap \mathbb{R}^p_+) \le \frac{2}{y^*} \int\limits_{S_1} \Big(\int\limits_{E_2(\sigma_0)} \frac{g'(t)}{\psi(\frac{g(t)}{2})} dt\Big) ds \le \frac{2}{y^*} \int\limits_{S_1} \Big(\int\limits_{g(0)}^{g(+\infty)} \frac{du}{\psi(\frac{u}{2})}\Big) ds \le$$
$$\le C \int\limits_0^{+\infty} \frac{dt}{\psi(t)} < +\infty.$$

Since $\tau_p(E_1 \cap \mathbb{R}^p_+) < +\infty$, we have $\tau_p(E \cap \mathbb{R}^p_+) < +\infty$. □

In [3] an analogue of Theorem 4 for the class $H(\Lambda_p)$ is proved.

Theorem 5 ([3])**.** *Let $\omega \in L_3 \cap L_4 \cap L_5$. For each function $F \in H(\Lambda_p)$ the relation*

$$\omega(\ln M(\sigma, F)) - \omega(\ln \mu(\sigma, F)) = o(1) \tag{22}$$

holds $|\sigma| \to +\infty$ $(\sigma \in K \setminus E, \mathrm{meas}_p(E \cap S_r) = O(r^{p-1})\ (r \to +\infty))$ if and only if

$$\int\limits_0^{+\infty} \frac{k(\ln n_0(t))}{t^2} dt < +\infty, \tag{23}$$

holds, where K is an arbitrary cone $K \subset \mathbb{R}^p$ with vertex in point such that $(\overline{K} \setminus O) \subset \{\sigma \in \mathbb{R}^p\colon \lim_{t\to+\infty} \frac{1}{t} \ln \mu(t\sigma, F) = +\infty\}$, S_r is a cylinder, which obtains from the cylinder $S'_r = \{x = (x_1, ..., x_p) \in \mathbb{R}^p\colon x_2^2 + ... + x_p^2 \le r^2\}$ by turning the coordinate system so that the axle Ox_1 moves in ray $\{x \in \mathbb{R}^p\colon x_1 = x_2 = ... + x_p\}$.

From the proof in [3] necessary condition (23) in Theorem 5 and from Theorem 4 we obtain the following theorem.

Theorem 6. *Let $F \in H(\Lambda_p), \omega \in L_2 \cap L_4 \cap L_5, k(t)$ be the inverse function to the function $\frac{1}{\omega'(t)}$. For each function $F \in H(\Lambda_p)$ relation* (22) *holds as $|\sigma| \to +\infty, \sigma \in K\setminus E$, where K is an arbitrary real cone in $\mathbb{R}^p_+$ with the vertex at the point O such that $\overline{K}\setminus\{O\} \subset \mathbb{R}^p_+$ and measurable set E satisfied* (10) *if and only if condition* (23) *holds.*

Proof. Sufficiency. We choose the measure ν of the form (7). Then condition (18) is equivalent to condition (12). It remains to apply Theorem 4.
Necessity. The necessity of condition (23) one can prove in a similar way to the proof of Corollary 3 ([3]), where for all $\sigma \in E = \{\sigma \in \mathbb{R}^p_+\colon \sigma_1 \ge t_0, \sigma_1 \ge \max\{\sigma_2, \sigma_3, ..., \sigma_p\}\}$ relation (22) holds, and that for this set $\tau_p(E) = +\infty$ (see. proof of Theorem 2). □

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Received 12.09.2010
Revised 20.03.2011