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O. B. Skaskiv, D. Yu. Zikrach

# ON THE BEST POSSIBLE DESCRIPTION OF AN EXCEPTIONAL SET IN ASYMPTOTIC ESTIMATES FOR LAPLACE–STIELTJES INTEGRALS

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For the Laplace–Stieltjes integrals new description of the exceptional set in asymptotic upper estimates in terms of the maximum of the integrand function is obtained.

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Для интегралов Лапласа-Стильтьеса получено новое описание исключительного множества в асимптотических оценках сверху через максимум подинтегральной функции.

Let $\mathbb{R}_+ = (0, +\infty)$. For $x, y \in \mathbb{R}^p_+$, we denote

$$\langle x, y\rangle = \sum_{i=1}^{p} x_i y_i,\ |x| = \Big(\sum_{i=1}^{p} x_i^2\Big)^{\frac{1}{2}},\ \|x\| = \sum_{i=1}^{p} x_i.$$

Let $\nu$ be a countably additive nonnegative measure on $\mathbb{R}^p_+$ with unbounded support $\operatorname{supp}\nu$, $f(x)$ an arbitrary nonnegative $\nu$-measurable function on $\mathbb{R}^p_+$. By $\mathcal{I}^p(\nu)$ we denote the class of function $F\colon \mathbb{R}^p \to [0, +\infty)$ of the form

$$F(\sigma) = \int_{\mathbb{R}^p_+} f(x) e^{\langle\sigma,x\rangle} \nu(dx),\ \sigma \in \mathbb{R}^p. \tag{1}$$

By $\nu(E)$ we denote the $\nu$-measure of a $\nu$-measurable set $E \subset \mathbb{R}^p$.

The class of nonnegative continuous functions $\psi(t)$ on $[0, +\infty)$ such that $\psi(t) \to +\infty$ as $t \to +\infty$ is denoted by $L$, the subclass of functions $\psi \in L$ such that $\psi(t) \nearrow +\infty$ as $t \to +\infty$ is denoted by $L^+$; $L_1$ is the subclass of $L$ consisting of functions $\psi \in L$ such that $\int^{+\infty} \frac{dt}{\psi(t)} < +\infty$; $L_1^+ = L_1 \cap L^+$; $L_2$ is the class differentiable concave functions $\omega \in L^+$ such that

$$\frac{1}{t} = O(\omega'(t))\ (t \to +\infty);$$

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$L_2^0$ is the subclass of $L$ which contains functions $\omega \in L_2$ such that for all function $\varepsilon(t) \to +0$ $(t \to +\infty)$

$$\omega'(t) \searrow 0 \ (t \to +\infty) \text{ i } \varlimsup_{t\to+\infty} \omega'((1-\varepsilon(t))t)/\omega'(t) = 1;$$

$L_3$ is the class of differentiable functions $\omega \in L$ such that $\omega'(t)\ln t = o(1)$ $(t \to +\infty)$ and for all function $\varepsilon(t) \to +0$ $(t \to +\infty)$

$$\frac{1}{\omega'(t\varepsilon(t))} = o\biggl(\frac{1}{\omega'(t)}\biggr) \ (t \to +\infty).$$

**Example.** For

$$\omega(t) = t^{\alpha}, \ 0 < \alpha < 1, \quad \omega(t) = (\ln t)^{1+\beta}, \ \beta \ge 0,$$

we have $\omega \in \bigcap_{j=2}^4 L_j$.

**1. Asymptotic relations with restrictions only to the measure: a new description of exceptional sets.** In papers [1, 2] the problem of obtaining asymptotic upper estimates of functions $F \in \mathcal{I}^p(\nu)$ (for $p = 1$ in [1]) via

$$\mu_*(\sigma, F) = \sup\{f(x)e^{\langle\sigma,x\rangle} \colon x \in \operatorname{supp}\nu\}$$

with restrictions only to the measure $\nu$ was considered.

Theorem 6 ([1]) implies that for each function $F \in \mathcal{I}^1(\nu)$ as soon as the function $\omega \in L_2$ condition

$$(\exists \psi_1 \in L_1^+, \psi_2 \in L_1) \colon \varlimsup_{t\to+\infty} \omega'(\psi_1^{-1}(t)) \ln \nu_1(t - \sqrt{\psi_2(t)}; t + \sqrt{\psi_2(t)}] \le d, \tag{2}$$

holds for $\nu_1(a,b] = \nu(\{t \in \mathbb{R} \colon a < t \le b\})$, then there exists a set $E \subset [0;+\infty)$ finite Lebesgue measure such that

$$\varlimsup_{\substack{\sigma\to+\infty,\\ \sigma\notin E}} (\omega(\ln F(\sigma)) - \omega(\ln \mu_*(\sigma, F))) \le d. \tag{3}$$

In the general case, according to [9, 10] the finiteness of the measure of an exceptional set $E$ in relation

$$\omega(\ln F(\sigma)) - \omega(\ln \mu_*(\sigma, F)) \le d + o(1) \tag{4}$$

with $d = 0$ is the best possible description. It is proved in [9, 10] for entire Dirichlet series and $\omega(x) = \ln x$, i.e. for integrals of the form (1) with an atomic measure $\nu = \sum \delta_{\lambda_n}$, where $\delta_{\lambda_n}$ is a unit measure concentrated at the point $\lambda_n$ (Dirac's measure). The such proposition is contained in Theorem 1 ([2]) in the case of the class $\mathcal{I}^p(\nu)$ $(p \ge 2)$.

**Theorem A. (Theorem 1([2]))** *Let $\nu_0(0,t] = \nu(\{x \in \mathbb{R}^p \colon \|x\| < t\}), t > 0$. Then for each function $F \in \mathcal{I}^p(\nu)$ satisfying (2) with $\nu_1(a,b] = \nu_0(a,b] = \nu(\{x \in \mathbb{R}^p \colon a < \|x\| \le b\})$, there exists a measurable set $E \subset \mathbb{R}^p$ such that*

$$\mathrm{meas_p}(E \cap \mathcal{C}(R)) = O(R^{p-1}) \ (R \to +\infty) \tag{5}$$

*and the relation* (4) *holds as* $|\sigma|\to+\infty$ $(\sigma\in K\setminus E)$, *where* $K\subset\mathbb{R}^p$ *is an arbitrary real cone with the vertex at the point* $O=(0,\dots,0)$ *such that*

$$\overline{K}\setminus\{O\}\subset\gamma(F)=\{\sigma\in\mathbb{R}^p\colon \varliminf_{t\to\infty}\frac{1}{t}\ln F(t\sigma)=+\infty\},$$

*and* $\mathcal{C}(R)$ *is a direct unbounded cylinder with the axis* $\{\sigma\in\mathbb{R}^p\colon \sigma_1=\sigma_2=\ldots=\sigma_p\}$ *and guide surface be a* $(p-1)$*-dimensional ball of radius* $R>0$ *centered at the point* $O$.

Note ([1]) that condition (2) with $d=0$ and $\omega\in L_3$ is equivalent to the condition

$$\int_{t_0}^{+\infty}\frac{k(\ln\nu_1(0,t])}{t^2}dt<+\infty,\quad t_0>0, \tag{6}$$

where $k(t)$ is the inverse function to $\frac{1}{\omega'(t)}$.

Therefore, choosing the measure $\nu$ such that for each bounded set $G\subset\mathbb{R}^p$

$$\nu(G)=\sum_{\|n\|=0}^{+\infty}\delta_{\lambda_n}(G), \tag{7}$$

where $\delta_\lambda(G)$ is a unit Dirac's measure concentrated at point $\lambda$, then Theorem A yields Theorem 3 [3] for entire multiple Dirichlet series

$$F(z)=\sum_{\|n\|=0}^{+\infty}a_ne^{\langle z,\lambda_n\rangle}, \tag{8}$$

where $\Lambda_p=(\lambda_n)_{\|n\|=1}^{+\infty}$ is a fixed sequence such that $\lambda_n=(\lambda_{n_1}^{(1)},\dots,\lambda_{n_p}^{(p)})$ for $n=(n_1,\dots,n_p)\in\mathbb{Z}_+^p$ and $0\le\lambda_k^{(j)}\uparrow+\infty$ $(k\to+\infty)$ for all $1\le j\le p$.

By $H(\Lambda_p)$ we denote the class of entire multiple Dirichlet series with a fixed sequence of the exponents $\Lambda_p=(\lambda_n)$. For $F\in H(\Lambda_p)$ and $\sigma\in\mathbb{R}_+^p$ we denote

$$M(\sigma,F)=\sup\{|F(\sigma+iy)|\colon y\in\mathbb{R}^p\},\quad \mu(\sigma,F)=\max\{|a_n|e^{\langle\sigma,\lambda_n\rangle}\colon n\in\mathbb{Z}_+^p\}.$$

For each measurable set $E\in\mathbb{R}^p$ and $\alpha>1$ we define

$$\tau_\alpha(E)=\int\limits_E\frac{d\sigma_1\dots d\sigma_p}{|\sigma|^{\alpha-1}}.$$

If we choose $\omega(x)=\ln x$, then condition (6) can be rewritten as

$$\int_0^{+\infty}\frac{d\ln\nu_1(0,t]}{t}<+\infty\quad \nu_1(a,b]=\nu\{x\in\mathbb{R}^p\colon a<\|x\|\le b\}.$$

In [4, 5, 6] it is proved that if the last condition is satisfied for $\nu_1(0,t]=n_\lambda(t)=\sum_{\|\lambda_n\|\le t}1$, then for each entire multiple Dirichlet series $F\in H(\Lambda_p)$ and for each cone $K$ with the vertex at the origin $O=(0,\dots,0)$ such that $\overline{K}\setminus\{O\}\subset\mathbb{R}_+^p$, Borel's relation

$$\ln M(\sigma,F)=(1+o(1))\ln\mu(\sigma,F) \tag{9}$$

is valid as $|\sigma| \to +\infty, \sigma \in K\backslash E$, where the set $E \subset \mathbb{R}^p_+$ such that

$$\tau_p(E) < +\infty. \tag{10}$$

And this description of exceptional set in Borel's relation is the best possible in a certain sense. For entire Dirichlet series $H(\Lambda_1)$ (that is the class $H(\Lambda_p)$ for $p = 1$) similar results was obtained in [8, 9, 10]. Note that this, in particular, implies that in the case of the class

$$\mathcal{I}^p = \bigcup_\nu \mathcal{I}^p(\nu)$$

the description of an exceptional set in relation

$$\ln F(\sigma) \le (1 + o(1)) \ln \mu_*(\sigma, F) \tag{11}$$

can not be improved considerable.

The aim of this paper is to prove that in the class $\mathcal{I}^p$ condition (6) implies relation (4) with $d = 0$ outside an exceptional set satisfying condition (10). The following theorem is true.

**Theorem 1.** *Let $F \in \mathcal{I}^p(\nu)$. If the condition*

$$\int_0^{+\infty} \frac{d \ln \nu_0(0, t]}{t} < +\infty, \tag{12}$$

*holds, then the relation*

$$\ln F(\sigma) \le (1 + o(1)) \ln \mu_*(\sigma, F) \tag{13}$$

*holds as $|\sigma| \to +\infty, \sigma \in K\backslash E$, where $K$ is an arbitrary real cone in $\mathbb{R}^p_+$ with the vertex at the point $O$ such that $\overline{K}\backslash\{O\} \subset \mathbb{R}^p_+$ and the set $E$ satisfies* (10).

*Proof.* For $\sigma_0 \in \mathbb{R}^p_+,\ |\sigma_0| = 1$, we define

$$\nu_{\sigma_0}(0, t] = \nu(\{x \in \mathbb{R}^p_+ \colon \langle \sigma_0, x\rangle \le t\}).$$

Let $F \in \mathcal{I}^p(\nu)$. Without loss of generality, we suppose that $F(0) = 1$.

For fixed $\sigma_0 \in \mathbb{R}^p_+, |\sigma_0| = 1$, we consider the function $g(t) = \ln F(t\sigma_0), t \in \mathbb{R}_+$. It is proved in [2] (Proposition 5) that $g(t)$ is a convex function for $t > 0$. Let us consider the probabilistic space $\Omega = \mathbb{R}^p_+$ with the probabilistic measure

$$P(dx) = f(x) e^{t\langle\sigma_0, x\rangle} \frac{\nu(dx)}{F(t\sigma_0)}.$$

and the random variable $\xi = \langle \sigma_0, x\rangle$. Similar to [2] we can prove that $\mathbf{M}\xi = g'(t)$.

It is proved in [2] (Proposition 5′) that for every $K$ real cone with the vertex at the point $O$ such that $\overline{K} \setminus \{O\} \subset \gamma(F)$

$$\lim_{|\sigma| \to +\infty,\ \sigma \in K} \frac{\ln F(\sigma)}{|\sigma|} = +\infty.$$

Since, $\mathbb{R}^p_+ \subset \gamma(F)$, we obtain

$$g'(t) \ge \frac{g(t) - g(0)}{t} = \frac{\ln F(t\sigma_0)}{t} \ge \inf\left\{\frac{\ln F(t\sigma)}{t} \colon |\sigma| = 1, \sigma \in K\right\} \to +\infty\ (t \to +\infty).$$

So, $g'(t)\to+\infty$ $(t\to+\infty)$.

By Markov's inequality $P\{\xi>a\}\le\frac{\mathbf{M}\xi}{a}$ $(a>0)$ for $a=2\mathbf{M}\xi=2g'(t)$ and $x=t\sigma_0$, we have $P\{\xi>2g'(t)\}\le\frac{1}{2}$. Thus,

$$F(t\sigma_0)=\int\limits_{\{x\in\mathbb{R}^p_+\colon\,\langle\sigma_0,x\rangle\le 2g'(t)\}} f(x)e^{t\langle\sigma_0,x\rangle}d\nu(x)+\int\limits_{\{x\in\mathbb{R}^p_+\colon\,\langle\sigma_0,x\rangle>2g'(t)\}} f(x)e^{t\langle\sigma_0,x\rangle}d\nu(x)\le$$
$$\le\mu(t\sigma_0,F)\nu(\{x\in\mathbb{R}^p_+\colon\langle\sigma_0,x\rangle\le 2g'(t)\})+F(t\sigma_0)P(\{x\in\mathbb{R}^p_+\colon\langle\sigma_0,x\rangle>2g'(t)\})\le$$
$$\le\mu(t\sigma_0,F)\nu_{\sigma_0}(0,2g'(t)]+\frac{1}{2}F(t\sigma_0).$$

Hence,

$$F(t\sigma_0)\le 2\mu(t\sigma_0,F)\nu_{\sigma_0}(0,2g'(t)]. \tag{14}$$

Let ([5]) $y^*:=\inf\{\inf\{y_j\colon y=(y_1,\dots,y_j,\dots,y_p),|y|=1,y\in\overline{K}\}\colon 1\le j\le p\}$. Since $\overline{K}\backslash\{O\}\subset\mathbb{R}^p_+$, we have $y^*>0$, and for $y\in\overline{K},|y|=1$, $t\in\mathbb{R}_+$, we obtain

$$\nu_y(0,t]=\nu(\{x\in\mathbb{R}^p_+\colon\langle y,x\rangle\le t\})\le\nu(\{x\in\mathbb{R}^p_+\colon y^*\|x\|\le t\})=\nu_0\Big(0,\frac{t}{y^*}\Big].$$

Applying the previous inequality to (14), we have

$$F(t\sigma_0)\le 2\mu(t\sigma_0,F)\sup\{\nu_y(0,2g'(t)]\colon y\in\overline{K},|y|=1\}\le 2\mu(t\sigma_0,F)\nu_0\Big(0,\frac{2g'(t)}{y^*}\Big]. \tag{15}$$

We prove that $\exists\psi\in L_1^+\colon\ \ln\nu_0(0,t]=o(\psi^{-1}(t))(t\to+\infty)$ holds. We denote

$$l(t)=\int\limits_t^{+\infty}\frac{\ln\nu_0(0,t]}{t^2}dt,\qquad C(t)=(l(t))^{-\frac{1}{2}}\quad(t>0).$$

Since

$$\int\limits_0^{+\infty}\frac{d\ln\nu_0(0,t]}{t}=\frac{\ln\nu_0(0,t]}{t}+\int\limits_0^{+\infty}\frac{\ln\nu_0(0,t]}{t^2}dt<+\infty,$$

we have $C(t)\nearrow+\infty(t\to+\infty)$. Now, we choose a positive function $\psi$ increasing to $+\infty$ as $t\to+\infty$ such that the inverse function has the form

$$\psi^{-1}(t)=\begin{cases}C(t)\ln\nu_0(0,t], & \text{if } t\ge t_0,\\ \frac{1}{2}C(t_0)\ln\nu_0(0,t_0](1+\frac{t}{t_0}), & \text{if } t\in[0;t_0],\end{cases}$$

where $t_0>0$ such that $C(t_0)\ln\nu_0(0,t_0]>0$. Therefore,

$$\int\limits_{t_0}^{+\infty}\frac{\psi^{-1}(t)}{t^2}dt=\int\limits_{t_0}^{+\infty}\frac{C(t)\ln\nu_0(0,t]}{t^2}dt=-\int\limits_{t_0}^{+\infty}\frac{dl(t)}{\sqrt{l(t)}}=$$
$$=2(l(t_0))^{1/2}=2\Bigg(\int\limits_{t_0}^{+\infty}\frac{\ln\nu_0(0,t]}{t^2}dt\Bigg)^{1/2}<+\infty.$$

It is clear that $\psi$ is nondecreasing, hence $\int\limits_A^{+\infty}\frac{\psi^{-1}(t)}{t^2}dt\ge\frac{\psi^{-1}(A)}{A}$. Thus, by Cauchy's criterion we have $t=o(\psi(t))$ $(t\to+\infty)$. Therefore, since

$$\int\limits_0^{+\infty}\frac{dt}{\psi(t)}=\int\limits_0^{t_0}\frac{dt}{\psi(t)}+\int\limits_{t_0}^{+\infty}\frac{dt}{\psi(t)},$$

$$\int\limits_{t_0}^{+\infty}\frac{dt}{\psi(t)}=\int\limits_{\psi(t_0)}^{+\infty}\frac{d\psi^{-1}(t)}{t}=\frac{\psi^{-1}(t)}{t}\Big|_{\psi(t_0)}^{+\infty}+\int\limits_{\psi(t_0)}^{+\infty}\frac{\psi^{-1}(t)}{t^2}dt<+\infty,$$

we obtain $\psi\in L_1^+$ and $\ln\nu_0(0,t]=o(\psi^{-1}(t))$ $(t\to+\infty)$.

We denote $E(\sigma_0)=\{\sigma=t\sigma_0\colon t>0,\frac{2}{y^*}g'(t)>\psi(g(t))\}$ for fixed $\sigma_0\in K$, and

$$E=\bigcup_{|\sigma_0|=1,\sigma_0\in\mathbb{R}^p_+}E(\sigma_0).$$

Then for $\sigma=t\sigma_0,\sigma\in K\backslash E$ we have

$$\ln F(\sigma)\le\ln 2+\ln\mu_*(\sigma,F)+\ln\nu_0\Big(0,\frac{2g'(t)}{y^*}\Big]=\ln\mu_*(\sigma,F)+o\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big)\le$$
$$\le\ln\mu_*(\sigma,F)+o(\psi^{-1}(\psi(\ln F(\sigma))))=\ln\mu_*(\sigma,F)+o(\ln F(\sigma))\quad(|\sigma|\to+\infty).$$

Hence, the relation (13) holds as $|\sigma|\to+\infty$ $(\sigma\in K\setminus E)$.

Let $S_1=\{\sigma\in K\colon|\sigma|=1\}$. Finally, we obtain the following estimate for the exceptional set $E$

$$\tau_p(E\cap\mathbb{R}^p_+)=\iint\limits_E\frac{d\sigma}{|\sigma|^{p-1}}=\int\limits_{S_1}\Big(\int\limits_{E(\sigma_0)}dt\Big)ds\le\frac{2}{y^*}\int\limits_{S_1}\Big(\int\limits_{E(\sigma_0)}\frac{g'(t)}{\psi(g(t))}dt\Big)ds\le$$
$$\le\frac{2}{y^*}\int\limits_{S_1}\Big(\int\limits_{g(0)}^{g(+\infty)}\frac{du}{\psi(u)}\Big)ds\le C\int\limits_0^{+\infty}\frac{du}{\psi(u)}<+\infty.$$

Theorem 1 is completely proved. □

Necessity of condition (12) in Theorem 1 for $p=1$ is proved in [11]. It follows from Theorem 3 ([11]) that if a measure $\nu$ is a countably additive measure on $\mathbb{R}_+$ such that

$$\int\limits_0^{+\infty}\frac{d\ln\nu(0,t]}{t}=+\infty,\quad\ln\nu(0,t]=O(t)\ (t\to+\infty),$$

where $\nu(0,t]=\nu(\{x\in\mathbb{R}_+\colon x\le t\})$, then there exists a nonnegative function $F\in\mathcal{I}^1(\nu)$, a constant $d>0$, and a fixed point $\sigma_0>0$ such that for all $\sigma\ge\sigma_0$

$$\ln F(\sigma)\ge(1+d)\ln\mu(\sigma,F).\tag{16}$$

If $p\ge 2$ and a measure $\nu$ on $\mathbb{R}^p_+$ is a direct product of countably additive measures $\nu_j$ on $\mathbb{R}_+$ then the necessity of condition (12) in Theorem 1 follows from the following theorem.

**Theorem 2.** *Let $\nu$ be a direct product of countably-additive measures $\nu_j$ on $\mathbb{R}_+$, $\nu = \nu_1 \times \ldots \times \nu_p$. If condition* (12) *does not hold and $\ln \nu_0(0,t] = O(t)$ $(t \to +\infty)$, then there exist a function $F \in \mathcal{I}^p(\nu)$, a constant $d > 0$, and a measurable set $E$ such that for all $\sigma \in E$ inequality* (16) *holds and $\tau_p(E) = +\infty$.*

*Proof.* If condition (12) does not hold and $\ln \nu_0(0,t] = O(t)$ $(t \to +\infty)$, then there exists $j \in \{1,\ldots,p\}$ such that

$$\int\limits_0^{+\infty} \frac{d\ln \nu_j(0,t]}{t} = +\infty, \quad \ln \nu_j(0,t] = O(t)\ (t \to +\infty), \tag{17}$$

where $\nu_j(0,t] = \nu\{x \in \mathbb{R}_+ \colon x \le t\}$.

Without loss of generality we may suppose that condition (17) holds for $j = 1$. Then by Theorem 3 ([11]) there exists a function

$$F_1(\sigma_1) = \int\limits_0^{+\infty} f_1(x) e^{\sigma_1 x} d\nu_1(x),$$

such that for $\sigma_1 \ge \sigma_0$ the inequality $\ln F_1(\sigma_1) \ge (1+d)\ln \mu_*(\sigma_1, F_1)$ holds.

Convexity of $\ln \mu(t, F_1)$ implies that $l(t) = \frac{1}{t}\ln \mu(t, F_1) \nearrow +\infty$ $(t \to +\infty)$. We choose $l_1(t) \equiv \ln l(t)$ and $l_2(t) = tl(t)/l_1(t)$. It is easy to see that $\dfrac{1}{t} l_2(t) \uparrow +\infty$ $(t_0 \le t \uparrow +\infty)$. Therefore, there exist positive functions $f_j(y), j \in \{2,\ldots,p\}$ such that for each $s \ge t_0$ and $j \in \{2,\ldots,p\}$

$$\sup\{\ln f_2(y) + ys \colon t_0 \le y < +\infty\} \le \frac{1}{p} l_2(s).$$

For each $\sigma \in \mathbb{R}_+^p$ we define the functions

$$F_j(s) = \int\limits_{t_0}^{+\infty} f_j(y) e^{sy} d\nu_j(y), \quad F(\sigma) = \int\limits_{\mathbb{R}_+^p} f_1(y_1) f_2(y_2) \cdots f_p(y_p) e^{\langle \sigma, y\rangle} d\nu(y).$$

Since for each $s \in \mathbb{R}$ and $j \in \{2,\ldots,p\}$ $F_j(s) < +\infty$, we have $F \in \mathcal{I}^p(\nu)$.

Let $t \ge t_0$. Then

$$\sum_{j=2}^{p} \ln \mu(s, F_j) = \sum_{j=2}^{p} \sup\{\ln f_j(y) + ys \colon y \in \operatorname{supp} \nu_j \cap [t_0; +\infty)\} \le$$

$$\le \sum_{j=2}^{p} \sup\{\ln f_j(y) + ys \colon t_0 \le y < +\infty\} \le \sum_{j=2}^{p} \frac{1}{p} l_2(s) = l_2(s) = o(\ln \mu(s, F_1))\ (s \to +\infty),$$

that is,

$$\left(1 + \frac{d}{2}\right) \sum_{j=2}^{p} \ln \mu(s, F_j) \le \frac{d}{2} \ln \mu(s, F_1)$$

for all sufficiently large $s$.

Since

$$\nu(0;t] = \nu\{y \in \mathbb{R}^p_+ \colon \|y\| \le t\} \le \prod_{j=1}^{p} \nu_j(0,t]$$

and $F_j(s) \ge 1$ for all $2 \le j \le p$, we obtain the following inequality

$$\ln F(\sigma) \ge \sum_{j=1}^{p} \ln F_j(\sigma_j) \ge \ln F_1(\sigma_1) \ge (1+d)\ln\mu(\sigma_1, F_1) \ge$$
$$\ge \left(1+\frac{d}{2}\right)\ln\mu(\sigma_1,F_1) + \frac{d}{2}\ln\mu(\sigma_1,F_1) \ge \left(1+\frac{d}{2}\right)\ln\mu(\sigma_1,F_1)+$$
$$+\left(1+\frac{d}{2}\right)\sum_{j=2}^{p}\ln\mu(\sigma_1,F_j) \ge \left(1+\frac{d}{2}\right)\sum_{j=1}^{p}\ln\mu(\sigma_j,F_j).$$

for $\sigma \in E = \{\sigma \in \mathbb{R}^p_+ \colon \sigma_1 \ge t_0, t_0 \le \sigma_j \le \sigma_1 \; j \in \{2,\dots,p\}\}$

It remains to note that for all $\sigma \in E$

$$\ln\mu(\sigma, F) = \sum_{j=1}^{p} \ln\mu(\sigma_j, F_j).$$

We show that $\tau_p(E) = +\infty$

$$\tau_p(E) = \int\limits_E \frac{d\sigma_1 \dots d\sigma_p}{|\sigma|^{p-1}} = \int_{t_0}^{+\infty} d\sigma_1 \int_{t_0}^{\sigma_1} d\sigma_2 \dots \int_{t_0}^{\sigma_1} \frac{d\sigma_p}{|\sigma|^{p-1}} \ge$$
$$\ge \int_{t_0}^{+\infty} d\sigma_1 \int_{t_0}^{\sigma_1} d\sigma_2 \dots \int_{t_0}^{\sigma_1} \frac{d\sigma_p}{(\sigma_1\sqrt{p})^{p-1}} = \int_{t_0}^{+\infty} \frac{(\sigma_1 - t_0)^{p-1}}{(\sigma_1\sqrt{p})^{p-1}} d\sigma_1 = +\infty.$$

Theorem 2 is completely proved. □

**Conjecture.** Condition (12) in Theorem 1 is necessary in the case when the measure $\nu$ on $\mathbb{R}^p_+$ is arbitrary. Is it true in general case?

For the class $H(\Lambda_p)$ the description (10) of a exceptional set in Borel's relation (9) can not improved in the following sense.

**Theorem 3** ([6]). *Let $h \in L^+$. There there exists a sequence $\Lambda_p = (\lambda_n)_{n\in\mathbb{Z}^p_+}$ satisfying the condition*

$$\int\limits_0^{+\infty} \frac{d\ln n_0(t)}{t} dt < +\infty, \tag{18}$$

*a function $F \in H_+(\Lambda_p)$, a constant $d > 0$ and a measurable set $E \subset \mathbb{R}^p_+$ such that:*

1. $(\forall\, x \in E)\colon\; \ln M(x,F) \ge (1+d)\ln\mu(x,F)$;
2. $\displaystyle\int\limits_E \frac{h(|x|)dx_1 \dots dx_p}{|x|^{p-1}} = +\infty.$

**Corollary 1.** *For each function $h \in L^+$ there exist a countably additive measure $\nu$ on $\mathbb{R}^p_+$, satisfying condition* (12)*, a function $F \in \mathcal{I}^p(\nu)$, a constant $d > 0$ and a measurable set $E \subset \mathbb{R}^p_+$ such that:*

1. $(\forall\ \sigma \in E)\colon\ \ln F(\sigma) \ge (1+d)\ln \mu_*(\sigma, F)$;
2. $\displaystyle\int\limits_E \frac{h(|x|)d\sigma_1 \dots d\sigma_p}{|\sigma|^{p-1}} = +\infty.$

*Proof.* We choose the measure $\nu$ of the form (7). Then condition (18) is equivalent to condition (12), and it remains to apply the previous theorem.

This completes the proof of Corollary 1. □

**Theorem 4.** *Let $F \in \mathcal{I}^p(\nu), \omega \in L_2, k(t)$ be the inverse function to $\frac{1}{\omega'(t)}$. If condition* (6) *holds for $\nu_1(0,t] = \nu_0(0,t]$, then relation* (4) *with $d = 0$ holds as $|\sigma| \to +\infty, \sigma \in K\backslash E$, where $K$ is an arbitrary real cone in $\mathbb{R}^p_+$ with the vertex at the point $O$ such that $\overline{K}\backslash\{O\} \subset \mathbb{R}^p_+$, and for the measurable set $E$* (10) *holds.*

*Proof.* Without loss of generality we may suppose that $F(0) = 1$. Repeating arguments similar to that in proof of Theorem 1 in the part of obtaining inequalities (15), and saving notation, we obtain

$$F(t\sigma_0) \le 2\mu(t\sigma_0, F)\nu_0\Big(0, \frac{2g'(t)}{y^*}\Big]. \tag{19}$$

We prove that $\exists \psi \in L_1^+\colon \ln \nu_0(0,t] = o(\psi^{-1}(t)) (t \to +\infty)$. As above we define the function

$$l(t) = \int\limits_t^{+\infty} \frac{k(\ln \nu_0(0,t])}{t^2} dt, \quad C(t) = (l(t))^{-\frac{1}{2}} \quad (t > 0).$$

As in the proof of Theorem 1 we have $\exists \psi \in L_1^+\colon k(\ln \nu_0(0,t]) = o(\psi^{-1}(t))\ (t \to +\infty)$. Since $k(t)$ is the inverse function to $\frac{1}{\omega'(t)}$ and $\omega \in L_2$ we obtain

$$\ln \nu_0(0,t] = o(k^{-1}(\psi^{-1}(t))) = o\Big(\frac{1}{\omega'(\psi^{-1}(t))}\Big) = o(\psi^{-1}(t))\ (t \to +\infty).$$

It now follows from the proof of Theorem 1 that inequality (13) holds as $\sigma = t\sigma_0, \sigma \in K\backslash E_1$, where

$$E_1 = \bigcup_{|\sigma_0|=1,\ \sigma_0 \in \mathbb{R}^p_+} E_1(\sigma_0), E_1(\sigma_0) = \Big\{\sigma = t\sigma_0\colon t > 0, \frac{2}{y^*}g'(t) > \psi(g(t))\Big\}.$$

Moreover $\tau_p(E \cap \mathbb{R}^p_+) < +\infty$.

Hence

$$\ln \mu_*(\sigma, F) \ge \frac{1}{2}\ln F(\sigma)\ (|\sigma| \to +\infty) \tag{20}$$

as $\sigma = t\sigma_0, \sigma \in K\backslash E_1$.

Since $\omega \in L_2$, $\omega'$ is a decreasing function. Then from (15) and by the mean value Lagrange's theorem of finite increments we obtain

$$\omega(\ln F(\sigma)) - \omega(\ln \mu_*(\sigma, F)) \le \omega'(\ln \mu_*(\sigma, F))(\ln F(\sigma) - \ln \mu_*(\sigma, F)) \le \\ \le \omega'(\ln \mu_*(\sigma, F))\Big(\ln 2 + \ln \nu_0\Big(0, \frac{2g'(t)}{y^*}\Big]\Big)$$

as $\sigma = t\sigma_0, \sigma \in K\backslash E_1$.

Let

$$E_2 = \bigcup_{|\sigma_0|=1,\ \sigma_0 \in \mathbb{R}^p_+} E_2(\sigma_0),\ \ E_2(\sigma_0) = \Big\{\sigma = t\sigma_0 \colon t > 0, \frac{2}{y^*} g'(t) > \psi\Big(\frac{g(t)}{2}\Big)\Big\}.$$

Then for $\sigma = t\sigma_0$, $\sigma \in K\backslash(E_1 \cup E_2)$,

$$\omega'(\ln \mu_*(\sigma, F)) \le \omega'\Big(\frac{1}{2}\ln F(\sigma)\Big) = \omega'\Big(\frac{1}{2} g(t)\Big) \le \omega'\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big)$$

as $|\sigma| \to +\infty$.

Since

$$\ln \nu_0(0, t] = o(k^{-1}(\psi^{-1}(t))) = o(1/\omega'(\psi^{-1}(t)))\ (t \to +\infty),$$

we have

$$\omega(\ln F(\sigma)) - \omega(\ln \mu_*(\sigma, F)) \le \omega'(\ln \mu_*(\sigma, F))\Big(\ln 2 + \ln \nu_0\Big(0, \frac{2g'(t)}{y^*}\Big]\Big) \le \\ \le \omega'\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big)\Big(\ln 2 + o\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big)\Big) = o(1) + \omega'\Big(\psi^{-1}\Big(\frac{2g'(t)}{y^*}\Big)\Big)\ln 2. \qquad (21)$$

Therefore $\psi^{-1}$ is nondecreasing and $\omega \in L_2$, we obtain (4).

Finally, we obtain the following estimate for the exceptional set $E = E_1 \cup E_2$

$$\tau_p(E_2 \cap \mathbb{R}^p_+) \le \frac{2}{y^*} \int\limits_{S_1} \Big( \int\limits_{E_2(\sigma_0)} \frac{g'(t)}{\psi(\frac{g(t)}{2})} dt \Big) ds \le \frac{2}{y^*} \int\limits_{S_1} \Big( \int\limits_{g(0)}^{g(+\infty)} \frac{du}{\psi(\frac{u}{2})} \Big) ds \le \\ \le C \int\limits_0^{+\infty} \frac{dt}{\psi(t)} < +\infty.$$

Since $\tau_p(E_1 \cap \mathbb{R}^p_+) < +\infty$, we have $\tau_p(E \cap \mathbb{R}^p_+) < +\infty$. □

In [3] an analogue of Theorem 4 for the class $H(\Lambda_p)$ is proved.

**Theorem 5** ([3])**.** *Let $\omega \in L_3 \cap L_4 \cap L_5$. For each function $F \in H(\Lambda_p)$ the relation*

$$\omega(\ln M(\sigma, F)) - \omega(\ln \mu(\sigma, F)) = o(1) \qquad (22)$$

*holds $|\sigma| \to +\infty$ ($\sigma \in K \setminus E, \mathrm{meas}_p(E \cap S_r) = O(r^{p-1})$ $(r \to +\infty)$) if and only if*

$$\int\limits_0^{+\infty} \frac{k(\ln n_0(t))}{t^2} dt < +\infty, \qquad (23)$$

*holds, where $K$ is an arbitrary cone $K \subset \mathbb{R}^p$ with vertex in point such that $(\overline{K} \setminus O) \subset \{\sigma \in \mathbb{R}^p\colon \lim_{t\to+\infty} \frac{1}{t}\ln\mu(t\sigma, F) = +\infty\}$, $S_r$ is a cylinder, which obtains from the cylinder $S'_r = \{x = (x_1, ..., x_p) \in \mathbb{R}^p\colon x_2^2 + ... + x_p^2 \le r^2\}$ by turning the coordinate system so that the axle $Ox_1$ moves in ray $\{x \in \mathbb{R}^p\colon x_1 = x_2 = ... + x_p\}$.*

From the proof in [3] necessary condition (23) in Theorem 5 and from Theorem 4 we obtain the following theorem.

**Theorem 6.** *Let $F \in H(\Lambda_p), \omega \in L_2 \cap L_4 \cap L_5, k(t)$ be the inverse function to the function $\frac{1}{\omega'(t)}$. For each function $F \in H(\Lambda_p)$ relation* (22) *holds as $|\sigma| \to +\infty, \sigma \in K\backslash E$, where $K$ is an arbitrary real cone in $\mathbb{R}^p_+$ with the vertex at the point $O$ such that $\overline{K}\backslash\{O\} \subset \mathbb{R}^p_+$ and measurable set $E$ satisfied* (10) *if and only if condition* (23) *holds.*

*Proof. Sufficiency.* We choose the measure $\nu$ of the form (7). Then condition (18) is equivalent to condition (12). It remains to apply Theorem 4.
*Necessity.* The necessity of condition (23) one can prove in a similar way to the proof of Corollary 3 ([3]), where for all $\sigma \in E = \{\sigma \in \mathbb{R}^p_+\colon \sigma_1 \ge t_0, \sigma_1 \ge \max\{\sigma_2, \sigma_3, ..., \sigma_p\}\}$ relation (22) holds, and that for this set $\tau_p(E) = +\infty$ (see. proof of Theorem 2). □

## REFERENCES


1. Скаскив О.Б. *О некоторых соотношениях между максимумом модуля и максимальным членом целого ряда Дирихле*// Матем. заметки. – 1999. – Т.66, №2. – С. 282–292.
2. Skaskiv O.B., Trakalo O.M. *Asymptotic estimates for Laplace integrals*// Mat. Stud. – 2002. – V.18, №2. – P. 125–146. (in Ukrainian)
3. Skaskiv O.B., Oryshchyn O.H. *Generalization of Borel's theorem for the multiple Dirichlet series*// Mat. Stud. – 1997. – V.8, №1. – P. 43–52. (in Ukrainian)
4. Skaskiv O.B., Trakalo O.M. *On Borel relation for entire multiple Dirichlet series*// Mat. Stud. – 2001. – V.15, №2. – P. 163–172. (in Ukrainian)
5. Skaskiv O.B., Zikrach D.Yu. *The best possible description of exceptional set in Borel's relation for multiple Dirichlet series*// Mat. Stud. – 2008. – V.30, №2. – P. 189–194.
6. Zikrach D.Yu., Skaskiv O.B. *On the description of an exceptional set in Borel's relation for multiple Dirichlet series with upper restriction on the growth*// Mat. Stud. – 2009. – V.32, №2. – P. 139–147. (in Ukrainian)
7. Скаскив О.Б. *О поведении максимального члена ряда Дирихле, задающего целую функцию*// Матем. заметки. – 1985. – V.37, №1. – C. 41–47.
8. Skaskiv O.B., Trakalo O.M. *Sharp estimate of exceptional set in Borel's relation for entire functions of several complex variables*// Mat. Stud. – 2002. – V.18, №1. – P. 53–56. (in Ukrainian)
9. Salo T.M., Skaskiv O.B., Trakalo O.M.*On the best possible description of exceptional set in Wiman–Valiron theory for entire function*// Mat. Stud. – 2001. – V.16, №2. – P. 131–140.
10. Filevych P.V. *Asymptotic relations between the means of Dirichlet series and their applications*// Mat. Stud. – 2002. – V.19, №2. – P. 127–140.
11. Скаскiв О.Б., Тракало О.М. *Про стiйкiсть максимального члена цiлого ряду Дiрiхле*// Укр. мат. журн. – 2005. – Т.57, №4. – С. 571–576.


Ivan Franko National University of L'viv,
matstud@franko.lviv.ua
Ukrainian Academy of Printing,
zikrach.dm@gmail.com

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