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ON THE LATTICE OF QUASI-FILTERS OF LEFT CONGRUENCE ON A PRINCIPAL LEFT IDEAL SEMIGROUPS

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The lattice structure on quasi-filters of left congruence on a principal left ideal semigroup are described.

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Описывается строение решетки квазифильтров левых конгруэнций над полугруппой главных левых идеалов.

Torsion theory, as mentioned by B. Stenstrom, is an important tool in the study of the quotient structure of rings. The concept of torsion theory for S-acts was introduced by J. K. Luedeman [4] in 1983. Recently, R. Zhang, W. Gao, F. Xu [8] introduced the concept of quasi-filter of right congruence on a semigroup S, and established a close relation between these quasi-filters and the hereditary torsion theory of S-acts. Their results have a new influence on the study of the torsion theories of S-acts [5], [7].

For rings the lattice structure on filters are considered in the book J. S. Golan [1]. We recall some necessary definitions.

Throughout this paper, S is always a multiplicative semigroup with 0 and 1. The terminology and definitions that are not given in this paper can be found in [3]. Denoted by $\mathrm{Con}(S)$ the set of all left congruence on S.

Definition 1. A quasi-filter (see [8]) of S is *defined to be the subset $\mathcal{E}$ of* $\mathrm{Con}(S)$ satisfying the following conditions:

1. If $\rho \in \mathcal{E}$ and $\rho \subseteq \tau \in \mathrm{Con}(S)$, then $\tau \in \mathcal{E}$.
2. $\rho \in \mathcal{E}$ implies $(\rho : s) \in \mathcal{E}$ for every $s \in S$.
3. If $\rho \in \mathcal{E}$ and $\tau \in \mathrm{Con}(S)$ such that $(\tau : s), (\tau : t)$ are in $\mathcal{E}$ for every $(s,t) \in \rho\backslash\tau$, then $\tau \in \mathcal{E}$.

Denoted by S-q-fil the set of all quasi-filters of left congruence on S. The set S-q-fil can be partially ordered by inclusion.

Since the intersection of an arbitrary family of quasi-filters is a quasi-filter, we may see that the set of all quasi-filters has the structure of a complete lattice, where the meet and the join of quasi-filters are defined in the usual way.

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The meet of quasi-filters $\mathcal{E}_1$ and $\mathcal{E}_2$ is the quasi-filter $\mathcal{E}_1 \wedge \mathcal{E}_2 = \{\rho | \rho \in \mathcal{E}_1 \text{ i } \rho \in \mathcal{E}_2\}$.

The join of quasi-filters $\mathcal{E}_1$ and $\mathcal{E}_2$ is the least quasi-filter $\mathcal{E}_1 \vee \mathcal{E}_2$ which contain both $\mathcal{E}_1$ and $\mathcal{E}_2$.

The unique minimal element in S-q-fil is $\omega = S \times S$ and the unique maximal element is $\mathcal{E}_{\triangle_S}$, which contains $\triangle_S$, when $\triangle_S = \{(s,s)|s \in S\}$. Also we call a quasi-filter $\mathcal{E}$ trivial if either it contains $\triangle_S$ or contains ω only.

Definition 2. A semigroup S is called a *principal left ideal* (p.l.i) semigroup if every left ideal L is principal, i.e. $L = Sf$ for some $f \in S$.

Theorem 1 ([6])**.** *If S is a p.l.i. semigroup then the left ideals form a chain under the set inclusion.*

Definition 3. A monoid S is called *left perfect* if every left S-act has a projective cover.

Theorem 2 ([3])**.** *A monoid S is left perfect if and only if S satisfies the following two conditions:*

(A) Every left S-act satisfies the ascending chain conditions for cyclic subacts.

(D) Every right unitary submonoid of S has a minimal left ideal generated by an idempotent.

Lemma 1 ([2])**.** *A monoid S satisfies condition A if and only if for any elements $a_1, a_2, ...$ of S there exists $n \in \mathbb{N}$ such that for any $i \in \mathbb{N}$, $i \geq j$ there exists $j_i, j_i \geq i+1$ with $Sa_i a_{i+1}...a_{j_i} = Sa_{i+1}...a_{j_i}$.*

Lemma 2. *If a monoid S has an infinite chain of left congruences then this chain form a nontrivial quasi-filter.*

Theorem 3. *All quasi-filters $\mathcal{E}$ are trivial if and only if S is a left perfect monoid.*

Proof. If all quasi-filters $\mathcal{E}$ are trivial then either it contains $\triangle_S$ or contains ω only, when $\triangle_S = \{(s,s)|s \in S\}$. So $(\triangle_S : s) \in \mathcal{E}$. Thus, there exists $n \in \mathbb{N}$ such that for any $i \in \mathbb{N}$, $i \geq j$ there exists $j_i, j_i \geq i+1$ with $Sa_i a_{i+1}...a_{j_i} = Sa_{i+1}...a_{j_i}$ and by Lemma 2: every right unitary submonoid of S has a minimal left ideal generated by an idempotent. Therefore, the monoid S is left perfect.

If a monoid S is a left perfect then by Lemma 1: there exists $n \in \mathbb{N}$ such that for any $i \in \mathbb{N}$, $i \geq j$ there exists $j_i, j_i \geq i+1$ with $Sa_i a_{i+1}...a_{j_i} = Sa_{i+1}...a_{j_i}$. Since $(\triangle_S : s) \in \mathcal{E}$ and by Lemma 2: monoid S has a finite chain of left congruences. Therefore, all quasi-filters $\mathcal{E}$ on the monoid S are trivial. □

Let I be a left principal ideal on S. We consider the set $\rho_{(1,a)} = \{s(1,a), s(a,1), s(1,a^2), s(a^2,1), ..., s(1,a^n), s(a^n,1), ...\} \bigcup \Delta$, where a is the element such that $I = Sa$ and $n \in \mathbb{N}$.

Lemma 3. *Let I be a left principal ideal on S. The set $\rho_{(1,a)}$, where a is the element such that $I = Sa$ and $n \in \mathbb{N}$, is a left congruence on the monoid S.*

For the proof see [9].

Lemma 4. *Let S be a p.l.i. semigroup that is not perfect. Then S-q-*fil *consists of three nontrivial quasi-filters.*

Proof. By Theorem 1: the left ideals form a chain under the set inclusion. Let $I_1 \subset I_2 \subset I_3 \subset ...$ be this chain. Any ideal I_i defines the Rees congruence ρ_{I_i} on S, by setting $a\rho_I b$ if $a, b \in I$ or $a = b$. Let $\rho_{I_1} \subset \rho_{I_2} \subset \rho_{I_3} \subset ...$ be a chain Rees congruence. By Lemma 2: this chain form a nontrivial quasi-filter, and we denoted it by $\mathcal{E}_1$.

Since S is a p.l.i. semigroup, there exist elements $a_1, a_2, a_3, ...$ such that $I_i = Sa_i$ and $i \in \mathbb{N}$. By Lemma 3: $\rho_{(1,a_1)} \subset \rho_{(1,a_2)} \subset \rho_{(1,a_3)} \subset ...$ is a chain left congruences on S, and by Lemma 2: this chain form a nontrivial quasi-filter, and we denoted it by $\mathcal{E}_2$.

It is clear that $\mathcal{E}_3 = \mathcal{E}_1 \vee \mathcal{E}_2$ and $\mathcal{E}_1 \wedge \mathcal{E}_2 = \varpi$. □

The next theorem is a corollary from the last lemma.

Theorem 4. *Let S be a p.l.i. semigroup that is not perfect. Then the lattice of S-q-*fil *has the following construction:*

$$\begin{tikzcd}[arrows=-]
 & \mathcal{E}_{\triangle_S} \arrow[d] & \\
 & \mathcal{E}_3 = \mathcal{E}_1 \vee \mathcal{E}_2 \arrow[dl] \arrow[dr] & \\
\mathcal{E}_2 \arrow[dr] & & \mathcal{E}_1 \arrow[dl] \\
 & \omega &
\end{tikzcd}$$

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