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PRERADICAL AND RADICAL FUNCTORS IN DIFFERENT CATEGORIES

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Usually preradicals in the categories of modules or rings are investigated. This notion for an arbitrary category are generalized.

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Исследуются обычные предрадикалы в категории модулей или колец. Эти понятия обобщаются на произвольные категории.

The paper is devoted to a study of usually preradicals in the categories of modules or rings. We generalize this notion for an arbitrary category.

All categories in our paper are assumed to be concrete. Recall that a category is called concrete if all objects are (structured) sets, morphisms from A to B are (structure preserving) mappings from A to B, composition of morphisms is the composition of mappings, and the identities are the identity mappings [1].

Let $\mathcal{A}$ be an arbitrary concrete category. (Though all these things we can do in an arbitrary category.)

Definition 1. A preradical functor on $\mathcal{A}$ is a subfunctor of the identity functor on $\mathcal{A}$. In other words, *a preradical functor* T assigns to each object A a subobject $T(A)$ as follows that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle i_1}\Big\uparrow & & {\scriptstyle i_2}\Big\uparrow \\ T(A) & \xrightarrow{T(\alpha)} & T(B) \end{array},$$

where i_1, i_2 are monomorphisms, is commutative.

Proposition 1. *Let T be a preradical functor on $\mathcal{A}$, C and D be objects of $\mathcal{A}$ such that $D \subset C$ (D is a subobject of C). If $T(D) = D$, then $D \subset T(C)$.*

Proof. By the definition of a preradical functor we have $T(D) \subset T(C)$. Since $T(D) = D$, it follows that $D \subset T(C)$. □

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If T_1 and T_2 are preradical functors, one defines the preradical functor T_1T_2 as

$$T_1T_2(C) = T_1(T_2(C)).$$

A preradical functor T is idempotent if $TT = T$.

An example of a nonidempotent preradical functor. Let R-Mod be a category of left R-modules. I is a left ideal of R. We define a preradical functor T_I as follows: $T_I(M) = IM \quad \forall M \in R$-Mod. If $I^2 \neq I$ then T_I is a nonidempotent preradical functor.

With a preradical functor T one can associate a class of objects, namely

$$\mathcal{T}_T = \{C |\ T(C) = C\}.$$

The objects belonging to $\mathcal{T}_T$ are called T-radical objects.

Proposition 2. *The class $\mathcal{T}$ is closed under epimorphic images.*

Proof. Let $A \in \mathcal{T}(T)$, and let $\varphi\colon A \to B$ be an epimorphism. By the definition of the preradical functor, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ A & \longrightarrow & T(B) \end{array},$$

where i_1, i_2 are monomorphisms, is commutative. Since φ is an epimorphism of the category, we obtain $T(B) = B$. So, $B \in \mathcal{T}(T)$. □

Proposition 3. *Let T be an idempotent preradical functor on $\mathcal{A}$. Then for every object C of $\mathcal{A}$ one has $T(C) = \bigcup_{i\in I} C_i$ (if the union exists), where all $C_i, i \in I$ are T–radical subobjects of $\mathcal{A}$.*

Proof. Since T is an idempotent preradical functor, it follows that $T(C) \in \mathcal{T}_T$. By Proposition 1, $\forall i \in I\ C_i \subset T(C)$, thus, $\cup_{i\in I} C_i \subset T(C)$ □

Throughout the paper, all preradical functors are considered to be idempotent.

Let $Nonset$ be a category such that the class of its objects is the class of nonempty sets and the class of its morphisms is the class of all maps between nonempty sets [2]. It is easy to verify that in $Nonset$ monomorphisms are injective mappings and epimorphisms are surjective ones. We will describe all idempotent preradical functors in $Nonset$.

The identity functor (in the arbitrary category) and the functor T, such that $T(A) = 0$ for every object (in a category with the zero object 0) are the preradical functors. These preradical functors will be called the *trivial* ones.

Theorem 1. *In the category $Nonset$ all idempotent preradical functors are trivial.*

Proof. To prove the theorem we must show that for every idempotent preradical functor T in $Nonset$ $T(M) = M$ for every object M of $Nonset$.

Suppose that $T(M) = M_1$, where M_1 is a subobject of M. Then $T(M_1) = TT(M) = T(M) = M_1$, i. e. M_1 belongs to $\mathcal{T}_T$. Consider an epimorphism $f\colon M_1 \to \{m\}$, where $\{m\}$ is a one-element set, $\{m\} \subset M$. Since the class $\mathcal{T}_T$ is closed under epimorphic images, $T(\{m\}) = \{m\}$ for every one-element set $\{m\}$. By Proposition 3, $T(M) = \cup_{i\in I}\{m\} = M$. □

We change the category $Nonset$ in such a way that it has zero objects and zero morphisms. For this purpose, we adjoin the special object, which will be denoted by 0 and adjoin the morphisms such that for every object S of the $Nonset$ with 0 there exists precisely one morphism $S \to 0$ and precisely one morphism $0 \to S$. A morphism $S \to S'$ will be a zero morphism if it factors through 0. We will denote it sometimes by $0_{SS'}$, but more often simply by 0. Such modified category with the zero object and the zero morphisms will be called Set_0.

Remark 1. In such a way we can make every category to have the zero object and zero morphisms.

Now consider categories $Mono\text{-}Set_0$ and $Epi\text{-}Set_0$ whose objects are all objects of Set_0 and morphisms are zero morphisms and all monomorphisms of Set_0 and all epimorphisms of Set_0 respectively.

Proposition 4. *In the category $Mono\text{-}Set_0$ all non-trivial preradical functors are defined as follows: $T_\beta(S) = \begin{cases} 0, |S| \le \beta \\ S, |S| > \beta \end{cases}$, where β is a cardinal number. Moreover, if $\beta = \beta + 1$ then also $T'_\beta(S) = \begin{cases} 0, |S| < \beta; \\ S, |S| \ge \beta. \end{cases}$*

Proof. Let $M \in Ob(Mono\text{-}Set_0)$, $|M| = \beta$, $T(M) = M$, where T is an idempotent preradical functor, $S \in Ob(Mono\text{-}Set_0)$, $|S| \ge |M|$. We can write:

$$S = \bigcup_{i \in I} M_i, \quad |M_i| = |M|.$$

Since $T(M_i) = M_i$ and $M_i \subset S$ we obtain by Proposition 3 $T(S) = \cup_{i\in I} M_i = S$.

On the other hand, if we have a morphism $S \to M$, then $|S| \le |M|$. So we can have $T(S) = 0$, $T(M) = M$ or $T(S) = S$, $T(M) = M$. In all these cases the diagram

$$\begin{array}{ccc} S & \longrightarrow & M \\ \uparrow & & \uparrow \\ T(S) & \longrightarrow & T(M) \end{array}$$

is commutative. So T (and T') is a preradical functor. □

Proposition 5. *In the category $Epi - Set_0$ all non-trivial preradical functors are defined as follows: $T_\beta(S) = \begin{cases} S, |S| \le \beta; \\ 0, |S| > \beta \end{cases}$, where β is a cardinal number. Moreover, if $\beta = \beta + 1$ then also $T'_\beta(S) = \begin{cases} S, |S| < \beta; \\ 0, |S| \ge \beta. \end{cases}$*

Proof. Let $M \in Ob(Epi\text{-}Set_0)$, $|M| = \beta$, $T(M) = M$, where T is an idempotent preradical functor, $S \in Ob(Epi\text{-}Set_0)$, $|S| \le |M|$. Consider an epimorphism $f\colon M \to S$. Since $M \in \mathcal{T}_T$ and $\mathcal{T}_T$ is closed under epimorphic images, one has that $T(S) = S$.

On the other hand, if we have a morphism $S \to M$, then $|S| \ge |M|$. So, we can obtain $T(S) = S$, $T(M) = 0$ or $T(S) = S$, $T(M) = M$. In all these cases the diagram

$$\begin{array}{ccc} S & \longrightarrow & M \\ \uparrow & & \uparrow \\ T(S) & \longrightarrow & T(M) \end{array}$$

is commutative. Hence, T (and T') is a preradical functor. □

Definition 2. Let T_1 and T_2 be functors from a category $\mathcal{A}$ to a category $\mathcal{B}$. The functor T_1 is called a *subfunctor* of the functor T_2 (denote $T_1 \leqslant T_2$) if $T_1(A)$ is a subobject of $T_2(A)$ (denote $T_1(A) \subseteq T_2(A)$) for every $A \in Ob(\mathcal{A})$ and the following diagram

$$\begin{array}{ccc} T_1(A_1) & \xrightarrow{T_1(\varphi)} & T_1(A_2) \\ {\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow \\ T_2(A_1) & \xrightarrow{T_2(\varphi)} & T_2(A_2) \end{array}$$

is commutative for every morphism $\varphi\colon A_1 \to A_2$, $A_1, A_2 \in Ob(\mathcal{A})$.

Definition 3. The functor T_1 is called a *normal subfunctor* of the functor T_2 if $T_1(A)$ is a normal subobject of $T_2(A)$ for every $A \in Ob(A)$.

Recall that A' is called a normal subobject of A (or an ideal) if $A' \to A$ is the kernel of some morphism [3, 4].

As a rule, we will consider the cases when the categories $\mathcal{A}$ and $\mathcal{B}$ coincide.

Definition 4. Let $\mathcal{A}$ be a category, T_1 and T_2 be functors on $\mathcal{A}$, such that T_1 is a normal subfunctor of T_2. A *factor-functor* T_2/T_1 is a functor such that $(T_2/T_1)(A) = T_2(A)/T_1(A)$ $\forall A \in Ob(\mathcal{A})$ and the next diagram is commutative

$$\begin{array}{ccc} T_1(A_1) & \xrightarrow{T_1(\varphi)} & T_1(A_2) \\ {\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow \\ T_2(A_1) & \xrightarrow{T_2(\varphi)} & T_2(A_2) \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ T_2(A_1)/T_1(A_1) & \longrightarrow & T_2(A_2)/T_1(A_2), \end{array}$$

where i_1, i_2 are normal monomorphisms, π_1, π_2 are canonical epimorphisms.

Definition 5. A preradical functor T on the category $\mathcal{A}$ is called a *radical functor* if $T(I/T) = 0$, where I is the identity functor.

Corollary (Of theorem 1)**.** *All idempotent radical functors in the category Set_0 are trivial.*

Proof. It is easy to verify that we can put $M/M = 0$ and $M/0 = M$. In this case all preradical functors in the category Set_0 will be radical functors. □

Now recall some definitions.

Definition 6. Let S be a monoid and let $A \neq \varnothing$ be a set. If we have a mapping

$$\mu\colon S \times A \to A,\ (s,a) \mapsto sa{:=}\mu(s,a)$$

such that

(a) $1 \cdot a = a$ and

(b) $(st)a = s(ta)$ for $s, t \in S$, $a \in A$,

we call A a *left S-act or* a *left act over S* and write ${}_SA$. More informally, we say that μ defines a left multiplication of elements from A by elements of S.

Analogously, we define a right S-act A and write A_S [1].

We denote the category of left S-acts by *S-Act* (objects of this category are left S-acts, and morphisms are S-homomorphisms).

Throughout the paper, all monoids and acts are assumed to have zeros [1].

Definition 7. A non-empty subset K of a semigroup S is called a *left (right) ideal of S* if $SK \subseteq K$ ($KS \subseteq K$), an ideal or a two-sided ideal of S if $KS \subseteq K$ and $SK \subseteq K$.

Definition 8. A subset A of a semigroup S is called *left* (resp., *right*) *T-nilpotent* if for any sequence of elements $\{a_1, a_2, a_3, \dots\} \subseteq A$, there exists an integer $n \geq 1$ such that $a_1 a_2 \cdots a_n = 0$ (resp., $a_n \cdots a_2 a_1 = 0$)[5].

Proposition 6. *Let I be a left ideal of the semigroup S. If I is a left T-nilpotent then $I_S A =_S A$ implies $A = 0$.*

Proof. Since $I_S A =_S A$, we have $\forall a_0 \in_S A \;\; \exists i_1 \in I, \exists a_1 \in_S A \;\; a_0 = i_1 a_1$; $\exists i_2 \in I, \exists a_2 \in_S A$ $a_1 = i_2 a_2$, so $a_0 = i_1 i_2 a_2$; $\dots$; $\exists i_n \in I, \exists a_n \in_S A \;\; a_{n-1} = i_n a_n$, so $a_0 = i_1 i_2 \cdots i_n a_n = 0 a_n = 0$. Hence, $A = 0$. □

A pair of mappings $(\varphi, \psi)\colon (S_1, A_1) \to (S_2, A_2)$, where $\psi\colon A_1 \to A_2$ is a map, and $\varphi\colon S_1 \to S_2$ is a surjective homomorphism of monoids is called a *semilinear transformation* if $\forall s \in S_1, \forall a \in A_1 \; \psi(sa) = \varphi(s)\psi(a)$.

We construct the category $\bigcup \mathcal{S}$-$\mathcal{A}ct$. The objects of the category $\bigcup \mathcal{S}$-$\mathcal{A}ct$ will be the pairs $(S, A) =_S A$, where S is a monoid and A is an S-act. The set of morphisms $H({}_{S_1}A_1, {}_{S_2}A_2)$ is defined as a quotient set of a collection of all semilinear transformations $(\varphi, \psi)\colon (S_1, A_1) \to$ $\to (S_2, A_2)$ by the equivalence relation $\sim$, such that $(\varphi, \psi) \sim (\varphi', \psi')$, if $\psi = \psi'$, and product of morphisms is defined naturally. The class, determined by the semilinear transformation (φ, ψ) will be denoted by $\widetilde{(\varphi, \psi)}$, or, more frequently, (φ, ψ). It is easy to verify that $\bigcup \mathcal{S}$-$\mathcal{A}ct$ is a category.

Theorem 2. *Let T be a preradical functor in the category $\bigcup \mathcal{S}$-$\mathcal{A}ct$. Then on every category S-act it induces a preradical functor T_S.*

Proof. Since $T(S, A)$ is a subobject of the object (S, A), it follows we can consider T on the category S-act. Put $T(S, A) = (S, T_S(A))$. But $T_S(A)$ is a subact of S-act A, so T_S is a preradical functor in the category S-act. □

However, it is not true in general.

Definition 9. The surjective homomorphism of monoids $\varphi\colon S_1 \to S_2$ is called essential in a subcategory $\mathcal{N}$ of a category $\mathcal{K}$ if every morphism (φ, ψ) belongs to $\mathcal{N}$.

Definition 10. A subcategory $\mathcal{N}$ of a category K is called essential if it has the following properties:

1) if (S, A) is an object of $\mathcal{N}$ then *S-act* belongs to $\mathcal{N}$;

2) if $\widetilde{(\varphi_0, \psi_0)}$ is a morphism of $\mathcal{N}$ then $\widetilde{(\varphi_0, \psi_0)} = \widetilde{(\varphi_1, \psi_1)}$, where φ_1 is a surjective homomorphism essential in the category $\mathcal{N}$.

Theorem 3. *Let $\mathcal{K}$ be an essential subcategory of $\bigcup \mathcal{S}$-$\mathcal{A}ct$, and let T_S be preradical functors on the categories S-Act, which belong to $\bigcup \mathcal{S}$-$\mathcal{A}ct$. Preradical functors T_S generate a preradical functor on $\mathcal{K}$ if and only if for every morphism $(\varphi, \psi)\colon (S_1, A_1) \to (S_2, A_2)$ of the category $\mathcal{K}$ one has that $\psi(T_{S_1}(A_1)) \subseteq T_{S_2}(A_2)$.*

Proof. ($\Rightarrow$) Let preradical functors T_S generate a preradical functor T. Then for every $(\varphi, \psi)\colon (S_1, A_1) \to (S_2, A_2)$ the following diagram is commutative

$$\begin{array}{ccc} (S_1, A_1) & \xrightarrow{(\varphi,\psi)} & (S_2, A_2) \\ {\scriptstyle i_1}\Big\uparrow & & {\scriptstyle i_2}\Big\uparrow \\ T(S_1, A_1) & \xrightarrow{T(\varphi,\psi)} & T(S_2, A_2) \end{array},$$

where i_1, i_2 are monomorphisms. But $T(S_1, A_1) = (S_1, T_{S_1}(A_1))$, $T(S_2, A_2) = (S_2, T_{S_2}(A_2))$, hence, $\psi(T_{S_1}(A_1)) \subseteq T_{S_2}(A_2)$

($\Leftarrow$) 1. We want to show that for every $(\varphi, \psi)\colon (S_1, A_1) \to (S_2, A_2)$ the following diagram is commutative

$$\begin{array}{ccc} (S_1, A_1) & \xrightarrow{(\varphi,\psi)} & (S_2, A_2) \\ {\scriptstyle i_1}\Big\uparrow & & {\scriptstyle i_2}\Big\uparrow \\ T(S_1, A_1) & \xrightarrow{T(\varphi,\psi)} & T(S_2, A_2) \end{array},$$

where i_1, i_2 are monomorphisms. Since $T(S_1, A_1) = (S_1, T_{S_1}(A_1))$, $T(S_2, A_2) = (S_2, T_{S_2}(A_2))$ and $\psi(T_{S_1}(A_1)) \subseteq T_{S_2}(A_2)$ we obtain the commutativity of the diagram. Thus, T is a preradical functor. □

Let I be a left ideal of the monoid S. Define $r_I(A) = \cup\{B \text{ is a subact of } A \mid IB = B\}$. An idempotent preradical r_I is called I–preradical.

It is clear that r_I is an idempotent preradical.

Lemma 1. *Let $I(S)$ be a left ideal of any monoid S, and let φ be essential in essential subcategory $\mathcal{N}$ of the category $\bigcup \mathcal{S}$-$\mathcal{A}ct$. If $\varphi(I(S_1)) \subseteq I(S_2)$ for all morphisms $(\varphi, \psi)\colon (S_1, A_1) \to (S_2, A_2)$ of the category $\bigcup \mathcal{S}$-$\mathcal{A}ct$, then $I(S)-$preradical functors generate a preradical functor T on the category $\mathcal{N}$ (which coincides with $I(S)$–preradical functor on every category S-Act).*

Proof. Define a functor T as follows: $T(S, A) = (S, T_S(A))$ and $T(\varphi, \psi) = (\varphi, \psi_{T_S(A)})$ for every (S, A) and (φ, ψ) belonging to the category $\mathcal{N}$, where $\psi_{T_S(A)}$ is the restriction of the homomorphism ψ on the act $T_S(A)$. It remains to show that inclusions $\psi(T_{S_1}(A_1)) \subseteq$ $\subseteq T_{S_2}(A_2)$ hold true for every morphism $(\varphi, \psi)\colon (S_1, A_1) \to (S_2, A_2)$ of the category $\mathcal{N}$. The surjective homomorphism φ can be considered as essential in $\mathcal{N}$, because the category $\mathcal{N}$ is essential. Since $\varphi(I(S_1)) \subseteq I(S_2)$, it follows by the definition of an $I(S_2)$-radical in S_2-act, $\psi(T_{S_1}(A_1)) = \psi(I(S_1)T_{S_1}(A_1)) = \varphi(I(S_1))\psi(T_{S_1}(A_1))$. □

Definition 11. A monoid S is called *monoid with ACC on left (resp., right, two-sided) ideals* if for every strictly ascending chain of left (resp., right, two-sided) ideals $I_1 \subset I_2 \subset \dots$ $\exists i \in \mathbb{N}$ $I_i = I_{i+1} = \dots$ [3].

Definition 12. A left (resp., right, two-sided) ideal J is called the *maximal* left (resp., right, two-sided) ideal for preradical r_I if $r_I = r_J$ implies $I \subseteq J$.

Theorem 4. *Let S be arbitrary monoid with ACC on two-sided ideals and let $I(S)$ be a two-sided ideal of the monoid S, which is maximal for preradical functor $T_I(S)$. Preradical functors $T_I(S)$ in S-Act generate the $I(S)$-preradical functor in every subcategory $\mathcal{K}$ of the category $\bigcup \mathcal{S}$-$\mathcal{A}ct$ if and only if for every morphism $(\varphi, \psi)\colon (S_1, A_1) \to (S_2, A_2)$ of the category $\mathcal{K}$ $\varphi(I(S_1)) \subseteq I(S_2)$ holds.*

Proof. ($\Leftarrow$) See the proof of Lemma 1.

($\Rightarrow$)$\psi(T_{S_1}(A_1)) = \varphi(I(S_1))\psi(T_{S_1}(A_1))$ (see the proof of Lemma 1). $T_{I(S_2)}(A_2) = I(S_2)$ $T_{I(S_2)}(A_2)$ implies $\psi(T_{S_1}(A_1)) = I(S_2) \times \psi(T_{S_1}(A_1))$. Since φ is a surjective monoid homomorphism, S_2-act $\subseteq$ S_1-act and since $I(S_2)$ is a maximal, we obtain that $\varphi(I(S_1)) \subseteq I(S_2)$. □

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