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VECTOR–VALUED FUNCTIONAL CALCULUS FOR A CONVOLUTION ALGEBRA OF DISTRIBUTIONS ON CONE

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For the Fourier image $\widehat{D}'_\Gamma$ of the algebra D'_Γ of the distributions with supports on a cone Γ the functional calculus for generators of n-parametric (C_o)-semigroups of operators is determined. For this purpose, we consider construction of the dual pair $\langle \widehat{D}'_\Gamma, \widehat{D}_\Gamma\rangle$, and provide some examples with respect to the formula of operator calculus.

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Для Фурье-образов $\widehat{D}'_\Gamma$ алгебры D'_Γ распределений с носителями в конусе Γ строится функциональное исчисление для генераторов n-параметрических (C_o)-полугрупп операторов. С этой целью рассматривается построение дуальной пары $\langle \widehat{D}'_\Gamma, \widehat{D}_\Gamma\rangle$. Относительно формулы операторного исчисления рассмотрены несколько примеров.

1. Introduction. In general, the idea of construction of the functional calculus is related to Poincare's work on the theory of continuous groups. Functional calculus in Sobolev spaces of generalized functions for selfajoint operators is developed in the paper [1]. A full mathematical verification of Heaviside operational calculus is constructed in [2]. Functional calculus for generators of the one-parametric (C_o)-semigroups of operators over Banach spaces in the convolution algebra of distributions on the semiaxis is defined in [3].

The purpose of this article is the construction of functional calculus for generators of n-parametric (C_o)-semigroups of operators in convolution algebra of Schwartz distributions with supports in any closed, acute-angled and convex cone. Special case of this functional calculus is considered in [4].

2. Construction of the duality $\langle \widehat{\mathbf{D}}'_{\boldsymbol\Gamma}, \widehat{\mathbf{D}}_{\boldsymbol\Gamma}\rangle$. Let us consider the classical Schwartz duality $\langle D'(\mathbb{R}^n), D(\mathbb{R}^n)\rangle$. As usually, $D(\mathbb{R}^n)$ — the space of infinitely differentiable functions with compact supports $\mathrm{supp}\varphi \subset \mathbb{R}^n$, $D'(\mathbb{R}^n)$ — the space of linear and continuous functionals over $D(\mathbb{R}^n)$, i.e. the space of Schwartz distributions.

We denote by Γ any closed, acute–angled and convex cone in $\mathbb{R}^n$, D'_Γ is the subspace of $D'(\mathbb{R}^n)$ of distributions f, such that $\mathrm{supp} f \subset \Gamma$ [5, Sect. I, §4].

The polar of subspace D'_Γ with respect to the duality $\langle D'(\mathbb{R}^n), D(\mathbb{R}^n)\rangle$ is given by

$$(D'_\Gamma)^o = \{\varphi \in D(\mathbb{R}^n)\colon \mathrm{supp}\varphi \subset \mathbb{R}^n \backslash \Gamma\}.$$

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The restriction of bilinear form $D'(\mathbb{R}^n) \times D(\mathbb{R}^n) \ni (f,\varphi) \longmapsto \langle f,\varphi\rangle \in \mathbb{C}$ onto the direct product $D'_\Gamma \times D(\mathbb{R}^n)$ is constant on any set $\{(f_o,\varphi)\}$, where $f_o \in D'_\Gamma$ is fixed functional and function φ runs through quotient class $[\varphi]$ in the quotient space $D(\mathbb{R}^n)/(D'_\Gamma)^o$. Thus, the bilinear form $D'_\Gamma \times D(\mathbb{R}^n)/(D'_\Gamma)^o \ni (f,[\varphi]) \longmapsto \langle f,\varphi\rangle \in \mathbb{C}$, $\varphi \in [\varphi]$ leads the spaces D'_Γ and $D(\mathbb{R}^n)/(D'_\Gamma)^o$ into duality [6, Sect.II].

The topology convergence in $D(\mathbb{R}^n)/(D'_\Gamma)^o$ is equivalent to sequential convergence, i.e. $[\varphi]_m \longrightarrow [\varphi]$ if for any ε–neighborhood $\mathcal{O}_{\varepsilon,K}$ of compactum $K \subset \Gamma$, for any representatives $\{\varphi_m \in [\varphi]_m\colon \mathrm{supp}\varphi_m \subset \mathcal{O}_{\varepsilon,K}\}$ and $\{\varphi \in [\varphi]\colon \mathrm{supp}\varphi \subset \mathcal{O}_{\varepsilon,K}\}$ the following convergence is fulfilled:

$$\lim_{m\to\infty} \sup_{t\in\mathcal{O}_{\varepsilon,K}} \left|\partial^k\varphi_m(t) - \partial^k\varphi(t)\right| = 0, \forall k \in \mathbb{Z}^n_+,$$

where $\partial^k = \partial_1^{k_1}\ldots\partial_n^{k_n}$, $\partial_j^{k_j} = \frac{\partial^{k_j}}{\partial t_j^{k_j}}$, $(j = 1,\ldots,n)$.

It is well known that, for any $\varepsilon > 0$ there exists $\rho_\varepsilon \in C^\infty(\mathbb{R}^n)$ such that

$$\rho_\varepsilon(t) = \begin{cases} 1, t \in \mathcal{O}_{\varepsilon/2,K}; \\ 0, t \notin \mathcal{O}_{\varepsilon,K}. \end{cases}$$

Therefore, the corresponding quotient class $[\varphi] \in D(\mathbb{R}^n)/(D'_\Gamma)^o$ can be identified with the germ of C^∞–functions of form $\rho_\varepsilon \cdot \varphi$, $\varphi \in [\varphi]$.

Let $\lambda_\Gamma(t) = \begin{cases} 1, t \in \Gamma \\ 0, t \notin \Gamma \end{cases}$ be the characteristic function of the cone Γ. Define the mapping

$$\varrho\colon D(\mathbb{R}^n) \ni \varphi \longmapsto \lambda_\Gamma\varphi =: \psi \in D_\Gamma$$

as multiplication operator by the characteristic function, and the space D_Γ is defined as follows $D_\Gamma\colon = \{\psi = \lambda_\Gamma\varphi\colon \varphi \in D(\mathbb{R}^n)\}$. It is obvious that $\mathrm{Ker}\varrho = (D'_\Gamma)^o$.

For any natural number ν we construct the set Γ_ν as intersection of cone Γ with the ball of radius ν and consider the space of functions

$$D_{\Gamma_\nu}\colon = \{\psi(t) = \lambda_\Gamma(t)\varphi(t)\colon \varphi(t) \in D(\mathbb{R}^n), \mathrm{supp}\varphi \cap \Gamma \subset \Gamma_\nu\}.$$

The topology of the space D_{Γ_ν} is defined by the set of norms

$$\|\psi\|_{m,\nu} = \sum_{|k|\le m} \frac{1}{k!} \sup_{t\in\Gamma_\nu} |\partial^k\psi(t)| < \infty.$$

For any numbers $\nu \le \mu$ inclusions $D_{\Gamma_\nu} \subset D_{\Gamma_\mu}$ are continuous. Thus, we can represent the space D_Γ as the following inductive limit

$$D_\Gamma \simeq \bigcup_{\Gamma_\nu\subset\Gamma} D_{\Gamma_\nu} = \limind_{\nu\to\infty} D_{\Gamma_\nu}. \tag{1}$$

We can now formulate some useful propositions, which were proved in [7].

Proposition 1. *The spaces D_Γ and $D(\mathbb{R}^n)/(D'_\Gamma)^o$ are topologically isomorphic and canonical bilinear form from $\langle D'(\mathbb{R}^n), D(\mathbb{R}^n)\rangle$ induces the duality $\langle D'_\Gamma, D_\Gamma\rangle$.*

Proposition 2. *The space D_Γ is (LF)–space, in addition it is barreled, bornological Montel space.*

Proposition 3. *Suppose that for an arbitrary compactum $K \subset \Gamma$ the space D'_K is conjugate to D_K endowed by the strong topology with respect to the duality $\langle D'_K, D_K\rangle$. Then the space D'_Γ in its strong topology $\beta(D'_\Gamma, D_\Gamma)$ is topologically isomorphic to the projective limit*

$$D'_\Gamma \simeq \lim_{K\subset\Gamma} \operatorname{pr} D'_K.$$

Let us observe that the space D'_Γ being strongly conjugate to the space D_Γ, is a Montel space. The space D'_Γ is a convolution algebra [5, §4, p.75]. The convolution $D'_\Gamma \times D'_\Gamma \ni (f,h) \longmapsto f * h \in D'_\Gamma$ is determined by the formula

$$\langle f * g, \psi\rangle = \langle f(x), \xi(x)\langle h(y), \eta(y)\psi(x+y)\rangle\rangle,$$

where $\psi \in D_\Gamma$ and $\xi(x), \eta(y)$ are arbitrary infinitely differentiable functions, which are equal to 1 in the neighborhood of supports of distributions f and h respectively and equal to 0 out of these neighborhoods.

Lemma 1. *The convolution in the algebra D'_Γ is continuous with respect to the strong topology $\beta(D'_\Gamma, D_\Gamma)$.*

Proof. The locally convex space D'_Γ is reflexive one in the topology $\beta(D'_\Gamma, D_\Gamma)$. Since the multiplication of reflexive locally convex algebra is continuous [8, Ch.5, Pr. 5.1], the convolution in algebra D'_Γ is a continuous mapping with respect to the strong topology. □

Lemma 2. *Inclusion $D_\Gamma \subset D'_\Gamma$ is dense with respect to the strong topology $\beta(D'_\Gamma, D_\Gamma)$.*

Proof. Let $\{\psi_n\}_{n=1}^\infty$ be any sequence of test functions from the space $D(\mathbb{R}^n)$, that approximate delta-function δ with respect to the strong topology of the space $D'(\mathbb{R}^n)$. The result of convolution $f * \psi_n$ belongs to the space $D(\mathbb{R}^n)$ for any $f \in D'(\mathbb{R}^n)$. Thus, $D(\mathbb{R}^n)$ is dense in $D'(\mathbb{R}^n)$. Then intersection $D'_\Gamma \cap D(\mathbb{R}^n)$ is dense in the subspace $D'_\Gamma \subset D'(\mathbb{R}^n)$. So, the mapping ϱ is the identity operator on the space $D'_\Gamma \cap D(\mathbb{R}^n)$ as the operator of multiplication by the characteristic function of the cone Γ. Thus, the subspace $D'_\Gamma \cap D(\mathbb{R}^n) = \varrho[D'_\Gamma \cap D(\mathbb{R}^n)]$ is dense in the space D'_Γ. □

The n-parametric semigroup of shifts along the cone Γ is defined as follows

$$\mathcal{T}_s\colon D(\mathbb{R}^n) \ni \varphi(t) \mapsto \varphi(t+s) \in D(\mathbb{R}^n), \forall s \in \Gamma, \forall t \in \mathbb{R}^n.$$

We define the (C_o)–semigroup T_s to be the unique semigroup satisfying the following relation $(T_s \circ \varrho)\varphi(t) = (\varrho \circ \mathcal{T}_s)\varphi(t)$. Let $L(D_\Gamma)$ denote the algebra of linear and continuous mappings over the space D_Γ with the composition instead of the multiplication. Note that $T_s \in L(D_\Gamma)$ and T_s is an equicontinuous semigroup on the space D_Γ (see [7]).

For every distribution $f \in D'_\Gamma$ and $\varphi \in D(\mathbb{R}^n)$ we define the operation

$$(\mathcal{M}_f\varphi)(t) = (f \star \varphi)(t) = \langle f(s), \mathcal{T}_s\varphi(t)\rangle, \forall s \in \Gamma, \forall t \in \mathbb{R}^n.$$

The operation of cross-correllation, denoted by M_f, is defined to be the unique operator satisfying the following relation $(M_f \circ \varrho)\varphi(t) = (\varrho \circ \mathcal{M}_f)\varphi(t)$. In [7] it is proved, that the mapping $D'_\Gamma \ni f \mapsto M_f \in L(D_\Gamma)$ realizes the topological isomorphism of convolution algebra of distributions D'_Γ onto the commutant of the semigroup T_s in the algebra $L(D_\Gamma)$. In particular, for any distributions $f, g \in D'_\Gamma$ the following equalities hold

$$M_{f*g} = M_f \circ M_g, M_\delta = I,$$

where δ is the Dirac function and I is the identity operator in $L(D_\Gamma)$.

Lemma 3. *The following properties of the operation of cross–correllation are valid* $\forall k \in \mathbb{Z}_+^n, |k| = k_1 + \cdots + k_n, \forall f, g \in D'_\Gamma, \forall \psi \in D_\Gamma$

1) $\partial^k (M_f \psi) = M_f \partial^k \psi = (-1)^{|k|} M_{\partial^k f} \psi$;

2) $M_{\partial^k f} \circ M_g = M_f \circ M_{\partial^k g}$;

3) $M_f \psi(0) = \langle f, \psi \rangle$.

Fourier transformation in the space $D(\mathbb{R}^n)$ is defined by

$$\mathcal{F} \colon D(\mathbb{R}^n) \ni \varphi \mapsto \widehat{\varphi}(\xi) \in \widehat{D}(\mathbb{R}^n),$$

where $\widehat{D}(\mathbb{R}^n) \colon = \left\{ \widehat{\varphi}(\xi) = \int\limits_{\mathbb{R}^n} \varphi(t) e^{-i(t,\xi)} dt \colon \varphi \in D(\mathbb{R}^n) \right\}$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

Let the Fourier transformation of functions from space D_Γ be given by

$$F \colon D_\Gamma \ni \psi(t) \mapsto \widehat{\psi}(\xi) = \int_\Gamma e^{-i(t,\xi)} \psi(t) dt, \tag{2}$$

Let $\widehat{D}_\Gamma$ denote the codomain of the space D_Γ, i.e. $\widehat{D}_\Gamma = \{\widehat{\psi}(\xi) \colon \psi(t) \in D_\Gamma\}$. Note that for any function $\psi \in D_\Gamma$ the equality $\mathcal{F}[\psi] = F[\psi]$ is valid because of the condition $\psi(t) \equiv 0$, $\forall t \in \mathbb{R}^n \setminus \Gamma$.

Since D_Γ has the form of inductive limit (1) of subspaces D_{Γ_ν} with norms $\{\| \cdot \|_{m,\nu}\}_{m \in \mathbb{N}}$, the injectivity of mapping (2) implies that we can define the topology on the Fourier–image $\widehat{D}_{\Gamma_\nu} = \{\widehat{\psi} \colon \psi \in D_{\Gamma_\nu}\}$ by norms $\|\widehat{\psi}\|_{m,\nu} = \|\psi\|_{m,\nu}$. Then $\widehat{D}_\Gamma = \bigcup_{\nu > 0} \widehat{D}_{\Gamma_\nu}$, and inclusion $\widehat{D}_{\Gamma_\nu} \subset \widehat{D}_{\Gamma_\mu}$ is continuous, when $\nu \le \mu$. Thus, we may endow the space $\widehat{D}_\Gamma$ with the locally convex topology of inductive limit $\widehat{D}_\Gamma = \liminf\limits_{\nu \to \infty} \widehat{D}_{\Gamma_\nu}$.

As results, the basic properties of the space D_Γ are transferred into the space $\widehat{D}_\Gamma$. That is, the Fourier–image $\widehat{D}_\Gamma$ is a Montel, barreled and bornologic (LF)–space.

Let us observe that the inverse Fourier transformation $F^{-1} \colon \widehat{D}_\Gamma \ni \widehat{\psi} \mapsto \psi \in D_\Gamma$ exists and has the following property

$$\psi(t) = F^{-1}(F[\psi]) = \lambda_\Gamma(t) \mathcal{F}^{-1}(F[\psi]),$$

where $\mathcal{F}^{-1} \colon \widehat{D}(\mathbb{R}^n) \to D(\mathbb{R}^n)$ is the inverse mapping to $\mathcal{F}$.

Let $\widehat{D}'_\Gamma$ be the conjugate space to $\widehat{D}_\Gamma$. We shall denote by

$$F^* = (2\pi)^n (F^{-1})' \colon D'_\Gamma \ni f \mapsto \widehat{f} \in \widehat{D}'_\Gamma \tag{3}$$

the conjugate mapping to the inverse Fourier transformation F^{-1}. We call F^* the generalized Fourier transformation of distributions from the space D'_Γ. The definition of F^* can be written as

$$\langle F^* f, \widehat{\psi} \rangle = (2\pi)^n \langle f, F^{-1} F[\psi] \rangle, \quad f \in D'_\Gamma, \widehat{\psi} \in \widehat{D}_\Gamma.$$

For similarity of notation, we write $\widehat{f}$ instead of $F^* f$. Then we may write

$$\langle \widehat{f}, \widehat{\psi} \rangle = (2\pi)^n \langle f, F^{-1} F[\psi] \rangle = (2\pi)^n \langle f, \psi \rangle. \tag{4}$$

Formula (4) with mapping (3) defines a new duality $\langle \widehat{D}'_\Gamma, \widehat{D}_\Gamma \rangle$.

3. The vector cross-correllation operation. In this part of the paper we shall prove the nuclearity of the spaces of duality $\langle D'_\Gamma, D_\Gamma \rangle$ and shall define the vector cross-correllation operation.

Lemma 4. *The spaces D_Γ and D'_Γ are nuclear.*

Proof. As known [9, §2, Theor. 7], the nuclearity of any locally convex space implies that its quotient space with respect to a close subspace and the strongly conjugate space are nuclear too. The space $D(\mathbb{R}^n)$ is nuclear (see [10, §10, p. 405]). Thus, the space $D_\Gamma \colon = D(\mathbb{R}^n)/(D'_\Gamma)^o$ and D'_Γ are nuclear. □

Let $\{\mathcal{Y}, \|\cdot\|\}$ be a complex Banach space. We consider the space $D(\mathbb{R}^n, \mathcal{Y})$ of infinitely smooth $\mathcal{Y}$-valued functions $x(t)$ with compact supports in $\mathbb{R}^n$. Topology of $D(\mathbb{R}^n, \mathcal{Y})$ is defined by the set of norms

$$\|x\|_m = \sum_{|k|\le m} \frac{1}{k!} \sup_{t\in\mathbb{R}^n} |\partial^k x(t)| < \infty.$$

The space $D^\nu(\mathbb{R}^n) = \{\varphi \in D(\mathbb{R}^n) \colon \operatorname{supp}\varphi \subset B^n_\nu\}$ is a Fréchet subspace of $D(\mathbb{R}^n)$. Here B^n_ν is the ball in $\mathbb{R}^n$ of radius ν. Let us note, that the space D_{Γ_ν} can be reintroduced as $D_{\Gamma_\nu} = \varrho(D^\nu(\mathbb{R}^n))$, where ϱ is the characteristic function of the cone Γ.

We shall denote by $D_\Gamma(\mathcal{Y})$ the space of infinitely smooth $\mathcal{Y}$–valued functions $x(t)$ with compact supports in Γ.

Theorem 1. *The following topological isomorphisms are realized*

$$D(\mathbb{R}^n, \mathcal{Y}) \simeq \mathcal{Y}\widetilde{\otimes} D(\mathbb{R}^n) \simeq \liminf_{\nu\to\infty} \mathcal{Y}\widetilde{\otimes} D^\nu(\mathbb{R}^n), \tag{5}$$

$$D_\Gamma(\mathcal{Y}) \simeq \mathcal{Y}\widetilde{\otimes} D_\Gamma \simeq D(\mathbb{R}^n, \mathcal{Y})/\mathcal{Y}\widetilde{\otimes} Ker\varrho \simeq \liminf_{\nu\to\infty} \mathcal{Y}\widetilde{\otimes} D_{\Gamma_\nu}, \tag{6}$$

where the sign $\widetilde{\otimes}$ denotes the completion of tensor product in the projective topology.

Proof. The topological isomorphism $D(\mathbb{R}^n, \mathcal{Y}) \simeq \mathcal{Y}\widetilde{\otimes} D(\mathbb{R}^n)$ is realized by virtue of well known Grothendieck's theorem [9, §3, theor. 13] about a representation of tensor product of two complete spaces one of which is nuclear. Then Proposition 1 implies

$$D(\mathbb{R}^n, \mathcal{Y})/\mathcal{Y}\widetilde{\otimes}\mathrm{Ker}\varrho \simeq \mathcal{Y}\widetilde{\otimes} D(\mathbb{R}^n)/\mathcal{Y}\widetilde{\otimes}\mathrm{Ker}\varrho \simeq \mathcal{Y}\widetilde{\otimes}[D(\mathbb{R}^n)/\mathrm{Ker}\varrho] \simeq \mathcal{Y}\widetilde{\otimes} D_\Gamma.$$

Let $\mathcal{Y}\widetilde{\widetilde{\otimes}} D_\Gamma$ be completion of tensor product in the uniform convergence topology on the equicontinuous subsets of the dual spaces $\mathcal{Y}'$ and D'_Γ. It is known [11, Sect. IV, 9.4] that the isomorphism relation $D_\Gamma(\mathcal{Y}) \simeq \mathcal{Y}\widetilde{\widetilde{\otimes}} D_\Gamma$ holds. The isomorphism relation $\mathcal{Y}\widetilde{\otimes} D_\Gamma \simeq \mathcal{Y}\widetilde{\widetilde{\otimes}} D_\Gamma$ is realized due to the nuclearity of D_Γ [11, Sect. IV, 9.4], i.e. $D_\Gamma(\mathcal{Y}) \simeq \mathcal{Y}\widetilde{\otimes} D_\Gamma$. Thus, the topological isomorphism relation (5) and (6) are proved. □

Lemma 5. *For any element $x = x(t) \in D_\Gamma(\mathcal{Y})$, $t \in \Gamma$ there exists a number $\nu > 0$ such that $x(t) \in \mathcal{Y}\widetilde{\otimes} D_{\Gamma_\nu}$ and $x(t)$ can be written as absolutely convergent series in the space $\mathcal{Y}\widetilde{\otimes} D_{\Gamma_\nu}$ in the following way*

$$x(t) = \sum_{m=1}^{\infty} \lambda_m x_m \otimes \psi_m(t), \tag{7}$$

where $\sum_m |\lambda_m| < \infty$, and the sequences of functions $\{\psi_m(t)\}$ and $\{x_m\}$ converge to zero in D_{Γ_ν} and $\mathcal{Y}$ respectively.

In addition, the equality

$$\partial^k x(t) = \sum_{m=1}^{\infty} \lambda_m x_m \otimes \partial^k \psi_m(t), \forall k \in \mathbb{Z}^n_+ \tag{8}$$

holds.

Proof. From definition of the space $D_\Gamma(\mathcal{Y})$ it follows that for every $x \in D_\Gamma(\mathcal{Y})$ there exists a number $\nu > 0$ such that $x \in \mathcal{Y} \otimes D_{\Gamma_\nu}$. Obviously, the spaces $\mathcal{Y}$ and D_{Γ_ν} are metrizable. Then for an arbitrary element $x \in \mathcal{Y}\widetilde{\otimes} D_{\Gamma_\nu}$ we can use the theorem about representation of elements of metrizable spaces, which competed in the projective tensor product topology [11, Sect.III, 6.4]. Thus, from this theorem we obtain expansion $x(t)$ into series (7).

Equality (8) follows from absolute convergence of series (7). □

Let $I_\mathcal{Y}$ be the identity operator in a Banach space $\mathcal{Y}$, $M_f \in L(D_\Gamma)$ the operator of cross-correllation. We define the vector cross–correllation operation by

$$(I_\mathcal{Y} \otimes M_f)x(t) = \begin{cases} \sum\limits_{m=1}^{\infty} \lambda_m x_m (M_f \psi_m)(t), t \in \Gamma, \\ \qquad\qquad 0, \qquad\qquad t \notin \Gamma. \end{cases} \tag{9}$$

Lemma 6. *For any distributions $f, g \in D'_\Gamma$ and function $x(t) \in D_\Gamma(\mathcal{Y})$ we have $(I_\mathcal{Y} \otimes M_f) \in L(D_\Gamma(\mathcal{Y}))$ and the following equalities hold*

$$(I_\mathcal{Y} \otimes M_{f*g})x(t) = (I_\mathcal{Y} \otimes (M_f \circ M_g))x(t),$$

$$\partial^k (I_\mathcal{Y} \otimes M_f)x(t) = (I_\mathcal{Y} \otimes M_f)\partial^k x(t) = (-1)^{|k|}(I_\mathcal{Y} \otimes M_{\partial^k f})x(t), \forall k \in \mathbb{Z}^n_+,$$

$$(I_\mathcal{Y} \otimes M_f)x(0) = (I_\mathcal{Y} \otimes f)x(t).$$

Proof. Let $f \in D'_\Gamma$. Then definition (9) implies that the operator $I_\mathcal{Y} \otimes M_f$ is a linear and continuous transformation from the space $D_\Gamma(\mathcal{Y})$ into itself.

From the definition of operator $I_\mathcal{Y} \otimes M_f$ we obtain

$$(I_\mathcal{Y} \otimes M_{f*g})x(t) = \sum_{m=1}^{\infty} \lambda_m x_m (M_{f*g}\psi_m)(t) = \sum_{m=1}^{\infty} \lambda_m x_m ((M_f \circ M_g)\psi_m)(t) = (I_\mathcal{Y} \otimes M_f \circ M_g)x(t).$$

Let us prove the second property. For all $k \in \mathbb{Z}^n_+$ we have

$$\partial^k (I_\mathcal{Y} \otimes M_f)x(t) = \partial^k \sum_{m=1}^{\infty} \lambda_m x_m (M_f \psi_m)(t) = \sum_{m=1}^{\infty} \lambda_m \partial^k x_m (M_f \psi_m)(t) = (I_\mathcal{Y} \otimes M_f)\partial^k x(t).$$

Besides,

$$(I_\mathcal{Y} \otimes M_f)\partial^k x(t) = \sum_{m=1}^{\infty} \lambda_m x_m (M_f \partial^k \psi_m)(t) = (-1)^{|k|} \sum_{m=1}^{\infty} \lambda_m x_m \langle \partial^k f, (T_s \psi_m)(t)\rangle$$

$$= (-1)^{|k|}(I_\mathcal{Y} \otimes M_{\partial^k f})x(t), \quad \forall s \in \Gamma.$$

The last property follows from the definition of vector cross-correllation operation. □

The vector operator of shifts, denoted by $I_\mathcal{Y} \otimes T_s$, is defined as follows

$$I_\mathcal{Y} \otimes T_s \colon D_\Gamma(\mathcal{Y}) \ni x(t) \mapsto \sum_{m=1}^{\infty} \lambda_m x_m (T_s \psi_m)(t) \in D_\Gamma(\mathcal{Y}).$$

Theorem 2. *For every distribution $f \in D'_\Gamma$ the operator $I_{\mathcal{Y}} \otimes M_f$ is nuclear and invariant with respect to the vector operator of shifts.*

Conversely, for an arbitrary operator $K \in L(D_\Gamma)$, which is invariant with respect to the vector operator of shifts, there exists a unique distribution $f \in D'_\Gamma$ such that $K = M_f$ and $(I_{\mathcal{Y}} \otimes K)x(t) = (I_{\mathcal{Y}} \otimes M_f)x(t)$ for all $x(t) \in D_\Gamma(\mathcal{Y})$.

Proof. We have that $I_{\mathcal{Y}} \otimes M_f$ is a linear and continuous mapping from the space $D_\Gamma(\mathcal{Y})$ into itself and

$$(I_{\mathcal{Y}} \otimes M_f)x(t) = \sum_{m=1}^{\infty} \lambda_m x_m (M_f \psi_m)(t) = \sum_{m=1}^{\infty} \lambda_m x_m \langle f, T_s \psi_m(t)\rangle,$$

where the sequences $\{x_m\}_{m\in\mathbb{N}}$ and $\{T_s\psi_m(t)\}_{m\in\mathbb{N}}$ converge to zero in $\mathcal{Y}$ and D_Γ respectively. Therefore from the known criterion of nuclearity [10, Sect.X, theor.1] we obtain that $I_{\mathcal{Y}} \otimes M_f$ is a nuclear operator.

Further,

$$(I_{\mathcal{Y}} \otimes M_f \circ T_s)x(t) = \sum_{m=1}^{\infty} \lambda_m x_m (M_f \circ T_s)\psi_m(t) = \sum_{m=1}^{\infty} \lambda_m x_m (T_s \circ M_f)\psi_m(t) = (I_{\mathcal{Y}} \otimes T_s \circ M_f)x(t).$$

Conversely, for any function $\psi(t) \in D_\Gamma$ the linear and continuous functional $f\colon \psi \mapsto (K\psi)(0)$ defines the distribution $f \in D'_\Gamma$. Then for any function $x(t) \in D_\Gamma(\mathcal{Y})$ we can write that $\langle f, x\rangle = (I_{\mathcal{Y}} \otimes K)x(0)$, since $(I_{\mathcal{Y}} \otimes K)x(0) = \langle f, x\rangle = (I_{\mathcal{Y}} \otimes M_f)x(0)$. If we replace $(I_{\mathcal{Y}} \otimes T_s)x(t)$ instead of $x(t)$ and use the condition of invariance for operator $I_{\mathcal{Y}} \otimes K$, we obtain $(I_{\mathcal{Y}} \otimes K)x(t) = (I_{\mathcal{Y}} \otimes M_f)x(t)$. □

4. Functional calculus for distributions on cone. Let $U_s\colon \Gamma \ni s \to U_s \in L(\mathcal{Y})$ be an n–parametric semigroups of class (C_o) over the space $\mathcal{Y}$. Generators of this n–parametric (C_o)–semigroup are determined by the following way

$$\left.\frac{\partial U_s x}{\partial s_j}\right|_{s=0} = -iA_j x, x \in \mathcal{D}(A_j), j = 1, \ldots, n.$$

We assume that each A_j is a closed and dense operator with the domain $\mathcal{D}(A_j)$. Throughout the article, A stands for $A\colon = (A_1, \ldots, A_n)$.

Let us define the mapping $\mathcal{F}_A$ as

$$\mathcal{F}_A\colon D_\Gamma(\mathcal{Y}) \ni x(s) \mapsto \widehat{x} \in \widehat{D_\Gamma(\mathcal{Y})}, \tag{10}$$

where the space $\widehat{D_\Gamma(\mathcal{Y})}$ is defined by

$$\widehat{D_\Gamma(\mathcal{Y})}\colon = \Big\{\widehat{x} = \int_\Gamma U_s x(s) ds\colon x(s) \in D_\Gamma(\mathcal{Y})\Big\}. \tag{11}$$

Theorem 3. *If $\{U_s\colon s \in \Gamma\}$ is an n-parametric (C_o)-semigroup of operators, then the subspace $\widehat{D_\Gamma(\mathcal{Y})}$ is dense in the Banach space $\mathcal{Y}$.*

Proof. Let $\mathcal{Y}'$ be the conjugate space to $\mathcal{Y}$ of linear and continuous functionals and $x' \in \mathcal{Y}'$ any functional. Then the properties of the Bochner integral [12, Sect.III, 3.5] imply that

$$\langle x', \widehat{x}\rangle = \int_\Gamma \langle x', U_s x(s)\rangle ds. \tag{12}$$

Now assume, that for some functional $x' \in \mathcal{Y}'$ the condition $\langle x', \widehat{x}\rangle = 0$ holds for all $\widehat{x} \in \widehat{D_\Gamma(\mathcal{Y})}$. Our next goal is to prove that $x' = 0$. Without restriction of generality it is sufficient to show $x' = 0$ for elements of the form $x(s) = y \otimes \psi(s)$, only, where $y \in \mathcal{Y}$ and $\psi(s) \in D_\Gamma$. In this case we have $\langle x', U_s x(s)\rangle = \langle x', U_s y\rangle \psi(s)$. Thus, the polar $(\widehat{D_\Gamma(\mathcal{Y})})^o = \{x' \colon \langle x', \widehat{x}\rangle = 0\}$ consists of the unique element $x' = 0$. Then the bipolar theorem, as a consequence of the Hahn–Banach theorem (see [11, Sect. IV, theor. 1.5]), yields that the space $\widehat{D_\Gamma(\mathcal{Y})}$ is dense in $\mathcal{Y}$. □

We endow the space $\widehat{D_\Gamma(\mathcal{Y})}$ with the weakest topology with respect to which the mapping $\mathcal{F}_A$ is continuous. Let $\widehat{D_{\Gamma_\nu}(\mathcal{Y})}$ be the image of $D_{\Gamma_\nu}(\mathcal{Y})$ by the mapping (10). Then on the space $\widehat{D_{\Gamma_\nu}(\mathcal{Y})}$ define a topology by the set of norms

$$\|\widehat{x}\|_{m,\nu} = \inf_{x \in \mathcal{F}_A^{-1}\widehat{x}} \|x\|_{m,\nu},$$

where the norms $\|x\|_{m,\nu}$ determine the topology of the space $D_{\Gamma_\nu}(\mathcal{Y})$.

Obviously, the mapping $D_{\Gamma_\nu}(\mathcal{Y}) \ni x(s) \mapsto \widehat{x} \in \widehat{D_{\Gamma_\nu}(\mathcal{Y})}$ is linear and continuous. Thus, $\widehat{D_{\Gamma_\nu}(\mathcal{Y})}$ is a Fréchet space. The diagrams

$$\begin{array}{ccc} D_{\Gamma_\nu}(\mathcal{Y}) & \longrightarrow & D_{\Gamma_\mu}(\mathcal{Y}) \\ \mathcal{F}_A \downarrow & & \mathcal{F}_A \downarrow \\ \widehat{D_{\Gamma_\nu}(\mathcal{Y})} & \longrightarrow & \widehat{D_{\Gamma_\mu}(\mathcal{Y})} \end{array}$$

and injection $\widehat{D_{\Gamma_\nu}(\mathcal{Y})} \subset \widehat{D_{\Gamma_\mu}(\mathcal{Y})}$ are continuous for any $\nu \le \mu$.

Since, $\widehat{D_\Gamma(\mathcal{Y})} = \bigcup_{\nu>0} \widehat{D_{\Gamma_\nu}(\mathcal{Y})}$, we may endow the space $\widehat{D_\Gamma(\mathcal{Y})}$ with the topology of inductive limit of Fréchet subspaces

$$\widehat{D_\Gamma(\mathcal{Y})} = \liminf_{\nu\to\infty} \widehat{D_{\Gamma_\nu}(\mathcal{Y})}.$$

So, the mapping $\mathcal{F}_A$ realizes the topological homomorphism of the respective spaces.

Let $L(\widehat{D_\Gamma(\mathcal{Y})})$ be the space of linear and continuous operator of $\widehat{D_\Gamma(\mathcal{Y})}$ into itself, endowed with the topology of uniform convergence on the bounded sets. For an n–parametric semigroup of shifts $\{I_\mathcal{Y} \otimes T_s \colon\ s \in \Gamma\} \subset L(D_\Gamma(\mathcal{Y}))$ we consider the n–parametric semigroup defined by

$$\{\widehat{\mathcal{T}}_s \colon\ s \in \Gamma\} \subset L(\widehat{D_\Gamma(\mathcal{Y})}), \qquad \widehat{\mathcal{T}}_s \colon = \mathcal{F}_A \circ (I_\mathcal{Y} \otimes T_s) \circ \mathcal{F}_A^{-1}.$$

Indeed, for any $s, t \in \Gamma$

$$\widehat{\mathcal{T}}_{s+t} = \mathcal{F}_A \circ (I_\mathcal{Y} \otimes T_{s+t}) \circ \mathcal{F}_A^{-1} = \mathcal{F}_A \circ (I_\mathcal{Y} \otimes T_s) \circ \mathcal{F}_A^{-1} \circ \mathcal{F}_A \circ (I_\mathcal{Y} \otimes T_t) \circ \mathcal{F}_A^{-1} = \widehat{\mathcal{T}}_s \circ \widehat{\mathcal{T}}_t$$

and

$$\widehat{\mathcal{T}}_0 = \mathcal{F}_A \circ (I_\mathcal{Y} \otimes T_0) \circ \mathcal{F}_A^{-1} = \mathcal{F}_A \circ \mathcal{F}_A^{-1}$$

is the identity operator over the space $\widehat{D_\Gamma(\mathcal{Y})}$.

Then by the formula

$$\widehat{F}_A \colon\ [T_s]^c \ni T \mapsto \widehat{\mathcal{T}} \in \mathcal{F}_A \circ \big(I_\mathcal{Y} \otimes [T_s]^c\big) \circ \mathcal{F}_A^{-1}.$$

we define an algebraic isomorphism of commutant $[T_s]^c$ on the commutative subalgebra $\mathcal{F}_A \circ \big(I_\mathcal{Y} \otimes [T_s]^c\big) \circ \mathcal{F}_A^{-1}$.

Theorem 4. *The mapping*

$$\Phi\colon\ \widehat{D}'_\Gamma \ni \widehat{f} \longrightarrow \widehat{f}(A) \in L(\widehat{D_\Gamma(\mathcal{Y})}), \tag{13}$$

is a continuous isomorphism of the algebra $\widehat{D}'_\Gamma$ into the closed subalgebra of operators $\big\{[\widehat{\mathfrak{T}}_s]^c := \mathcal{F}_A \circ \big(I_\mathcal{Y} \otimes [T_s]^c\big) \circ \mathcal{F}_A^{-1}\colon s \in \Gamma\big\}$ of the algebra $L(\widehat{D_\Gamma(\mathcal{Y})})$, where the operator $\widehat{f}(A)$ is defined by the formula

$$\widehat{f}(A)\colon\ \widehat{D_\Gamma(\mathcal{Y})} \ni \widehat{x} \to \widehat{f}(A)\widehat{x} = \int\limits_\Gamma (U_s \otimes M_f)x(s)ds. \tag{14}$$

In particular,

$$\Phi(\widehat{f}\cdot\widehat{g}) = \widehat{f}(A)\circ\widehat{g}(A), \qquad \widehat{f},\widehat{g} \in \widehat{D}'_\Gamma$$

and we can extend the operator $\widehat{\delta}(A)$ to the identity operator $I_\mathcal{Y}$ of the space $\mathcal{Y}$.

Proof. The bijective mapping of commutant $[T_s]^c$ on commutative subalgebra $[\widehat{\mathfrak{T}}_s]^c$ by

$$\widehat{F}_A\colon\ T \longrightarrow \widehat{\mathfrak{T}}, \qquad \widehat{\mathfrak{T}}\colon\ = \mathcal{F}_A \circ (I_\mathcal{Y} \otimes T) \circ \mathcal{F}_A^{-1}$$

realizes an algebraic isomorphism. In [7] it is proved that the mapping $M\colon D'_\Gamma \longrightarrow [T_s]^c$ is an algebraic isomorphism. Besides, the generalized Fourier transformation $F^*\colon D'_\Gamma \longrightarrow \widehat{D}'_\Gamma$ realizes an algebraic isomorphism too. Thus, the next commutative diagram

$$\begin{array}{ccc} D'_\Gamma & \xrightarrow{M} & [T_s]^c \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle \widehat{F}_A} \\ \widehat{D}'_\Gamma & \xrightarrow{\widehat{M}_A} & [\widehat{\mathfrak{T}}_s]^c \end{array}$$

identifies the algebraic isomorphism $\widehat{M_A}\colon \widehat{D}'_\Gamma \longrightarrow [\widehat{\mathfrak{T}}_s]^c$.

The mapping Φ is continuous as a composition of continuous mappings. Let us show, that Φ is a homomorphism of algebras. For any $x(s) = x \otimes \varphi(s)$ we can obtain

$$\big[\Phi(\widehat{f})\circ\Phi(\widehat{g})\big]\widehat{x} = \iint_\Gamma U_{s+t}x \otimes (M_f \circ M_g)\varphi(s+t)dsdt$$

$$= \int_\Gamma U_p x \otimes M_{f*g}\varphi(p)dp = \Phi(\widehat{f*g})\widehat{x} = \Phi(\widehat{f}\cdot\widehat{g})\widehat{x}.$$

In particular,

$$\Phi(\widehat{\delta})\widehat{x} = \int_\Gamma (U_s \otimes M_\delta)x(s)ds = \widehat{x}.$$

By Lemma 5, any element $x \in D_\Gamma(\mathcal{Y})$ can be written as absolutely convergence series $x = \sum_{m=1}^\infty \lambda_m x_m \otimes \varphi_m(s)$. This fact implies that

$$\big[\Phi(\widehat{f})\circ\Phi(\widehat{g})\big]\widehat{x} = \sum_{m=1}^\infty \lambda_m \int\limits_\Gamma U_p x_m \otimes M_{f*g}\varphi_m(p)dp = \sum_{m=1}^\infty \lambda_m \Phi(\widehat{f}\cdot\widehat{g})\widehat{x_m\otimes\varphi_m} = \Phi(\widehat{f}\cdot\widehat{g})\widehat{x}$$

for any $x \in D_\Gamma(\mathcal{Y})$. □

Note that the partial differentiation operators $\partial_j^m x, (j = 1, \dots, n)$ are defined in the space $D_\Gamma(\mathcal{Y})$ of the test $\mathcal{Y}$–valued functions $x(s) = x(s_1, \dots, s_n)$ on the cone Γ. For the interior of Γ it is the usual differentiation and on the limit of cone it is one-sided differentiation.

Lemma 7. *For any $f \in D'_\Gamma$, $x \in D_\Gamma(\mathcal{Y})$ and $m \in \mathbb{N}$ the following formula holds*

$$\widehat{f}(A)\widehat{\partial_j^m x} = (iA_j)^m \widehat{f}(A)\widehat{x} - \sum_{k_j=0}^{m-1} (iA_j)^{m-k_j-1} \langle f, \partial_j^{k_j} x\rangle, \quad (j = 1, \dots, n). \tag{15}$$

Proof. Without restriction of generality it is sufficient to show (15) for elements of the form $x(s) = x \otimes \varphi(s) \in D_\Gamma(\mathcal{Y})$. only. So, for any function $x(s) = x \otimes \varphi(s)$ we have

$$\widehat{f}(A)\widehat{\partial_j^m x} = \int_\Gamma (U_s \otimes M_f)\partial_j^m x(s)ds = \int_\Gamma U_s x \partial_j^m M_f \varphi(s)ds.$$

Last integral we integrate by parts l–times. Then from the property of the cross–correllation operation $\partial_j^m (M_f \varphi)(0) = \langle f, \partial_j^m \varphi\rangle$ and the definition of generator $\left.\frac{\partial U_s x_k}{\partial s_j}\right|_{s=0} = -iA_j x_k$ we obtain formula (15). □

Note that mapping (13) we designate as the functional calculus in the algebra $\widehat{D}'_\Gamma$ over the Banach space $\mathcal{Y}$.

Let us consider examples, where we use constructed functional calculus.

Example 1. The value of the Dirac function for the generator $A = (A_1, \dots, A_n)$ of the n-parametric (C_0)-semigroup defined by

$$U_s = e^{-i(s,A)} = e^{-i(s_1A_1 + \dots + s_nA_n)}, \ s = (s_1, \dots, s_n) \in \Gamma$$

we can evaluate by the formula

$$\delta(A)\widehat{x} = \frac{1}{(2\pi)^n} \int_\Gamma \widehat{T_t x}dt, \qquad x \in D_\Gamma(\mathcal{Y}).$$

Note that any n–parametric (C_0)–semigroup can be written as a product of one–parametric (C_0)–semigroups [12, Sect. IX, 9.7]. Thus,

$$\delta(A)\widehat{x} = \frac{1}{(2\pi)^n} \int_\Gamma (U_s \otimes M_{\lambda_\Gamma})x(s)ds = \frac{1}{(2\pi)^n} \int_\Gamma e^{-i(s,A)} \int_\Gamma x(t+s)dtds$$

$$= \frac{1}{(2\pi)^n} \int_\Gamma \int_\Gamma e^{-i(s,A)} T_t x(s)dsdt = \frac{1}{(2\pi)^n} \int_\Gamma \widehat{T_t x}dt.$$

Example 2. Now we use generalized Fourier transformation to the known formula

$$\lambda_\Gamma * (\partial_1 \dots \partial_n \delta) = (\partial_1 \dots \partial_n \delta) * \lambda_\Gamma = \delta.$$

Then we obtain

$$\delta \cdot (\widehat{\partial_1 \dots \partial_n \delta}) = (\widehat{\partial_1 \dots \partial_n \delta}) \cdot \delta = \frac{1}{(2\pi)^n}.$$

Thus, the Dirac function δ has inverse element $(2\pi)^n \widehat{\partial_1 \dots \partial_n \delta}$ in the algebra $\widehat{D}'_\Gamma$.

Let $A=(A_1,\dots,A_n)$ be the generator of the n-parametric (C_0)-semigroup $U_s=e^{-i(s,A)}$. Then from Lemma 7 we conclude that the equation

$$(\widehat{\partial_1\dots\partial_n}\delta)(A)\widehat{x}=\frac{\widehat{y}}{(2\pi)^n}$$

has a unique solution $\widehat{x}=\delta(A)\widehat{y}=\frac{1}{(2\pi)^n}\int_\Gamma \widehat{T_t x}dt$ for any $\widehat{y}\in\widehat{D_\Gamma(\mathcal{Y})}$.

Example 3. Let $\Gamma=\mathbb{R}^n_+$, $\mathcal{Y}=L^1(\mathbb{R}_+)\widetilde{\otimes}\cdots\widetilde{\otimes}L^1(\mathbb{R}_+)=L^1(\mathbb{R}^n_+)$. The semigroup of fractional integration $[0,+\infty)\ni t\longrightarrow f_t*\in L\big[L^1(\mathbb{R}_+)\big]$ is defined over the space $L^1(\mathbb{R}_+)$, where element $f_t\in D'_{\mathbb{R}_+}$ is defined by formula

$$f_t(s)=\frac{\theta(s)s^{t-1}}{\Gamma(t)},\ s\in[0,+\infty). \tag{16}$$

Determine the operator f_t* as the convolution with the function $\psi\in L^1(\mathbb{R}_+)$, then $f_t*\psi\in L^1(\mathbb{R}_+)$. In formula (16) $\theta(s)$ is the Heaviside function with support $[0,+\infty)$ and $\Gamma(t)=\displaystyle\int_0^{+\infty}y^{t-1}e^{-y}dy$ is the gamma function. The generator of fractional integration semigroup we designate by $\mathcal{G}$. Thus, $f_t*=e^{t\mathcal{G}}$.

Let us consider the n–parametric semigroup of convolution operators

$$\mathbb{R}^n_+\ni t=(t_1,\dots,t_n)\mapsto F_t=(f_{t_1}*)\otimes\cdots\otimes(f_{t_n}*)\in L\big[L^1(\mathbb{R}^n_+)\big].$$

Every operator $G_j:=I_1\otimes\cdots\otimes I_{j-1}\otimes\mathcal{G}_j\otimes I_{j+1}\otimes\cdots\otimes I_n$ generates one-parametric (C_0)-semigroup

$$e^{t_jG_j}:=I_1\otimes\cdots\otimes I_{j-1}\otimes e^{t_j\mathcal{G}_j}\otimes I_{j+1}\otimes\cdots\otimes I_n$$

over the space $L^1(\mathbb{R}^n_+)$, where the identity operator $I_j=\delta_j*$ is the convolution with the Dirac function in the variable $t_j\in[0,+\infty)$. Semigroups $\{e^{t_jG_j}\colon t\in[0,+\infty)\}$ belong to the algebra $L\big[L^1(\mathbb{R}^n_+)\big]$. So, the generator G of the n–parametric semigroup can be written as $G=(G_1,\dots,G_n)$.

Let $t\mapsto\psi_\tau(t)$ be an $L^1(\mathbb{R}^n_+)$–valued function of a variable $\tau\in\mathbb{R}^n_+$ from the space $D_{\mathbb{R}^n_+}\big(L^1(\mathbb{R}^n_+)\big)$. Then $\widehat{\psi}_\tau\in D_{\mathbb{R}^n_+}\widehat{\big(L^1(\mathbb{R}^n_+)\big)}$ and the subspace $D_{\mathbb{R}^n_+}\widehat{\big(L^1(\mathbb{R}^n_+)\big)}$ is dense in $L^1(\mathbb{R}^n_+)$.

It is well known that the degrees of the Dirac function in the algebra $\widehat{D}'_{\mathbb{R}_+}$ can be written as

$$\delta^m=\frac{1}{(2\pi)^m}\widehat{\theta*\cdots*\theta}=\frac{1}{(2\pi)^m}\widehat{\theta^{*m}},\qquad \delta=\frac{1}{2\pi}\widehat{\theta},\qquad m\in\mathbb{N}.$$

So, the degrees of the Dirac function in the algebra $\widehat{D}'_{\mathbb{R}^n_+}$ we determine by the formulae

$$\delta^{|k|}(s)=\frac{1}{(2\pi)^{|k|}}\widehat{\theta^{*k_1}}(s_1)\cdot\ldots\cdot\widehat{\theta^{*k_n}}(s_n)=\frac{1}{(2\pi)^{|k|}}\widehat{\theta^{*|k|}}(s),\quad s=(s_1,\dots,s_n)\in\mathbb{R}^n.$$

The Fourier transformation of functions from $D_{\mathbb{R}^n_+}$ is defined by the formula

$$F\colon\ D_{\mathbb{R}^n_+}\ni\theta^{*|k|}\longrightarrow\widehat{\theta^{*|k|}},\quad\text{where}\quad \widehat{\theta^{*|k|}}(s)=\int_{\mathbb{R}^n_+}e^{-i(\xi,s)}\theta^{*|k|}(\xi)d\xi.$$

We use formula (14) of constructing functional calculus and compute

$$
\begin{aligned}
\delta^{|k|}(G)\widehat{\psi}_\tau &= \frac{1}{(2\pi)^{|k|}}\int_{\mathbb{R}^n_+} F_t(\widehat{\theta^{*|k|}}\star\psi_\tau)(t)dt = \frac{1}{(2\pi)^{|k|}}\int_{\mathbb{R}^n_+}\theta^{*|k|}(\xi)\int_{\mathbb{R}^n_+}F_t(e^{-i(\xi,\cdot)}\star\psi_\tau)(t)dtd\xi \\
&= \frac{1}{(2\pi)^{|k|}}\int_{\mathbb{R}^n_+}\theta^{*|k|}(\xi)\int_{\mathbb{R}^n_+}e^{-i(\xi,s)}\int_{\mathbb{R}^n_+}F_t\psi_\tau(s+t)dtdsd\xi \\
&= \frac{1}{(2\pi)^{|k|}}\int_{\mathbb{R}^n_+}\theta^{*|k|}(\xi)\int_{\mathbb{R}^n_+}e^{-i(\xi,s)}\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{\theta(\zeta)\zeta^{|t-1|}}{\Gamma(t)}T_t\psi_\tau(s-\zeta)d\zeta dtdsd\xi \\
&= \frac{1}{(2\pi)^{|k|}}\int_{\mathbb{R}^n_+}\theta^{*|k|}(\xi)\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{\theta(\zeta)\zeta^{|t-1|}}{\Gamma(t)}\int_{\mathbb{R}^n_+}e^{-i(\xi,s)}T_t\psi_\tau(s-\zeta)dsdtd\zeta d\xi \\
&= \frac{1}{(2\pi)^{|k|}}\int_{\mathbb{R}^n_+}\theta^{*|k|}(\xi)\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{\theta(\zeta)\zeta^{|t-1|}}{\Gamma(t)}\widehat{(T_t\psi_\tau)}(\xi-\zeta)d\zeta dtd\xi,
\end{aligned}
$$

where $\zeta^{|t-1|}=\zeta_1^{t_1-1}\cdot\ldots\cdot\zeta_n^{t_n-1}$. Using the formula

$$\theta^{*|k|}(\xi)=\frac{\xi_1^{k-1}\ldots\xi_n^{k_n-1}}{(k_1-1)!\ldots(k_n-1)!}:\ =\frac{\xi^{|k-1|}}{(k-1)!}$$

we obtain

$$\delta^{|k|}(G)\widehat{\psi}_\tau=\frac{1}{(2\pi)^{|k|}}\int_{\mathbb{R}^n_+}\frac{\xi^{|k-1|}}{(k-1)!}\int_{\mathbb{R}^n_+}\big(F_t*\widehat{(T_t\psi_\tau)}\big)(\xi)dtd\xi.$$

For derivatives of the Dirac function the following formulae is known

$$\partial^k\delta=\frac{1}{(2\pi)^n}\widehat{\chi_k},\qquad \chi_k\colon\mathbb{R}^n_+\ni t\longrightarrow\lambda_\Gamma(t)(it)^{|k|},$$

where $t=(t_1,\ldots,t_n)\in\mathbb{R}^n$, $|k|=k_1+\ldots+k_n$, $\partial^k=\partial_1^{k_1}\cdot\ldots\cdot\partial_n^{k_n}$, $\partial_j^{k_j}:\ =(-i)^{k_j}\dfrac{\partial^{k_j}}{\partial t_j^{k_j}}$, $t^{|k|}=t_1^{k_1}\cdot\ldots\cdot t_n^{k_n}$ $(j=1,\ldots,n)$.

Now we calculate derivatives of the Dirac function for the generator of fractional integration. We continue in this fashion obtaining

$$
\begin{aligned}
\partial^k\delta(G)\widehat{\psi}_\tau &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+}F_t(\widehat{\chi_k}\star\psi_\tau)(t)dt=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+}\chi_k(\xi)\int_{\mathbb{R}^n_+}F_t(e^{-i(\xi,\cdot)}\star\psi_\tau)(t)dtd\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+}\chi_k(\xi)\int_{\mathbb{R}^n_+}e^{-i(\xi,s)}\int_{\mathbb{R}^n_+}F_t\psi_\tau(s+t)dtdsd\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+}\chi_k(\xi)\int_{\mathbb{R}^n_+}e^{-i(\xi,s)}\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{\theta(\zeta)\zeta^{|t-1|}}{\Gamma(t)}T_t\psi_\tau(s-\zeta)d\zeta dtdsd\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+}\chi_k(\xi)\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{\theta(\zeta)\zeta^{|t-1|}}{\Gamma(t)}\int_{\mathbb{R}^n_+}e^{-i(\xi,s)}T_t\psi_\tau(s-\zeta)dsdtd\zeta d\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+}\chi_k(\xi)\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{\theta(\zeta)\zeta^{|t-1|}}{\Gamma(t)}\widehat{(T_t\psi_\tau)}(\xi-\zeta)d\zeta dtd\xi.
\end{aligned}
$$

Thus, derivatives of the Dirac function can be written by the formula

$$\partial^k\delta(G)\widehat{\psi}_\tau=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_+}(i\xi)^{|k|}\int_{\mathbb{R}^n_+}\big(F_t*\widehat{(T_t\psi_\tau)}\big)(\xi)dtd\xi.$$

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