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SUBHARMONIC FUNCTIONS AND ELECTRIC FIELDS IN BALL LAYERS. II

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In this sequel to [1] we study a special case $BL(\frac{1}{r}, r)$, $r > 1$. Also the explicit representation of a subharmonic extension for a subharmonic function $u(x)$ near a removable point is obtained. Moreover, the diverse Nevanlinna characteristics are compared.

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В этом продолжении к [1] мы изучаем особый случай $BL(\frac{1}{r}, r)$, $r > 1$. Также получено представление в явной форме субгармонического продолжения субгармонической функции $u(x)$ в окрестности устранимой точки. Кроме того сравниваются различные Неванлинновские характеристики.

Introduction. Let D be an open set in the Euclidian space $\mathbb{R}^m (m \geq 3)$ and F a compact subset of D. It is a classical result (see [2], Theorem 5.18, p. 255) that if u is subharmonic in $D \setminus F$ and bounded from above and moreover F is polar, then u has a subharmonic extension to the whole of D. Gardiner [3] has shown that, in the case of a compact exceptional set the above boundedness condition can be relaxed by imposing certain smoothness and Hausdorff measure conditions on the set. Riihentaus [4] has replaced the smoothness and Hausdorff measure conditions with one sole condition on Minkowski upper content. It has been established in [5] that if u is a subharmonic function in $D \setminus F$ and bounded from above, where F is a closed polar set, then the function

$$\widetilde{u}(x) = \begin{cases} u(x), & (x \in D\setminus F), \\ \overline{\lim\limits_{y\to x,\ y\notin F}} u(y), & (x \in F), \end{cases}$$

is subharmonic in D. In this sequel to [1] we give an explicit representation of a subharmonic extension of u near a removable point. The approach presented here appears to be new. It allows us to give a comparison of $T(+0, r; u)$ and $T(r, u)$ for subharmonic functions in a ball. In this paper we also investigate subharmonic functions on symmetric ball layers.

We use results of the first part [1].

4. Subharmonic functions on symmetric ball layers. Consider a subharmonic function u on $\overline{BL(\frac{1}{r}, r)}$, non identical $-\infty$, and define

$$N_0(r; u) := (m-2)\int\limits_1^r \frac{n(t)}{t^{m-1}}dt - \frac{(m-2)}{r^{m-2}}\int\limits_{\frac{1}{r}}^1 \frac{n(t)}{t^{m-1}}dt,$$

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where $n(t)$ is the distribution function of the Riesz measure μ of the function u (see Definition 1 from [1]).

Corollary 1. *Let u be a subharmonic function, non identical $-\infty$, in $\overline{BL(\frac{1}{r}, r)}$ and μ be its Riesz measure. Then*

$$N_0(r,u) = \frac{1}{c_m r^{m-1}} \int\limits_{S(0,r)} u(x)d\sigma(x) + \frac{r^{2-m}}{c_m \left(\frac{1}{r}\right)^{m-1}} \int\limits_{S(0,\frac{1}{r})} u(x)d\sigma(x) -$$

$$-(1+r^{2-m})\frac{1}{c_m} \int\limits_{S(0,1)} u(x)d\sigma(x), \ r>1 \tag{1}$$

where c_m is the area of the unit sphere and $d\sigma(x)$ is an element of the surface area.

Definition 1. Let u be a subharmonic function in $BL(\frac{1}{r_0}, r_0)$, non identical $-\infty$. The function

$$T_0(r,u){:=}T\left(\frac{1}{r}, r; u\right) = \frac{1}{c_m r^{m-1}} \int\limits_{S(0,r)} u^+(x)d\sigma(x) + \frac{r^{2-m}}{c_m \left(\frac{1}{r}\right)^{m-1}} \int\limits_{S(0,\frac{1}{r})} u^+(x)d\sigma(x) -$$

$$-(1+r^{2-m})\frac{1}{c_m} \int\limits_{S(0,1)} u^+(x)d\sigma(x), \qquad 1<r<r_0 \tag{2}$$

is called the *Nevanlinna characteristic* of u, where $u^+ = \max\{u;0\}$.

Like to its counterpart for subharmonic functions in a ball (see [2], p. 146), the Nevanlinna characteristic $T_0(r,u)$ has elementary properties that have been collected in the following theorem (see [6]):

Theorem 1. *Let u, u_1, u_2 be subharmonic functions in $BL(\frac{1}{r_0}, r_0)$, non identical $-\infty$. Then*

a) *$T_0(r,u)$ is nonnegative, nondereasing and convex with respect to r^{2-m}, $1<r<r_0$ and $T_0(1,u)=0$.*

b) *if u is constant then $T_0(r,u)$, $1<r<r_0$ is identical zero;*

c) *$T_0(r,u_1+u_2) \le T_0(r,u_1)+T_0(r,u_2)+O(1)$, $r \to r_0$.*
$T_0(r,\lambda u) = \lambda T_0(r,u)$ for $\lambda>0$, $1<r<r_0$.

Proof. Since u^+ is subharmonic and according to (1), we can rewrite (2) as follows $T_0(r;u) = N_0(r;u^+)$. The function $N_0(r;u^+)$ is nonnegative, nondecreasing and convex with respect to r^{2-m} [2]. Hence, $T_0(r,u)$ satisfies property a). The statement b) immediately follows from Definition 1. Property c) follows from the inequality $(u_1+u_2)^+ \le u_1^+ + u_2^+$. □

Define

$$m_0(r,u) = \frac{1}{c_m r^{m-1}} \int\limits_{S(0,r)} u^-(x)d\sigma(x) + \frac{r^{2-m}}{c_m \left(\frac{1}{r}\right)^{m-1}} \int\limits_{S(0,\frac{1}{r})} u^-(x)d\sigma(x), \qquad 1<r<r_0,$$

where $u^- = -\min\{u;0\}$. Now we can rewrite (2) as follows

Theorem 2. *If u is a subharmonic function in $BL(\frac{1}{r_0}, r_0)$ then*

$$T_0(r,u) = m_0(r,u) + N_0(r,u) - (1+r^{2-m})\frac{1}{c_m}\int\limits_{S(0,1)} u^-(x)d\sigma(x), \qquad 1<r<r_0.$$

This is a counterpart of the first fundamental theorem for subharmonic functions on symmetric ball layers. Moreover, we give a comparison of $T_0(r,u)$ with $T(r,u)$ for functions subharmonic in a ball. If a subharmonic function u in $BL(\frac{1}{r_0}, r_0)$ has a subharmonic continuation into $B(0,r_0) = \{x\colon |x| \le r_0\}$, $r_0 > 1$, then its classical Nevanlinna characteristic $T(r,u)$ is also determined. By a subharmonic continuation we mean that there exists a subharmonic function u_1 in $B(0,r_0)$ with Riesz measure μ_1 such that $u_1 = u$ on $BL(\frac{1}{r_0}, r_0)$ and the restriction of μ_1 on $BL(\frac{1}{r_0}, r_0)$ is equal to μ, i.e. $\mu_1\left(\{x\colon \frac{1}{t} < |x| \le t\}\right) = \mu\left(\{x\colon \frac{1}{t} < |x| \le t\}\right)$, $1<t<r_0$, where μ is the Riesz measure of u. Then, using (1), we have

$$N_0(r,u_1) = N(r,u_1) + r^{2-m}N\left(\frac{1}{r}, u_1\right) - (1+r^{2-m})N(1,u_1), \quad 1<r<r_0, \tag{3}$$

where $N(r,u_1)$ is defined in [2]. Moreover,

$$T_0(r,u_1) = T(r,u_1) + r^{2-m}T\left(\frac{1}{r}, u_1\right) - (1+r^{2-m})T(1,u_1), \quad 1<r<r_0. \tag{4}$$

Equality (4) follows from the definitions of $T_0(r,u_1)$ and $T(r,u_1)$ (see [2]) immediately.

Corollary 2. *Let $u(x)$ be a subharmonic function, non identical $-\infty$, in $\overline{BL(\frac{1}{r}, r)}$ and let μ be the Riesz measure of $u(x)$. Then for $\xi \in BL(\frac{1}{r}, r)$*

$$u(\xi) = \frac{1}{c_m r^{m-1}}\int\limits_{S(0,r)} u(x)\mathcal{P}_r(\xi,x)d\sigma(x) +$$
$$+\frac{1}{c_m\left(\frac{1}{r}\right)^{m-1}}\int\limits_{S(0,\frac{1}{r})} u(x)\mathcal{P}_{\frac{1}{r}}(\xi,x)d\sigma(x) - \int\limits_{BL(\frac{1}{r},r)} \mathcal{G}(\xi,x)d\mu(x),$$

where

$$\mathcal{P}_r(\xi,x) = \sum_{n=0}^{\infty}\frac{2n+m-2}{m-2}\left(\frac{r}{|\xi|}\right)^{n+m-2}\frac{(r|\xi|)^{2n+m-2}-1}{r^{2(2n+m-2)}-1}p_n^{\nu}(\cos\phi),$$
$$\mathcal{P}_{\frac{1}{r}}(\xi,x) = (r|\xi|)^{2-m}\sum_{n=0}^{\infty}\frac{2n+m-2}{m-2}\left(\frac{r}{|\xi|}\right)^{n}\frac{r^{2n+m-2}-|\xi|^{2n+m-2}}{r^{2(2n+m-2)}-1}p_n^{\nu}(\cos\phi),$$

and

$$\mathcal{G}(\xi,x) = G_r(\xi,x) - \sum_{n=0}^{\infty}\frac{1}{r^{2(2n+m-2)}-1}\left(\left(\frac{r}{|x|}\right)^{n+m-2} - \left(\frac{|x|}{r}\right)^{n}\right)\left(\left(\frac{r}{|\xi|}\right)^{n+m-2} - \left(\frac{|\xi|}{r}\right)^{n}\right)$$

where $G_r(x,\xi)$ is the Green function in the ball of radius r centered at the origin. The convergence is uniform on compact subsets of $BL(\frac{1}{r}, r)$.

Set $B_0(r,u) = B(\frac{1}{r}, r; u)$, i.e, $B_0(r,u) = \max\{M(\frac{1}{r}; u); M(r;u)\}$, $r>1$ where $M(t;u) = \sup\{u(x)\colon |x| = t\}$, $t \ge 1$.

Corollary 3. *If $u(x)$ is a subharmonic function in $\overline{BL(\frac{1}{r}, r)}$, then for $1 < \rho < r$ we have*

$$\frac{1}{1+\rho^{2-m}}T_0(\rho;u) \le B_0(\rho;u^+) \le \frac{r^{m-2}(r+\rho)}{(r-\rho)^{m-1}}\Big(T_0(r;u) + \frac{1+r^{2-m}}{c_m}\int\limits_{S(0,1)} u^+(x)d\sigma(x)\Big). \quad (5)$$

Let $S(r)$ be a real and nonnegative function increasing for $r_0 < r < \infty$, where $r_0 > 0$. The the order λ and the lower order μ of the function $S(r)$ are defined as

$$\lambda = \lim_{r\to\infty}\sup\frac{\log S(r)}{\log r},\ \ \mu = \lim_{r\to\infty}\inf\frac{\log S(r)}{\log r},$$

Obviously, the order and the lower order of the function satisfy the relation $0 \le \mu \le \lambda \le \infty$.

Definition 2. If $u(x)$ is a subharmonic function in $\mathbb{R}^m \setminus \{0\}$, then the order λ and the lower order μ of u are called the *order* and the *lower order of* $T_0(r,u)$.

Theorem 3. *If $u(x)$ is a subharmonic function in $\mathbb{R}^m \setminus \{0\}$, the order λ and the lower order μ of the functions $T_0(r;u)$ and $B_0(r;u)$ are the same, i.e., $\lambda[u] = \lambda_0[u]$, $\mu[u] = \mu_0[u]$, where*

$$\lambda[u] = \lim_{r\to\infty}\sup\frac{\log T_0(r,u)}{\log r},\ \ \lambda_0[u] = \lim_{r\to\infty}\sup\frac{\log B_0(r,u)}{\log r},$$
$$\mu[u] = \lim_{r\to\infty}\inf\frac{\log T_0(r)}{\log r},\ \ \mu_0[u] = \lim_{r\to\infty}\inf\frac{\log B_0(r)}{\log r}.$$

Proof. The conclusion follows from Corollary 3. Indeed, setting in (5) $r = \gamma\rho$, $\gamma > 1$, provided u^+ is positive in $|x| = \rho$ for the certain ρ, we get

$$\frac{T_0(r;u)}{1+\rho^{2-m}} \le B_0(\rho;u) \le \frac{\gamma^{m-2}(1+\gamma)}{(\gamma-1)^{m-1}}\Big(T_0(\gamma\rho;u) + \frac{1+(\gamma\rho)^{2-m}}{c_m}\int\limits_{S(0,1)} u^+(x)d\sigma(x)\Big). \quad (6)$$

From (6) we deduce at once that $\lambda[u] = \lambda_0[u]$ and $\mu[u] = \mu_0[u]$, since the order and the lower order of $T_0(r;u)$ is not greater than that of $B_0(r;u)$ and also that the order and the lower order of $B_0(r;u)$ is not greater than that of $T_0(r;u)$, completing the proof. □

Corollary 4. *If $u(x)$ is a subharmonic function of finite lower order μ in $\mathbb{R}^m \setminus \{0\}$, then*

$$\lim_{r\to\infty}\inf\frac{B_0(r;u)}{T_0(r;u)} \le K(\mu,m),$$

where

$$K(\mu,m) \le 3\left(\frac{2\exp(\mu-1)}{m-1}\right)^{m-1},\ \mu \ge m, \quad K(\mu,m) \le \left(\frac{2m\exp}{\mu}\right)^{\mu},\ 0<\mu<m$$

and $K(0,m) = 1$.

The proof is similar to that of Theorem 4.3 in ([2], p.166).

5. Representation of a subharmonic extension of a subharmonic function near a removable point. Let $BL(s,1)$ be the annular region $\{x \in \mathbb{R}^m \colon s < |x| < 1\}$, $0 < s < 1$.

Definition 3. Let D be an open subset of $\mathbb{R}^m$. If $x_0 \in D$ and u is a subharmonic function in $D\backslash\{x_0\}$ then u is said to *have an isolated singularity at the point* x_0.

Definition 4. The singular point x_0 of a subharmonic function u is *removable* if there exists a subharmonic function v in D that coincides with u for all $x \in D\backslash\{x_0\}$. We say that v is a *subharmonic extension of* u.

Theorem 4. *Subharmonic function u in D has a removable singularity at $x_0 \in D$ if and only if there exists a subset $\{x\colon 0 < |x - x_0| < r\} \subset D$ on which u is bounded from above. Furthermore, the extension v of u can be represented near the removable point in the following way*

$$v(x) = \frac{1}{c_m} \int\limits_{S(x_0,1)} u(\xi) P_1(x,\xi) d\sigma(\xi) - \int\limits_{0<|\xi|<1} G(x,\xi) d\mu(\xi) - \gamma(|x|^{2-m} - 1) \tag{7}$$

where $P_1(x,\xi)$ and $G(x,\xi)$ are Poisson's kernel and Green's function for the unit ball respectively,

$$-\gamma = \lim_{s\to x_0} s^{m-2} I(s;u), \quad 0 \le \gamma < \infty, \quad I(s;u) = \frac{1}{c_m s^{m-1}} \int\limits_{S(x_0,s)} u(\xi) d\sigma(\xi). \tag{8}$$

Recall that $B(x_0, r) = \{x\colon |x - x_0| < r\}$.

Proof. By Definition 4 and the property of a subharmonic function, the extension v of u is bounded from above on some compact subset of $B(x_0, r)$. Since u coincides with v on $\{x\colon 0 < |x - x_0| < r\}$, u is bounded from above there.

Using linear transformation that preserves subharmonicity,

$$x \mapsto x_0 + \frac{r}{2} x$$

one can get a subharmonic function u in $BL(0, 2)$. Since u is bounded from above, there exists a constant C such that $u \le C$. Without loss of generality, we can consider the function $u - C$ instead of u.

Let μ be the Riesz measure of u on $BL(0, 2)$. We extend μ to $\overline{B(0, 1)}$ in such a way

$$\nu(E) = \begin{cases} \mu(E) & (0 \notin E) \\ \mu_1 + \mu_2 & (0 \in E), \end{cases} \tag{9}$$

where $\mu_1 = \mu(E\backslash\{0\})$ and $\mu_2 = \gamma\delta(0)$, $\delta(0)$ is the Dirac delta-function, and E is a Borel set.

In the case $0 \notin E$ the measure $\mu(E)$ is finite as the Riesz measure. In the case $0 \in E$ we have the sum of two measures. Since $\gamma < \infty$, the second measure is finite . Indeed, if we have a convex function $f(t)$ on such an interval $(a, +\infty)$, then there exists such a limit (see [2], p.31)

$$-\infty < \lim_{t\to\infty} \frac{f(t)}{t} \le +\infty.$$

Using the substitution $t = s^{2-m}$, we obtain (8). Note that $I(s; u)$ is a convex function with respect to s^{2-m} ([2], Theorem 2.12, p.81).

Now we prove that $\mu(E\backslash\{0\}) < \infty$. It will be enough to prove that $\mu(BL(0, 1)) < \infty$, because $E\backslash\{0\} \subset BL(0, 1)$.

Using Definition 1 from [1], we can set $n(1) = 0$. Hence, we have $\mu(BL(0,1) = n(1) - n(0) = -n(0)$. Fixing $r = r_0$ in Theorem 3 from [1] and considering the limit as s tends to 0, we get

$$\lim_{s\to 0} s^{m-2} \int_s^1 \frac{n(t)}{t^{m-1}} dt \ge -\frac{C}{m-2} \tag{10}$$

for some constant C that does not depend on s.

Suppose that $n(0) \to -\infty$; then there exists a sequence $\{s_k\}$ of $k \in \mathbb{N}$. In view of this and integrating by parts, we obtain

$$s^{m-2} \int_s^1 \frac{n(t)}{t^{m-1}} dt \le s^{m-2} \int_s^{s_k} \frac{n(t)}{t^{m-1}} dt \le -s^{m-2} k \int_s^1 \frac{dt}{t^{m-1}} = -\frac{k}{m-2} + \frac{k}{m-2} \frac{s^{m-2}}{s_k^{m-2}}.$$

As $s \downarrow 0$, we get

$$\lim_{s\to 0} s^{m-2} \int_s^1 \frac{n(t)}{t^{m-1}} \le -\frac{k}{m-2}.$$

That contradicts (10). Hence, $n(0) > -\infty$, i.e., $\mu(BL(0,1)) < \infty$.

Next we consider the function

$$v = u \star P_1 - \int_{B_1} G d\nu,$$

that is subharmonic in $\overline{B(0,1)}$ and $u \star P_1$ is the convolution of u with the Poisson kernel on $\partial B(0,1)$.

Using (9) for all $x \in B(0,1)$, we have

$$v(x) = \frac{1}{c_m} \int_{S(x_0,1)} u(\xi) P_1(x,\xi) d\sigma(\xi) - \int_{0<|\xi|<1} G(x,\xi) d\mu(\xi) - \gamma(|x|^{2-m} - 1).$$

It remains to show that $u = v$ for all $x \in BL(0,1)$. Let us choose s such that $0 < s < 1$ and fix x. Applying Poisson-Jensen's Theorem 2 from [1] to a subharmonic function u in $BL(s,1)$, we obtain

$$u(x) = \frac{1}{c_m} \int_{S(0,1)} u(\xi) P_1(x,\xi) d\sigma(\xi) - \frac{s^{m-2}}{1-s^{m-2}} \left(|x|^{2-m} - 1\right) \frac{1}{c_m} \int_{S(0,1)} u(\xi) d\sigma(\xi) -$$
$$-\frac{1}{c_m} \int_{S(0,1)} u(\xi) \sum_{n=1}^{\infty} \frac{(2n+m-2)s^{2n+m-2}}{(m-2)(1-s^{2n+m-2})} \left(|x|^{2-m-n} - |x|^n\right) p_n^{\nu}(\cos\phi) d\sigma(\xi) +$$
$$+\frac{1}{(m-2)c_m} \int_{S(0,s)} u(\xi) \left[\frac{|x|\cos\phi - s}{(s^2 + |x|^2 - 2s|x|\cos\phi)^{\frac{m}{2}}} - |x| \frac{\cos\phi - s|x|}{(s^2|x|^2 + 1 - 2s|x|\cos\phi)^{\frac{m}{2}}} \right] d\sigma$$
$$-\frac{1}{s(s^{m-2}-1)} \left(|x|^{2-m} - 1\right) \frac{1}{c_m} \int_{S(0,s)} u(\xi) d\sigma(\xi) +$$

$$+\frac{1}{c_m}\int\limits_{S(0,s)} u(\xi)\sum_{n=1}^{\infty}\frac{s^{n-1}+ns^{3n+m-3}}{(m-2)(1-s^{2n+m-2})}\left(|x|^{2-m-n}-|x|^n\right)p_n^\nu(\cos\phi)d\sigma(\xi)-$$
$$-\int\limits_{s<|\xi|<1} G(x,\xi)d\mu(\xi)+\frac{s^{m-2}}{1-s^{m-2}}\left(|x|^{2-m}-1\right)\frac{1}{c_m}\int\limits_{s<|\xi|<1}\left(|\xi|^{2-m}-1\right)d\mu(\xi)+$$
$$+\int\limits_{s<|\xi|<1}\sum_{n=1}^{\infty}\frac{s^{2n+m-2}}{(1-s^{2n+m-2})}\left(|x|^{2-m-n}-|x|^n\right)\left(|\xi|^{2-m-n}-|\xi|^n\right)p_n^\nu(\cos\phi)d\mu(\xi). \tag{11}$$

Let us calculate the limit of the right-hand side of (11) as s tends to 0. The first summand does not depend on s. The next one

$$\left(|x|^{2-m}-1\right)\lim_{s\to 0}\frac{s^{m-2}}{1-s^{m-2}}\frac{1}{c_m}\int\limits_{S(0,1)} u(\xi)d\sigma(\xi)=0,$$

because u is bounded from above.

Let us estimate the third term. Since $u\le 0$ and

$$\max\{p_n^\nu(x):\ -1\le x\le 1\}=C_{n+m-3}^m=\frac{(n+m-3)!}{n!(m-3)!},$$

we get

$$\left|\frac{1}{c_m}\int\limits_{S(0,1)} u(\xi)\sum_{n=1}^{\infty}\frac{(2n+m-2)s^{2n+m-2}}{(m-2)(1-s^{2n+m-2})}\left(|x|^{2-m-n}-|x|^n\right)p_n^\nu(\cos\phi)d\sigma(\xi)\right|\le$$
$$\le-\Big(\frac{s}{|x|}\Big)^{m-2}\sum_{n=1}^{\infty}\frac{(2n+m-2)s^n}{(m-2)(1-s^{2n+m-2})}\Big(\frac{s}{|x|}\Big)^n C_{n+m-3}^n\frac{1}{c_m}\int\limits_{S(0,1)} u(\xi)d\sigma(\xi)\le$$

choosing $s<\frac{|x|}{2}$, we obtain

$$\le-\Big(\frac{s}{|x|}\Big)^{m-2}\sum_{n=1}^{\infty}\frac{(2n+m-2)!}{n!(m-2)!}\Big(\frac{1}{2}\Big)^{2n-1}\frac{1}{c_m}\int\limits_{S(0,1)} u(\xi)d\sigma(\xi).$$

The last series converges uniformly. Thus, the third term is vanishing as $s\downarrow 0$. Let us estimate the first summand of the forth term

$$\left|\frac{1}{(m-2)c_m}\int\limits_{S(0,s)} u(\xi)\left[\frac{|x|\cos\phi-s}{(s^2+|x|^2-2s|x|\cos\phi)^{\frac{m}{2}}}\right]d\sigma\right|\le$$
$$\le-\frac{s}{(m-2)}s^{m-2}\frac{1}{c_m s^{m-1}}\int\limits_{S(0,s)} u(\xi)\frac{|x|+s}{(|x|-s)^m}d\sigma(\xi).$$

Since γ is finite, the limit of the summand is 0. The analogous reasonings deal with the other summand of the forth term. Consider the limit of the next addend

$$\left(|x|^{2-m}-1\right)\lim_{s\to 0}\frac{1}{s(s^{m-2}-1)}\frac{1}{c_m}\int\limits_{S(0,s)} u(\xi)d\sigma(\xi)=\left(|x|^{2-m}-1\right)\lim_{s\to 0}\frac{s^{m-2}}{(s^{m-2}-1)}I(s;u)=$$
$$=\gamma\left(|x|^{2-m}-1\right).$$

It can be proved in the same manner as it was done for the third term that the limit of the fifth one is 0.

Now we prove that

$$\lim_{s\to 0}\int\limits_{s<|\xi|<1} G(x,\xi)d\mu(\xi)=\int\limits_{0<|\xi|<1} G(x,\xi)d\mu(\xi). \tag{12}$$

As we know, Green's function has a singularity at the point $x=\xi$, that is why we set $s<s_0<|x|<1$, and by the additivity property of Lebesgue's integrals, we have

$$\int\limits_{s<|\xi|<1} G(x,\xi)d\mu(\xi)=\int\limits_{s<|\xi|<s_0} G(x,\xi)d\mu(\xi)+\int\limits_{s_0<|\xi|<1} G(x,\xi)d\mu(\xi).$$

By Proposition from ([7], p. 20),

$$\int\limits_{s<|\xi|<s_0} G(x,\xi)d\mu(\xi)=\mu\{\xi\colon s<|\xi|<s_0\}$$

and by the property of measure continuity (see [7], p. 16), there exists a sequence $\{s_n\}$ with $n\in\mathbb{N}$ such that

$$\lim_{n\to\infty}\int\limits_{s_n<|\xi|<s_0} G(x,\xi)d\mu(\xi)=\int\limits_{0<|\xi|<s_0} G(x,\xi)d\mu(\xi).$$

Hence, we have proved (12).

Now we consider the limit of the eighth addend

$$\left(|x|^{2-m}-1\right)\lim_{s\to 0}\frac{s^{m-2}}{1-s^{m-2}}\int\limits_{s<|\xi|<1}\left(|\xi|^{2-m}-1\right)d\mu(\xi)=$$

$$=\left(|x|^{2-m}-1\right)\left[\lim_{s\to 0}\frac{s^{m-2}}{1-s^{m-2}}\int\limits_{s<|\xi|<1}\frac{d\mu(\xi)}{|\xi|^{m-2}}-\lim_{s\to 0}\frac{s^{m-2}}{1-s^{m-2}}\int\limits_{s<|\xi|<1} d\mu(\xi)\right].$$

Since the measure $\mu(BL(0,1))$ is finite, the last summand is vanishing.

Consider

$$s^{m-2}\int\limits_{s<|\xi|<1}|\xi|^{2-m}d\mu(\xi)=s^{m-2}\int\limits_{s}^{1}\frac{dn(t)}{t^{m-2}}. \tag{13}$$

Integrating the right-hand side of (13) by parts, we obtain

$$s^{m-2}\Big(n(1)-\frac{n(s)}{s^{m-2}}+(m-2)\int\limits_{s}^{1}\frac{n(t)}{t^{m-1}}dt\Big). \tag{14}$$

The limit of (14) as $s\downarrow 0$ is

$$-n(0)+(m-2)\lim_{s\to 0}s^{m-2}\int\limits_{s}^{1}\frac{n(t)}{t^{m-1}}dt. \tag{15}$$

Since

$$(m-2)s^{m-2}\int\limits_s^1 \frac{n(t)}{t^{m-1}}dt = (m-2)s^{m-2}\int\limits_s^{\sqrt{s}} \frac{n(t)}{t^{m-1}}dt + (m-2)s^{m-2}\int\limits_{\sqrt{s}}^1 \frac{n(t)}{t^{m-1}}dt \le$$

$$\le (m-2)s^{m-2}n(\sqrt{s})\ \frac{t^{2-m}}{2-m}\Big|_s^{\sqrt{s}} + (m-2)s^{m-2}n(1)\ \frac{t^{2-m}}{2-m}\Big|_{\sqrt{s}}^1 = n(0).$$

We conclude that (15) is 0.

Now we estimate the last term of (11)

$$\left|\int\limits_{s<|\xi|<1}\sum_{n=1}^{\infty}\frac{s^{2n+m-2}}{(1-s^{2n+m-2})}\left(|x|^{2-m-n}-|x|^n\right)\left(|\xi|^{2-m-n}-|\xi|^n\right)p_n^{\nu}(\cos\phi)d\mu(\xi)\right|\le$$

$$\le \frac{s}{|x|^{m-2}}\int\limits_{0<|\xi|<1}\sum_{n=1}^{\infty}\frac{C^n_{n+m-3}}{1-s^{2n+m-2}}\left(\frac{s}{|x|}\right)^n\left(\frac{s}{|\xi|}\right)^n\frac{d\mu(\xi)}{|\xi|^{m-2}}\le$$

$$\le \frac{s}{|x|^{m-2}}\sum_{n=1}^{\infty}C^n_{n+m-3}s_1^n\left(\frac{1}{2}\right)^{2n-1}\int\limits_{0<|\xi|<1}\frac{d\mu(\xi)}{|\xi|^{m-2}},\ s_1<1.$$

The last series converges uniformly and the last term is vanishing. The proof is completed. □

6. Comparison of $T(+0,r;u)$ and $T(r,u)$ for subharmonic functions extended into a ball. We consider bounded from above subharmonic functions in $BL(0,r)$. According to Theorem 3, such functions are extended to $B(0,r)$.

Using Theorem 3 from [1] and (8), we get

$$\lim_{s\to0}N(s,r;u) = \frac{1}{1-r^{2-m}}\Big(\frac{1}{c_m r^{m-1}}\int\limits_{S(0,r)}u(x)d\sigma(x) - \frac{1}{c_m}\int\limits_{S(0,1)}u(x)d\sigma(x)\Big) -$$

$$-\lim_{s\to0}\frac{s^{m-2}}{1-s^{m-2}}\frac{1}{c_m}\int\limits_{S(0,1)}u(x)d\sigma(x) + \lim_{s\to0}\frac{s^{m-2}}{1-s^{m-2}}\frac{1}{c_m s^{m-1}}\int\limits_{S(0,s)}u(x)d\sigma(x) =$$

$$= \frac{1}{1-r^{2-m}}\Big(\frac{1}{c_m r^{m-1}}\int\limits_{S(0,r)}u(x)d\sigma(x) - \frac{1}{c_m}\int\limits_{S(0,1)}u(x)d\sigma(x)\Big) - \gamma.$$

Hence,

$$(1-r^{2-m})(N(+0,r;u)+\gamma) = \frac{1}{c_m r^{m-1}}\int\limits_{S(0,r)}u(x)d\sigma(x) - \frac{1}{c_m}\int\limits_{S(0,1)}u(x)d\sigma(x).$$

Suppose $u(0)\ne-\infty$. According to [2], we have

$$(1-r^{2-m})(N(+0,r;u)+\gamma) = N(r,u) - N(1,u).$$

Moreover (see, definition 2 from [1]),

$$\lim_{s\to0}T(s,r;u) = \frac{1}{1-r^{2-m}}\frac{1}{c_m r^{m-1}}\int\limits_{S(0,r)}u^+(x)d\sigma(x) + \lim_{s\to0}\frac{s^{m-2}}{1-s^{m-2}}\frac{1}{c_m s^{m-1}}\int\limits_{S(0,s)}u^+(x)d\sigma(x) -$$

$$-\frac{1}{1-r^{2-m}}\frac{1}{c_m}\int\limits_{S(0,1)} u^+(x)d\sigma(x) - \lim_{s\to 0}\frac{s^{m-2}}{1-s^{m-2}}\frac{1}{c_m}\int\limits_{S(0,1)} u^+(x)d\sigma(x). \tag{16}$$

Since u^+ is a subharmonic and bounded from above function, we apply Theorem 3 to u^+. Denote

$$\lim_{s\to 0} s^{m-2}\frac{1}{c_m s^{m-1}}\int\limits_{S(0,s)} u^+(x)d\sigma(x) = -\gamma_1.$$

As far as $0 \le \gamma_1 < \infty$ and $u^+ = \max\{u; 0\}$, we have that $\gamma_1 = 0$. Thus,

$$(1-r^{2-m})T(+0,r;u) = T(r,u) - T(1,u). \tag{17}$$

Equality (17) follows from (16) and from the definition of $T(r,u)$ (see [2]) immediately. Note that (17) is true in both cases $u(0) = -\infty$ and $u(0) \ne -\infty$.

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