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A VERSION OF CARLEMAN’S FORMULA AND SUMMATION OF THE Riemann $\zeta$-FUNCTION LOGARITHM ON THE CRITICAL LINE


A version of Carleman’s formula for functions holomorphic in a rectangle is proved. It is applied to the evaluation of the integral of $\zeta$-function logarithm with the summing factor $\exp(-t)$ along the critical line. This allowed to obtain a new statement equivalent to the Riemann hypothesis.


Для функцій, голоморфних в прямокутнику, доказан один варіант формулі Карлемана. Он призначений на нахождение інтеграла з логарифмі $\zeta$-функції з сумируючим множителем $\exp(-t)$ вдоль критичної прямой. Це призвело установити нове утвер- ждение, еквівалентнє гіпотезі Рімана.

1. Introduction. A Carleman type formula for functions meromorphic in a rectangle with the remainder term in an explicit form was obtained in [1]. Here we prove its modification for the functions which are real on a side of a rectangle, and apply it to the study of the Riemann $\zeta$-function.

The integrals of the Riemann $\zeta$-function logarithm along the vertical lines with diverse summing factors were studied by many authors [2]–[6]. These results generate new statements equivalent to the well known Riemann hypothesis (RH) for $\zeta$-function. In particular, due to a theorem of Balazard-Saias-Your [4], the RH is true if and only if

$$\int_{\Re s = \frac{1}{2}} \log \left| \frac{\zeta(s)}{|\zeta|^2} \right| ds = 0.$$ 

In this paper applying the modified Carleman formula we evaluate the integral

$$I = \int_0^\infty e^{-t} \log \left| \zeta \left( \frac{1}{2} + it \right) \right| dt$$

in terms of non-trivial $\zeta$-function zeros and obtain a new statement equivalent to the RH. Namely, the RH is true if and only if $I = C$ where $C$ is indicated below.

2. A version of Carleman formula. Let $f$ be a meromorphic function on the closure of the rectangle $R_z = \{ z = t + iy : x_0 < t < x, 0 < y < \pi \}$. Denote by $\{a_q\}$, $a_q = \alpha_q + i\delta_q$, and

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\{\omega_q, \omega_q = \xi_q + i\eta_q,\} its zeros and poles in $R_x$, respectively. Choosing some $z^* \in R^*_x$ and some value $\log f(z^*)$, we define the function $\log f(z)$ in the domain

$$R^*_x = R_x \setminus \bigcup_j \{\{t\beta_j + i\gamma_j : t \geq 1\} \cup \{t\xi_j + i\eta_j : t \geq 1\}\}$$

and on the boundary $\partial R_x$ except the zeros and the poles that lie on the $\partial R_x$ by the relation

$$\log f(z) = \log f(z^*) + \int_{z^*}^z \frac{f'(\zeta)}{f(\zeta)} d\zeta.$$  \hspace{1cm} (1)

The integral is taking along a path in $R^*_x$ with the ends at $z^*$ and $z$.

We recall the following version of Carleman’s formula for a rectangle with the explicit remainder from term in the form Theorem 1 will be received.

\textbf{Theorem A (11).} Let $f, f \neq 0$, be a meromorphic function on the closure of the rectangle $R_x = \{z = t + iy : x_0 < t < x, 0 < y < \pi\}$. Let $\log f(z)$ be defined on

$$R^*_x = R_x \setminus \bigcup_j \{\{t\beta_j + i\gamma_j : t \geq 1\} \cup \{t\xi_j + i\eta_j : t \geq 1\}\}$$

and on $\partial R_x$ except the zeros and the poles, that lie on $\partial R_x$, by relation (1). Put also $\arg f(z) = \text{Im} \log f(z)$. Then

$$\sum_{\alpha_q \in R_x} \left( \frac{1}{e^{\alpha_q} - e^{2\pi}} \sin \delta_q - \sum_{\omega_p \in R_x} \left( \frac{1}{e^{\omega_p} - e^{2\pi}} \sin \eta_p = \frac{1}{2\pi} \int_{x_0}^x (\log |f(t)| + \log |f(t + i\pi)|) \times \right. \right.$$ 

$$\times \left( \frac{1}{e^t} - \frac{e^t}{e^{2\pi}} \right) dt + \frac{1}{\pi e^x} \int_0^\pi \log |f(x + iy)| \sin ydy + Q(x; x_0, f),$$  \hspace{1cm} (2)

where

$$Q(x; x_0, f) = \frac{\text{sh}(x - x_0)}{\pi e^x} \int_0^\pi \arg f(x_0 + iy) \cos ydy - \frac{\text{ch}(x - x_0)}{\pi e^x} \int_0^\pi \log |f(x_0 + iy)| \sin ydy.$$  \hspace{1cm} (3)

The main result of this section follows from Theorem A.

\textbf{Theorem 1.} Let $f, f(z) \not\equiv 0$, be a holomorphic function on the closure of the rectangle $R_x = \{z = t + iy : x_0 < t < x, 0 < y < \pi\}$ and real-valued on the segment $I_0 = \{z : z = x_0 + iy, 0 \leq y \leq \pi\}$. Then

$$\sum_{\alpha_q \in R_x} \left( \frac{1}{e^{\alpha_q} - e^{2\pi}} \sin \delta_q + \frac{1}{2} \sum_{\Re \alpha_q = x_0} \left( \frac{1}{e^{x_0} - e^{2\pi}} \sin \delta_q = \right. \right.$$ 

$$= \frac{1}{2\pi} \int_{x_0}^x (\log |f(t)| + \log |f(t + i\pi)|) \left( \frac{1}{e^t} - \frac{e^t}{e^{2\pi}} \right) dt + \frac{1}{\pi e^x} \int_0^\pi \log |f(x + iy)| \sin ydy.$$  \hspace{1cm} (4)

$$+ \frac{1}{\pi e^x} \int_0^\pi \log |f(x + iy)| \sin ydy - \frac{\text{ch}(x - x_0)}{\pi e^x} \int_0^\pi \log |f(x + iy)| \sin ydy.$$
Proof. Let us calculate the integral \( \frac{1}{\pi} \int_0^\pi \arg f(x_0 + iy) \cos ydy \). For this purpose consider the function

\[
g(z) = \frac{f(z)}{\prod_{q=1}^m (z - z_q)},
\]

(5)

where \( z_q = x_0 + i\delta_q \) are zeros of the function \( f \), \( 0 \leq \delta_q \leq \pi \). In the case of absence of such zeros, we suppose that the product (5) is equal to unity.

The function \( g \) does not have zeros on \( I_0 \) and

\[
g(x_0 + iy) = \frac{f(x_0 + iy)}{\prod_{q=1}^m i(y - \delta_q)}.
\]

If \( m \) is an even number, then \( g \) is real on \( I_0 \) and does not change the sign, since \( f \) is real, the denominator is real and \( g(z) \neq 0 \). If \( m \) is an odd number, then the function \( g \) obtains imaginary values without changing of the sign. So, \( \arg g(x_0 + iy) = C = \text{const} \), \( 0 \leq y \leq \pi \).

Hence, from previous equality and (5) we deduce

\[
\arg f(x_0 + iy) = \sum_{q=1}^m \arg (i(y - \delta_q)) + C.
\]

Consequently,

\[
\frac{1}{\pi} \int_0^\pi \arg f(x_0 + iy) \cos ydy = \frac{1}{\pi} \sum_{q=1}^m \int_0^\pi \arg (i(y - \delta_q)) \cos ydy.
\]

For \( y < \delta_q \) the following equality holds

\[
\arg (i(y - \delta_q)) = \arg(z^* - z_p) + \text{Im} \int_{z^*}^{x_q} \frac{dz}{z - z_q} = \arg(-i\delta_q).
\]

We fix \( \arg(-i\delta_q) = -\pi/2 \). Then for \( y > \delta_q \) we get \( \arg (i(y - \delta_q)) = \pi/2 \). Thus,

\[
\frac{1}{\pi} \int_0^\pi \arg (i(y - \delta_q)) \cos ydy = \frac{1}{\pi} \left( \int_0^{\delta_q} \left( -\frac{\pi}{2} \right) \cos ydy + \int_{\delta_q}^\pi \left( \frac{\pi}{2} \right) \cos ydy \right) = -\sin \delta_q
\]

and

\[
\frac{1}{\pi} \int_0^\pi \arg f(x_0 + iy) \cos ydy = -\sum_{q=1}^m \sin \delta_q.
\]

(6)

Taking into account (6) and (3) we obtain (4).

3. Summation of the Riemann \( \zeta \)-function logarithm on the critical line. Denote by \( \rho_q = \beta_q + i\gamma_q \) zeros of the Riemann \( \zeta \)-function on the strip \( R_\infty = \{ s : 1/2 < \text{Re } s < 1 \} \).
Theorem 2. The following equality holds

$$\frac{1}{2} \int_{0}^{+\infty} e^{-t} \log \left| \zeta \left( \frac{1}{2} + it \right) \right| \, dt = - \sum_{\rho \in R_{\infty}} \frac{\cos(\pi \beta)}{e^{\pi \gamma}} + C,$$

(7)

where $C = -(c_{1} + c_{2} + c_{3})/2$,

$$c_{1} = \int_{1/2}^{3/2} \log |(\sigma - 1)\zeta(\sigma)| \cos \pi \sigma d\sigma, \quad c_{2} = \int_{0}^{+\infty} \log \left( \frac{1}{4} + t^{2} \right) \frac{dt}{e^{t}}, \quad c_{3} = \frac{1}{2} \int_{0}^{+\infty} \log \left| \zeta \left( \frac{3}{2} + it \right) \right| \frac{dt}{e^{t}}.$$

The Riemann hypothesis holds if and only if

$$\frac{1}{2} \int_{0}^{+\infty} e^{-t} \log \left| \zeta \left( \frac{1}{2} + it \right) \right| \, dt = C.$$

(8)

Proof. In the complex plane of the variable $s = \sigma + it$ consider the rectangle $R_{T} = \{s : 1/2 < \sigma < 3/2, 0 < t < T\}$.

The function $(s - 1)\zeta(s)$ is holomorphic in $R_{T}$ and the transformation $z = i\pi(3/2 - s)$ maps $R_{T}$ to $R_{x}$ with $x_{0} = 0$. The inverse mapping is $s = iz/\pi + 3/2$.

The function

$$f(z) = \left( \frac{i}{\pi} + \frac{1}{2} \right) \zeta \left( \frac{iz}{\pi} + \frac{3}{2} \right), \quad z \in R_{x}$$

is holomorphic in $R_{x}$, where $x_{0} = 0$ and satisfy the conditions of Theorem 1.

Taking into account that $f$ has no zeros on $I_{0}$, we obtain (4) in the form

$$\sum_{\alpha \in R_{x}} \left( \frac{1}{e^{\alpha^{2}}} - \frac{e^{\alpha^{2}}}{2e^{2x}} \right) \sin \delta_{q} = \frac{1}{2\pi} \int_{0}^{x} \left( \log |f(t)| + \log |f(t + i\pi)| \right) \left( \frac{1}{e^{t}} - \frac{e^{t}}{e^{2x}} \right) \, dt +$$

$$+ \frac{1}{\pi e^{x}} \int_{0}^{\pi} \log |f(x + iy)| \sin ydy - \frac{c x}{\pi e^{x}} \int_{0}^{\pi} \log |f(iy)| \sin ydy = I_{1} + I_{2} + I_{3} + I_{4}.$$

(9)

We have

$$I_{4} = -\frac{c x}{\pi e^{x}} \int_{0}^{\pi} \log |f(iy)| \sin ydy = \frac{e^{\pi T} + e^{-\pi T}}{2e^{\pi T}} \int_{1/2}^{3/2} \log |(\sigma - 1)\zeta(\sigma)| \cos \pi \sigma d\sigma =$$

$$= \frac{1}{2} \int_{1/2}^{3/2} \log |(\sigma - 1)\zeta(\sigma)| \cos \pi \sigma d\sigma + \frac{1}{2e^{2\pi T}} \int_{1/2}^{3/2} \log |(\sigma - 1)\zeta(\sigma)| \cos \pi \sigma d\sigma.$$

Both integrals are convergent. The last addend vanishes as $T \to +\infty$. Hence,

$$\lim_{T \to +\infty} I_{4} = \frac{1}{2} \int_{1/2}^{3/2} \log |(\sigma - 1)\zeta(\sigma)| \cos \pi \sigma d\sigma = \frac{1}{2} c_{1}.$$
Further,

\[ I_1 = \frac{1}{2\pi} \int_0^x \log |f(u)| \left( \frac{1}{e^u} - \frac{e^u}{e^{2x}} \right) du = \frac{1}{2} \int_0^T \log \left| \frac{1}{2} + it \right| \frac{dt}{e^t} + \frac{1}{2} \int_0^T \log \left| \frac{3}{2} + it \right| \frac{dt}{e^t}. \]

Thus,

\[ \lim_{T \to +\infty} I_1 = \frac{1}{4} \int_0^\infty \log \left( \frac{1}{4} + t^2 \right) \frac{dt}{e^t} + \frac{1}{2} \int_0^\infty \log \left| \zeta \left( \frac{3}{2} + it \right) \right| \frac{dt}{e^t} = \frac{1}{4} c_2 + \frac{1}{2} c_3. \]

Similarly,

\[ I_2 = \frac{1}{2\pi} \int_0^x \log |f(u + i\pi)| \left( \frac{1}{e^u} - \frac{e^u}{e^{2x}} \right) du = \frac{1}{2} \int_0^T \log \left| (s - 1)\zeta(s) \right| \left( \frac{1}{e^t} - \frac{e^t}{e^{2T}} \right) dt, \]

\[ \lim_{T \to +\infty} I_2 = \frac{1}{2} \int_0^\infty \log \left( \frac{1}{4} + t^2 \right) \frac{dt}{e^t} + \frac{1}{2} \int_0^\infty \log \left| \zeta \left( \frac{3}{2} + it \right) \right| \frac{dt}{e^t} = \frac{1}{4} c_2 + \frac{1}{2} \int_0^\infty \log \left| \zeta \left( \frac{1}{2} + it \right) \right| \frac{dt}{e^t}. \]

Further, we have

\[ I_3 = \frac{1}{\pi e^x} \int_0^\pi \log |f(x + iy)| \sin y dy = -\frac{1}{e^{\pi T}} \int_{1/2}^{3/2} \log \left| (s - 1)\zeta(s) \right| \cos \pi s ds = \frac{1}{e^{\pi T}} \int_{1/2}^{3/2} \log |s - 1| \cos \pi s ds - \frac{1}{e^{\pi T}} \int_{1/2}^{3/2} \log |\zeta(s)| \cos \pi s ds \]

and \( \lim_{T \to +\infty} I_3 = 0. \) The left-hand side of (9) can be written as follows

\[ -\sum_{\rho_q \in R_T} \frac{\cos \pi \beta_q}{e^{\pi \gamma_q}} + \sum_{\rho_q \in R_T} \frac{e^{\pi \gamma_q} \cos \pi \beta_q}{e^{2\pi T}}. \]

The limit value of the first sum is

\[ \lim_{T \to +\infty} \sum_{\rho_q \in R_T} \frac{\cos \pi \beta_q}{e^{\pi \gamma_q}} = \sum_{\rho_q \in R_{\infty}} \frac{\cos \pi \beta_q}{e^{\pi \gamma_q}}. \]
Taking into account that the number \( N(T) \) of zeros of \( \zeta(s) \) in \( \mathbb{R}_T \) has the property \([7]\)
\[ N(T) \sim T \log T, \]
we obtain
\[
\lim_{T \to +\infty} \left| \sum_{\rho_q \in \mathbb{R}_T} e^{\pi \gamma_q} \cos \pi \beta_q \frac{e^{2 \pi T}}{e^{2 \pi T}} \right| \leq \lim_{T \to +\infty} \frac{1}{e^{2 \pi T}} \sum_{\rho_q \in \mathbb{R}_T} | \cos \pi \beta_q | \leq \lim_{T \to +\infty} \frac{T \log T}{e^{2 \pi T}} = 0.
\]

Therefore,
\[
- \sum_{\rho_q \in \mathbb{R}_T} \frac{\cos \pi \beta_q}{e^{2 \pi T}} = \frac{1}{2} \int_0^{+\infty} e^{-t} \log |\zeta(1/2 + it)| dt - C,
\]
where \( C = -(c_1 + c_2 + c_3)/2 \), i.e. \((7)\).

Let us prove the second statement of Theorem 1.

If the Riemann hypothesis holds then \( \beta_q = 1/2 \) for each \( q \), the sum at the right-hand side of \((7)\) vanishes and we have \((8)\).

Conversely, if \((8)\) fulfils then the sum in \((7)\) equals zero. But each its addend is nonpositive because \( 1/2 \leq \beta < 3/2 \). Thus, \( \beta_q = 1/2 \) for all \( q \) and the Riemann hypothesis holds.

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