УДК 517.5

A. M. Brydun, P. A. Yatsulka

A VERSION OF CARLEMAN'S FORMULA AND SUMMATION OF THE RIEMANN ζ-FUNCTION LOGARITHM ON THE CRITICAL LINE

A. M. Brydun, P. A. Yatsulka. *A version of Carleman's formula summation of the Riemann ζ-function on the critical line*, Mat. Stud. **35** (2011), 3–9.

A version of Carleman's formula for functions holomorphic in a rectangle is proved. It is applied to the evaluation of the integral of ζ-function logarithm with the summing factor $\exp(-t)$ along the critical line. This allowed to obtain a new statement equivalent to the Riemann hypothesis.

А. М. Бридун, П. А. Яцулка. *Вариант формулы Карлемана и суммирование ζ-функции Римана на критичной прямой* // Мат. Студії. – 2011. – Т.35, №1. – C.3–9.

Для функций, голоморфных в прямоугольнике, доказан один вариант формулы Карлемана. Он применен к нахождению интеграла от логарифма ζ-функции с сумирующим множителем $\exp(-t)$ вдоль критичной прямой. Это позволило установить новое утверждение, эквивалентное гипотезе Римана.

1. Introduction. A Carleman type formula for functions meromorphic in a rectangle with the remainder term in an explicit form was obtained in [1]. Here we prove its modification for the functions which are real on a side of a rectangle, and apply it to the study of the Riemann ζ-function.

The integrals of the Riemann ζ-function logarithm along the vertical lines with diverse summing factors were studied by many authors [2]– [6]. These results generate new statements equivalent to the well known Riemann hypothesis (RH) for ζ-function. In particular, due to a theorem of Balazard-Saias-Your [4], the RH is true if and only if

$$\int\limits_{\mathrm{Re}\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|\zeta|^2}|ds| = 0.$$

In this paper applying the modified Carleman formula we evaluate the integral

$$I = \int\limits_0^\infty e^{-t}\log\left|\zeta\left(\frac{1}{2}+it\right)\right| dt$$

in terms of non-trivial ζ-function zeros and obtain a new statement equivalent to the RH. Namely, the RH is true if and only if $I = C$ where C is indicated below.

2. A version of Carleman formula. Let f be a meromorphic function on the closure of the rectangle $R_x = \{z = t + iy : x_0 < t < x, 0 < y < \pi\}$. Denote by $\{a_q\}$, $a_q = \alpha_q + i\delta_q$, and

2010 *Mathematics Subject Classification:* 11M26, 11M41.

$\{\omega_q\}$, $\omega_q = \xi_q + i\eta_q$, its zeros and poles in R_x, respectively. Choosing some $z^* \in R_x^*$ and some value $\log f(z^*)$, we define the function $\log f(z)$ in the domain

$$R_x^* = R_x \backslash \bigcup_j \left(\{t\beta_j + i\gamma_j : t \geq 1\} \cup \{t\xi_j + i\eta_j : t \geq 1\}\right)$$

and on the boundary ∂R_x except the zeros and the poles that lie on the ∂R_x by the relation

$$\log f(z) = \log f(z^*) + \int\limits_{z^*}^{z} \frac{f'(\zeta)}{f(\zeta)} d\zeta. \tag{1}$$

The integral is taking along a path in R_x^* with the ends at z^* and z.

We recall the following version of Carleman's formula for a rectangle with the explicit remainder from term in the form Theorem 1 will be received.

Theorem A ([1]). *Let $f, f \not\equiv 0$, be a meromorphic function on the closure of the rectangle $R_x = \{z = t + iy : x_0 < t < x, 0 < y < \pi\}$. Let $\log f(z)$ be defined on*

$$R_x^* = R_x \backslash \bigcup_j \left(\{t\beta_j + i\gamma_j : t \geq 1\} \cup \{t\xi_j + i\eta_j : t \geq 1\}\right)$$

and on ∂R_x except the zeros and the poles, that lie on ∂R_x, by relation (1). Put also $\arg f(z) = \operatorname{Im} \log f(z)$. Then

$$\sum_{a_q \in R_x} \left(\frac{1}{e^{\alpha_q}} - \frac{e^{\alpha_q}}{e^{2x}}\right) \sin \delta_q - \sum_{\omega_p \in R_x} \left(\frac{1}{e^{\xi_p}} - \frac{e^{\xi_p}}{e^{2x}}\right) \sin \eta_p = \frac{1}{2\pi} \int\limits_{x_0}^{x} (\log |f(t)| + \log |f(t + i\pi)|) \times$$
$$\times \left(\frac{1}{e^t} - \frac{e^t}{e^{2x}}\right) dt + \frac{1}{\pi e^x} \int\limits_0^{\pi} \log |f(x + iy)| \sin y dy + Q(x; x_0, f), \tag{2}$$

where

$$Q(x; x_0, f) = \frac{\operatorname{sh}(x - x_0)}{\pi e^x} \int\limits_0^{\pi} \arg f(x_0 + iy) \cos y dy - \frac{\operatorname{ch}(x - x_0)}{\pi e^x} \int\limits_0^{\pi} \log |f(x_0 + iy)| \sin y dy. \tag{3}$$

The main result of this section follows from Theorem A.

Theorem 1. *Let $f, f(z) \not\equiv 0$, be a holomorphic function on the closure of the rectangle $R_x = \{z = t + iy : x_0 < t < x,\ 0 < y < \pi\}$ and real-valued on the segment $I_0 = \{z : z = x_0 + iy,\ 0 \leq y \leq \pi\}$. Then*

$$\sum_{a_q \in R_x} \left(\frac{1}{e^{\alpha_q}} - \frac{e^{\alpha_q}}{e^{2x}}\right) \sin \delta_q + \frac{1}{2} \sum_{\operatorname{Re} a_q = x_0} \left(\frac{1}{e^{x_0}} - \frac{e^{x_0}}{e^{2x}}\right) \sin \delta_q =$$
$$= \frac{1}{2\pi} \int\limits_{x_0}^{x} (\log |f(t)| + \log |f(t + i\pi)|) \left(\frac{1}{e^t} - \frac{e^t}{e^{2x}}\right) dt + \tag{4}$$
$$+ \frac{1}{\pi e^x} \int\limits_0^{\pi} \log |f(x + iy)| \sin y dy - \frac{\operatorname{ch}(x - x_0)}{\pi e^x} \int\limits_0^{\pi} \log |f(x_0 + iy)| \sin y dy.$$

Proof. Let us calculate the integral $\frac{1}{\pi}\int_0^\pi \arg f(x_0+iy)\cos y dy$. For this purpose consider the function

$$g(z) = f(z) \Big/ \prod_{q=1}^{m}(z - z_q), \tag{5}$$

where $z_q = x_0 + i\delta_q$ are zeros of the function f, $0 \le \delta_q \le \pi$. In the case of absence of such zeros, we suppose that the product (5) is equal to unity.

The function g does not have zeros on I_0 and

$$g(x_0 + iy) = f(x_0 + iy) \Big/ \prod_{q=1}^{m} i(y - \delta_q).$$

If m is an even number, then g is real on I_0 and does not change the sign, since f is real, the denominator is real and $g(z) \neq 0$. If m is an odd number, then the function g obtains imaginary values without changing of the sign. So, $\arg g(x_0 + iy) = C = \text{const}$, $0 \le y \le \pi$. Hence, from previous equality and (5) we deduce

$$\arg f(x_0 + iy) = \sum_{q=1}^{m} \arg\left(i(y - \delta_q)\right) + C.$$

Consequently,

$$\frac{1}{\pi}\int_0^\pi \arg f(x_0 + iy)\cos y dy = \frac{1}{\pi}\sum_{q=1}^{m}\int_0^\pi \arg\left(i(y - \delta_q)\right)\cos y dy.$$

For $y < \delta_q$ the following equality holds

$$\arg\left(i(y - \delta_q)\right) = \arg(z^* - z_p) + \operatorname{Im}\int_{z^*}^{x_0} \frac{dz}{z - z_q} = \arg(-i\delta_q).$$

We fix $\arg(-i\delta_q) = -\pi/2$. Then for $y > \delta_q$ we get $\arg\left(i(y - \delta_q)\right) = \pi/2$. Thus,

$$\frac{1}{\pi}\int_0^\pi \arg\left(i(y - \delta_q)\right)\cos y dy = \frac{1}{\pi}\left(\int_0^{\delta_q}\left(-\frac{\pi}{2}\right)\cos y dy + \int_{\delta_q}^{\pi}\left(\frac{\pi}{2}\right)\cos y dy\right) = -\sin\delta_q$$

and

$$\frac{1}{\pi}\int_0^\pi \arg f(x_0 + iy)\cos y dy = -\sum_{q=1}^{m}\sin\delta_q. \tag{6}$$

Taking into account (6) and (3) we obtain (4). $\square$

3. Summation of the Riemann ζ-function logarithm on the critical line. Denote by $\rho_q = \beta_q + i\gamma_q$ zeros of the Riemann ζ-function on the strip $R_\infty = \{s : 1/2 < \operatorname{Re} s < 1\}$.

Theorem 2. *The following equality holds*

$$\frac{1}{2}\int_{0}^{+\infty} e^{-t}\log\left|\zeta\left(\frac{1}{2}+it\right)\right|dt = -\sum_{\rho_q\in R_\infty}\frac{\cos(\pi\beta_q)}{e^{\pi\gamma_q}}+C, \tag{7}$$

where $C=-(c_1+c_2+c_3)/2$,

$$c_1=\int_{1/2}^{3/2}\log|(\sigma-1)\zeta(\sigma)|\cos\pi\sigma d\sigma,\ c_2=\int_{0}^{+\infty}\log\left(\frac{1}{4}+t^2\right)\frac{dt}{e^t},\ c_3=\frac{1}{2}\int_{0}^{+\infty}\log\left|\zeta\left(\frac{3}{2}+it\right)\right|\frac{dt}{e^t}.$$

The Riemann hypothesis holds if and only if

$$\frac{1}{2}\int_{0}^{+\infty} e^{-t}\log\left|\zeta\left(\frac{1}{2}+it\right)\right|dt = C. \tag{8}$$

Proof. In the complex plane of the variable $s=\sigma+it$ consider the rectangle $R_T=\{s : 1/2<\sigma<3/2, 0<t<T\}$.

The function $(s-1)\zeta(s)$ is holomorphic in $\overline{R_T}$ and the transformation $z=i\pi(3/2-s)$ maps R_T to R_x with $x_0=0$. The inverse mapping is $s=iz/\pi+3/2$.

The function

$$f(z)=\left(\frac{i}{\pi}z+\frac{1}{2}\right)\zeta\left(\frac{i}{\pi}z+\frac{3}{2}\right),\quad z\in R_x$$

is holomorphic in $\overline{R_x}$, where $x_0=0$ and satisfy the conditions of Theorem 1.

Taking into account that f has no zeros on I_0, we obtain (4) in the form

$$\sum_{a_q\in R_x}\left(\frac{1}{e^{\alpha_q}}-\frac{e^{\alpha_q}}{e^{2x}}\right)\sin\delta_q=\frac{1}{2\pi}\int_0^x(\log|f(t)|+\log|f(t+i\pi)|)\left(\frac{1}{e^t}-\frac{e^t}{e^{2x}}\right)dt+$$
$$+\frac{1}{\pi e^x}\int_0^\pi\log|f(x+iy)|\sin ydy-\frac{\operatorname{ch}x}{\pi e^x}\int_0^\pi\log|f(iy)|\sin ydy=I_1+I_2+I_3+I_4. \tag{9}$$

We have

$$I_4=-\frac{\operatorname{ch}x}{\pi e^x}\int_0^\pi\log|f(iy)|\sin ydy=\frac{e^{\pi T}+e^{-\pi T}}{2e^{\pi T}}\int_{1/2}^{3/2}\log|(\sigma-1)\zeta(\sigma)|\cos\pi\sigma d\sigma=$$
$$=\frac{1}{2}\int_{1/2}^{3/2}\log|(\sigma-1)\zeta(\sigma)|\cos\pi\sigma d\sigma+\frac{1}{2e^{2\pi T}}\int_{1/2}^{3/2}\log|(\sigma-1)\zeta(\sigma)|\cos\pi\sigma d\sigma.$$

Both integrals are convergent. The last addend vanishes as $T\to+\infty$. Hence,

$$\lim_{T\to+\infty}I_4=\frac{1}{2}\int_{1/2}^{3/2}\log|(\sigma-1)\zeta(\sigma)|\cos\pi\sigma d\sigma:=\frac{1}{2}c_1.$$

Further,

$$I_1 = \frac{1}{2\pi} \int\limits_0^x \log|f(u)| \left(\frac{1}{e^u} - \frac{e^u}{e^{2x}} \right) du = \frac{1}{2} \int\limits_0^T \log\left|\frac{1}{2} + it\right| \frac{dt}{e^t} +$$

$$+\frac{1}{2} \int\limits_0^T \log\left|\frac{3}{2} + it\right| \frac{dt}{e^t} = \frac{1}{4} \int\limits_0^T \log\left(\frac{1}{4} + t^2\right) \frac{dt}{e^t} + \frac{1}{2} \int\limits_0^T \log\left|\frac{3}{2} + it\right| \frac{dt}{e^t}.$$

Thus,

$$\lim_{T\to+\infty} I_1 = \frac{1}{4} \int\limits_0^{+\infty} \log\left(\frac{1}{4} + t^2\right) \frac{dt}{e^t} + \frac{1}{2} \int\limits_0^{+\infty} \log\left|\zeta\left(\frac{3}{2} + it\right)\right| \frac{dt}{e^t} := \frac{1}{4} c_2 + \frac{1}{2} c_3.$$

Similarly,

$$I_2 = \frac{1}{2\pi} \int\limits_0^x \log|f(u+i\pi)| \left(\frac{1}{e^u} - \frac{e^u}{e^{2x}} \right) du = \frac{1}{2} \int\limits_0^T \log|(s-1)\zeta(s)| \left(\frac{1}{e^t} - \frac{e^t}{e^{2T}} \right) dt,$$

$$\lim_{T\to+\infty} I_2 = \frac{1}{4} \int\limits_0^{+\infty} \log\left(\frac{1}{4} + t^2\right) \frac{dt}{e^t} + \frac{1}{2} \int\limits_0^{+\infty} \log\left|\zeta\left(\frac{1}{2} + it\right)\right| \frac{dt}{e^t} =$$

$$= \frac{1}{4} c_2 + \frac{1}{2} \int\limits_0^{+\infty} \log\left|\zeta\left(\frac{1}{2} + it\right)\right| \frac{dt}{e^t}.$$

Further, we have

$$I_3 = \frac{1}{\pi e^x} \int\limits_0^{\pi} \log|f(x+iy)| \sin y dy = -\frac{1}{e^{\pi T}} \int\limits_{1/2}^{3/2} \log|(s-1)\zeta(s)| \cos \pi\sigma d\sigma =$$

$$= \frac{1}{e^{\pi T}} \int\limits_{1/2}^{3/2} \log|s-1| \cos \pi\sigma d\sigma - \frac{1}{e^{\pi T}} \int\limits_{1/2}^{3/2} \log|\zeta(s)| \cos \pi\sigma d\sigma$$

and $\lim\limits_{T\to+\infty} I_3 = 0$. The left-hand side of (9) can be written as follows

$$-\sum_{\rho_q \in R_T} \frac{\cos \pi\beta_q}{e^{\pi\gamma_q}} + \sum_{\rho_q \in R_T} \frac{e^{\pi\gamma_q} \cos \pi\beta_q}{e^{2\pi T}}.$$

The limit value of the first sum is

$$\lim_{T\to+\infty} \sum_{\rho_q \in R_T} \frac{\cos \pi\beta_q}{e^{\pi\gamma_q}} = \sum_{\rho_q \in R_\infty} \frac{\cos \pi\beta_q}{e^{\pi\gamma_q}}.$$

Taking into account that the number $N(T)$ of zeros of $\zeta(s)$ in $\overline{R_T}$ has the property [7] $N(T) \sim T\log T$, we obtain

$$\lim_{T\to+\infty}\left|\sum_{\rho_q\in R_T}\frac{e^{\pi\gamma_q}\cos\pi\beta_q}{e^{2\pi T}}\right|\leq \lim_{T\to+\infty}\frac{1}{e^{\pi T}}\sum_{\rho_q\in R_T}|\cos\pi\beta_q|\leq$$
$$\leq \lim_{T\to+\infty}\frac{1}{e^{\pi T}}\sum_{\rho_q\in R_T}1\leq \lim_{T\to+\infty}\frac{T\log T}{e^{\pi T}}=0.$$

Therefore,

$$-\sum_{\rho_q\in R_T}\frac{\cos\pi\beta_q}{e^{\pi\gamma_q}}=\frac{1}{2}\int_0^{+\infty}e^{-t}\log|\zeta(1/2+it)|dt-C, \tag{10}$$

where $C=-(c_1+c_2+c_3)/2$, i.e. (7).

Let as prove the second statement of Theorem 1.

If the Riemann hypothesis holds then $\beta_q = 1/2$ for each q, the sum at the right-hand side of (7) vanishes and we have (8).

Conversely, if (8) fulfils then the sum in (7) equals zero. But each its addend is nonpositive because $1/2 \leq \beta < 3/2$. Thus, $\beta_q = 1/2$ for all q and the Riemann hypothesis holds. □

Acknowledgement. The authors are grateful to Prof. A. Kondratyuk for useful suggestions.

REFERENCES

1. Brydun A.M. *Nevanlinna characteristic and first main theorem for meromorphic functions in a half-strip//* Visnyk Lviv Univ. Ser. Mech-Math. – 2004. – V.63. – P. 32–43. (Ukrainian)
2. Volchkov V.V. *On an equality equivalent to the Riemann hypothesis//* Ukrain. Mat. Zh. – 1995. – V.47, №3. – P. 422–423. (Russisan)
3. Kondratyuk A.A. *A Carleman-Nevanlinna theorem and summation of the Riemann zeta-function logarithm//* Comput. Methods Funct. Theoryl. – 2004. – V.4. – P. 391–403.
4. Balazard M., Saias E., Yor M. *Notes sur la fonction ζ de Riemann, 2//* Adv. Math. – 1999. – V.143. – P. 284–287.
5. Kondratyuk A.A., Yatsulka P.A. *Summation of the Riemann ζ-function logarithm on the critical line//* Voronoi's Impact on Modern Science, Book 4, Vol. 1: Proceedings of the Fourth International Conference on Analytic Number Theory and Spatial Tesselations. Kyiv. 2008. P. 59–62.
6. Yatsulka P. *Summation of the Riemann zeta-function mod logarithm on the critical line//* Visnyk Lviv Univ. Ser. Mech-Math. – 2008. – V.68. – P. 276–280. (Ukrainian)
7. Titchmarsh E.C. The theory of the Riemann zeta-function. – Moscow: Izdat. inost. liter., 1953. – 408 p. (Russian)

Chair of KGM,
Institute of Geodesy,
Lviv Polytechnic National University,
a_brydun@yahoo.com

Chair of Mathematical and Functional Analysis,
Faculty of Mechanics and Mathematics,
Lviv National University,
petroandr@rambler.ru

Received 12.06.2010