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B. N. Khabibullin

# GENERALIZATIONS OF NEVANLINNA'S THEOREMS

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We give a short survey on generalizations of Nevanlinna theorems on zero distribution of bounded holomorphic functions and representation of meromorphic functions in multiply connected domains. It is a part of our report on the conference on complex analysis dedicated to the memory of Anatolii Asirovich Goldberg in Lviv, May 31-June 5, 2010.

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Мы даем краткий обзор обобщений теорем Неванлинны о распределении нулей ограниченных голоморфных функций и представлении мероморфных функций в многосвязных областях. Это – часть нашего доклада на конференции по комплексному анализу, посвященной памяти Анаполия Асировича Гольдберга во Львове 31 мая–5 июня 2010 г.

**1. Definitions, agreements, and basic notions.** Let $\Omega$ be a domain in the complex plane $\mathbb{C}$ or in the Riemann sphere $\mathbb{C}_\infty \neq \Omega$. For $S \subset \Omega \subset \mathbb{C}_\infty$, we denote by $\overline{S}$ and $\partial\Omega$ the *closure* and the *boundary* of $S$ relative to $\mathbb{C}_\infty$. We write $S \Subset \Omega$ if $\overline{S} \subset \Omega$.

Denote by $\mathrm{Hol}(\Omega)$, $\mathrm{Mer}(\Omega)$, $\mathrm{sbh}(\Omega)$, and $\mathrm{harm}(\Omega)$ the classes of all holomorphic, meromorphic, subharmonic, and harmonic functions on $\Omega$.

We are concerned with finite or infinite sequences $\Lambda = \{\lambda_k\}$, $k = 1, 2, \dots$ of not necessarily distinct points from the domain $\Omega$, without limit points in $\Omega$. Let $n_\Lambda$ be an integer-valued *counting measure* of sequence $\Lambda$ defined by

$$n_\Lambda(S) := \sum_{\lambda_k \in S} 1, \quad S \subset \Omega.$$

Let $S \subset \Omega$. $\Lambda \subset S \iff \operatorname{supp} n_\Lambda \subset S$.

A sequence $\Lambda$ coincides with a sequence $\Gamma = \{\gamma_n\}$ (or is equal to $\Gamma$, or $\Lambda = \Gamma$) iff $n_\Lambda = n_\Gamma$. $\Lambda \subset \Gamma$ means $n_\Lambda \leq n_\Gamma$. $\Lambda \cap \Gamma$ and $\Lambda \cup \Gamma$ are defined by $n_{\Lambda\cap\Gamma} := \min\{n_\Lambda, n_\Gamma\}$ and $n_{\Lambda\cup\Gamma} := n_\Lambda + n_\Gamma$.

Given $f\colon A \to B$ and $b \in B$, we write $f \equiv b$ *on* $A'$ if $f$ is identically equal to $b$ on $A' \subset A$; in the opposite case, $f \not\equiv b$ on $A'$.

Let $A, B \subset [-\infty, +\infty]$. A function $f\colon A \to B$ is *increasing* (*decreasing* resp.) if, for any $x_1, x_2 \in A$, $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$ $\quad (f(x_1) \geq f(x_2)$ resp.).

Given $a \in \mathbb{R}$, and $f\colon A \to [-\infty, +\infty]$, we set $a^+ := \max\{0, a\}$, $f^+ := \max\{0, f\}$.

The term "positive" ("negative" resp.) means "$\geq 0$" ("$\leq 0$" resp.).

Let $f \in \mathrm{Hol}(\Omega)$ or $f \in \mathrm{Mer}(\Omega)$, $f \not\equiv 0, \infty$ on $\Omega$. Write $\mathrm{Zero}_f$ for the zero set of $f$ (counting multiplicities). Evidently, $\mathrm{Zero}_f$ is a sequence of not necessarily distinct points from the domain $\Omega$, without limit points in $\Omega$.

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A sequence $\Lambda$ is a *zero sequence* for a subspace $H \subset \operatorname{Hol}(\Omega)$ (further we write $\Lambda \in \operatorname{Zero}(H)$) if and only if there exists a function $f \in H$ such that $\Lambda = \operatorname{Zero}_f$.

A function $f \in \operatorname{Hol}(\Omega)$ *vanishes on* $\Lambda$ if and only if $\Lambda \subset \operatorname{Zero}_f$ (we write $f(\Lambda) = 0$).

A sequence $\Lambda$ is a *zero subsequence* or a *non-uniqueness sequence* for the space $H$ if there exists a nonzero function $f \in H$ such that $f(\Lambda) = 0$.

**2. Problems.** Let $H$ be a subspace of $\operatorname{Hol}(\Omega)$. We consider the following five problems.

1. What point sequences $\Lambda$ can be zero sequences for $H$?
2. What point sequences $\Lambda$ can be zero subsequences for $H$?
3. In what cases there is a zero subsequence for $H$ which is simultaneously a zero sequence for $H$ or for some, preferably minimal, extension space $\widehat{H} \supset H$ of holomorphic functions on $\Omega$?
4. When a meromorphic function on $\Omega$ can be represented as a ratio of holomorphic functions from $H$?
5. When a meromorphic function on $\Omega$ can be represented as a ratio of holomorphic functions from $H$ without common zeros?

**3. Green's function.** Denote by $g_\Omega(\cdot, z)\colon \mathbb{C}_\infty \setminus \{z\} \to [0, +\infty)$ the *extended Green function* for $\Omega$ with a pole at $z \in \Omega$, i. e., $g_\Omega(\zeta, z) \equiv 0$ for points $\zeta \in \mathbb{C}_\infty \setminus \overline{\Omega}$, $g_\Omega(\cdot, z) \in \operatorname{sbh}(\mathbb{C}_\infty \setminus \{z\})$, $g_\Omega(\cdot, z) \in \operatorname{harm}(\Omega \setminus \{z\})$, and also

$$g_\Omega(\zeta, z) = -\log|\zeta - z| + O(1), \quad \zeta \to z.$$

Given a continuous function $\phi\colon \partial\Omega \to \mathbb{R}$, by $H_\Omega\phi$ we denote the *solution of the Dirichlet problem* for $\Omega$ with boundary function $\phi$ or the associated *Perron function*

$$H_\Omega\phi := \sup\{u \in \operatorname{sbh}(\Omega) : \limsup_{z\to\zeta} u(z) \le \phi(\zeta),\ \forall\zeta \in \partial\Omega\}.$$

**4. Harmonic measure.** Denote by $\mathcal{B}(\partial\Omega)$ the $\sigma$-algebra of Borel subset of $\partial\Omega$.

Denote by $\omega_\Omega(z, \cdot)$ the *harmonic measure* for $\Omega$ at the point $z \in \Omega$, i. e.,

$$\omega_\Omega(\cdot, \cdot)\colon \Omega \times \mathcal{B}(\partial\Omega) \to [0, 1]$$

such that

a) the map $B \mapsto \omega_\Omega(z, B)$ is a Borel probability measure on $\partial\Omega$;

b) if $\phi\colon \partial\Omega \to \mathbb{R}$ is continuous function, then

$$H_\Omega\phi(z) = \int_{\partial\Omega} \phi(\zeta)\, d\omega_\Omega(z, \zeta).$$

**5. The unit disk.** Starting points of our research are Nevalinna's theorems (1929).

Denote by $\operatorname{Hol}^\infty(\mathbb{D}) \subset \operatorname{Hol}(\mathbb{D})$ the space of holomorphic bounded functions on $\mathbb{D}$.

Denote by $f_\Lambda$ a holomorphic function $\not\equiv 0$ on $\mathbb{D}$ with zero sequence $\operatorname{Zero}_{f_\Lambda} = \Lambda \subset \Omega$.

**Theorem 1. (Nevanlinna)** *The following statements are equivalent.*

1) *$\Lambda$ is a zero sequence for $\operatorname{Hol}^\infty(\mathbb{D})$;*

2) *$\Lambda$ is a zero subsequence for $\operatorname{Hol}^\infty(\mathbb{D})$;*

3) *for each or some function* $f_\Lambda \in \operatorname{Hol}(\mathbb{D})$

$$\sup_{r<1}\bigl(H_{r\mathbb{D}} \log|f_\Lambda|\bigr)(0) = \sup_{r<1}\int_{r\partial\mathbb{D}} \log\bigl|f_\Lambda(z)\bigr|d\omega_{r\mathbb{D}}(0,z) = \sup_{r<1}\frac{1}{2\pi}\int_0^{2\pi}\log\bigl|f_\Lambda(re^{i\theta})\bigr|\,\mathrm{d}\theta < +\infty;$$

4) $\sum\limits_k g_{\mathbb{D}}(\lambda_k, 0) = \sum\limits_k \log\dfrac{1}{|\lambda_k|} < +\infty \iff \sum\limits_k (1-|\lambda_k|) < +\infty.$

Let $z \in \Omega$, $p, q \in \operatorname{Hol}(\Omega)$, and let

$$f = \frac{p}{q} \in \operatorname{Mer}(\Omega),\ p(z) = q(z) = 1. \tag{1}$$

We set

$$u_f := \max\{\log|p|, \log|q|\} \in \operatorname{sbh}(\Omega) \tag{2}$$

$$\text{or } u_f := \log\sqrt{|p|^2+|q|^2} \in \operatorname{sbh}(\Omega). \tag{3}$$

Let $D$ be a subdomain of $\Omega$, $z \in D \Subset \Omega$. The integral

$$T_D(f;z) := \int_{\partial D} u_f\,\mathrm{d}\omega_D(z,\cdot) \tag{4}$$

is *Nevanlinna's characteristic* of $f$ on $D$ (in the form of Ahlfors–Shimizu for (3)) at the point $z$ relative to representation (1).

If $\operatorname{Pol}_f = \{\gamma_k\}_{k=1}^\infty \subset \Omega$ is the *pole sequence* of $f$ in $\Omega$, i. e., zero sequence of $1/f$ in $\Omega$, and $\operatorname{Zero}_p \cap \operatorname{Zero}_q = \varnothing$, then, by (2),

$$T_D(f;z) = \sum_k g_D(\gamma_k, z) + \int_{\partial D} \log^+|f|\,\mathrm{d}\omega_D(z,\cdot).$$

If $z = 0 \in D$, then $T_D(f) := T_D(f;0)$.

**6. Multiply connected domains.** There are generalizations of Nevanlinna's theorems to classes of holomorphic and meromorphic functions on special finitely connected domains $\Omega$.

Given $z \in \mathbb{C}$ and $0 < t < +\infty$, we write

$$D(z,t) := \{w \in \mathbb{C}\colon |w-z| < t\}, \quad \overline{D}(z,t) := \overline{D(z,t)}.$$

We consider now results from [1] (Andriy Kondratyuk and Ilpo Laine, 2006). The authors developed the Nevalinna theory and combined topics for meromorphic and holomorphic functions to $(m+1)$-connected domains $\Omega$, $m \in \mathbb{N}$, for following cases.

1) An $(m+1)$-connected domain $\Omega = \mathbb{C} \setminus \bigcup_{j=1}^m \{c_j\}, c_j \in \mathbb{C}, m \in \mathbb{N}$, is called a *m-punctured plane*. For example, such a 2-connected domain is a *punctured plane* $\mathbb{C} \setminus \{0\}$.

2) A bounded[1] $(m+1)$-connected domain

$$\Omega = D(0,R) \setminus \Bigl(\bigcup_{j=1}^m D(z_j, r_j)\Bigr),\ m \in \mathbb{N}, \quad D(z_j, t_j) \Subset D(0,R),$$

$\overline{D}(z_j, t_j) \cap \overline{D}(z_{j'}, t_{j'}) = \varnothing$ for all $j \neq j'$, is called a *strictly circular domain*. 2-connected annuli $A_R = D(0,R) \setminus D(0, 1/R)$, $1 < R < +\infty$, are examples of such domains.

[1] always bounded relative to $\mathbb{C}$

3) Let $\Omega\subset\mathbb{C}$ be a bounded $(m+1)$-connected domain, $m\in\mathbb{N}$,

$$\Omega=\bigcup_{j=0}^{m} G_j,$$

where $G_j$ are simply connected domains (relative to $\mathbb{C}_\infty$), $\partial G_j$ are simple (Jordan) closed paths, $G_0$ is bounded, and $\infty\in G_j$ for $j=1,\dots,m$, $\mathbb{C}\setminus G_j\Subset G_0$,

$$(\mathbb{C}\setminus G_j)\cap(\mathbb{C}\setminus G_{j'})=\varnothing,\ j\neq j',\ j,j'\geq 1.$$

Such domains $\Omega$ are called in [1] *admissible*.

Let $\varphi_j$ be conformal mappings of $\mathbb{D}$ onto $G_j$ realizing a conformal equivalence of $\mathbb{D}$ and $G_j$ with $\varphi_0(0)=0$, and $\varphi_j(0)=\infty$ as $j=1,\dots,m$. Then every $\varphi_j$ can be extended to a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{G}_j$ (the Carathéodory theorem). For a simple closed path $\Gamma$, $\operatorname{int}\Gamma$ denote the bounded domain with the boundary $\Gamma$ (the Jordan theorem).

Let $\Gamma_{jr}(s){:=}\varphi_j(e^{is}),\ 0\leq s\leq 2\pi,\ j=0,\dots,m,\quad \Gamma^*_{jr}{:=}\Gamma_{jr}\bigl([0,2\pi)\bigr),\ j=0,\dots,m$, and

$$\Omega_r{:=}\bigl(\operatorname{int}\Gamma^*_{0r}\bigr)\setminus\bigcup_{j=1}^{k}\overline{\operatorname{int}\Gamma^*_{jr}},\ r_0\leq r<1.$$

Let $\Lambda=\{\lambda_k\}$ be a sequence in $\Omega$, and let

$$N_0(r,\Lambda){:=}\int_{r_0}^{r}\frac{n_\Lambda(\Omega_t)}{t}\,\mathrm{d}t,\ r_0\leq r<1,$$

where $r_0<1$ is a constant sufficiently close to 1.

For a meromorphic function $f$ on an admissible bounded domain $\Omega$ denote

$$m_0(r,f){:=}\frac{1}{2\pi}\sum_{j=1}^{m}\biggl(\int_0^{2\pi}\log^+\bigl|f\bigl(\varphi_j(re^{is})\bigr)\bigr|\,\mathrm{d}s-\int_0^{2\pi}\log^+\bigl|f\bigl(\varphi_j(r_0e^{is})\bigr)\bigr|\,\mathrm{d}s\biggr),$$

$r_0\leq r<1$. The function $T_0(r,f){:=}N_0(r,\operatorname{Pol}_f)+m_0(r,f),\ r_0\leq r<1$, is the *Nevalinna characteristic* of $f$ (in the sense of A. Kondratyuk and I. Laine).

**Theorem 2 (1, Theorem 43.2).** *Let $f\in\operatorname{Mer}(\Omega)$, where $\Omega\subset\mathbb{C}$ is a finitely connected bounded admissible domain. If $T_0(r,f)=O(1),\ r\to 1$, then there are bounded functions $h_1,h_2\in\operatorname{Hol}(\Omega)$ such that $f=h_1/h_2$.*

Let $f_\Lambda\in\operatorname{Hol}(\Omega)$ be a function with a zero sequence $\Lambda\subset\Omega$. If we apply this theorem to the function $f_\Lambda\in\operatorname{Hol}(\Omega)$, then we get

**Theorem 3.** *Let $\Omega\subset\mathbb{C}$ be a finitely connected admissible bounded domain. If*

$$\sup_{r_0\leq r<1} m_0(r,f_\Lambda)<+\infty,$$

*then there is a bounded holomorphic on $\Omega$ function $f$ such that $f(\Lambda)=0$.*

We consider Problems 1–5 in a general form in [2], [3].

A function (weight) $M\colon \Omega \to \mathbb{R}$ define a weighted space

$$\operatorname{Hol}(\Omega; M) := \left\{ f \in \operatorname{Hol}(\Omega)\colon \sup_{z\in\Omega} \frac{|f(z)|}{\exp M(z)} < +\infty \right\}.$$

If $M \equiv 0$ on $\Omega$, then $\operatorname{Hol}(\Omega; 0) = \operatorname{Hol}^\infty(\Omega)$.

Let $\Omega$ be a domain in $\mathbb{C}_\infty$.

We use the following classification of domains $\Omega$ such that $0 \in \Omega \not\ni \infty$.

**I.** This domain $\Omega$ is

a) *simply connected* (for example, $\Omega = \mathbb{C}$ or $\Omega = \mathbb{D}$) or

b) *finitely connected* and $\overline{\Omega} \neq \mathbb{C}_\infty$ (for example, each bounded finitely connected domain $\Omega$).

**II.** This domain $\Omega$ is $(m+1)$-*connected* with $m \in \mathbb{N}$ and $\overline{\Omega} = \mathbb{C}_\infty$.

**III.** This domain $\Omega$ is

a) *bounded* or

b) *unbounded*.

Let $\operatorname{Exh}\Omega = \{D\}$ be an *exhaustion* of $\Omega$, i.e., $\bigcup_{D\in\operatorname{Exh}\Omega} D = \Omega$, where $0 \in D$ and every $D \in \operatorname{Exh}\Omega$ is a regular domain for the Dirichlet problem. Such an exhaustion always exists (countable increasing and such that all $D$ have smooth boundary).

**Theorem 4.** *Let $\Lambda \subset \Omega$, and let $\Omega$ be a domain of type* **I**. *Then the following statements are equivalent.*

1) *$\Lambda$ is a zero sequence for $\operatorname{Hol}^\infty(\Omega)$;*

2) *$\Lambda$ is a zero subsequence for $\operatorname{Hol}^\infty(\Omega)$;*

3) *for each or some function $f_\Lambda \in \operatorname{Hol}(\Omega)$ with $\operatorname{Zero}_{f_\Lambda} = \Lambda$*

$$\sup_{D\in\operatorname{Exh}\Omega} \bigl(H_D \log|f_\Lambda|\bigr)(0) := \sup_{D\in\operatorname{Exh}\Omega} \int_D \log\bigl|f_\Lambda(z)\bigr| d\omega_D(0,z) < +\infty;$$

4) $\sup\limits_{D\in\operatorname{Exh}\Omega} \sum\limits_k g_D(\lambda_k, 0) < +\infty$.

*If $\Omega$ possesses the Green function, then we can remove $\sup\limits_{D\in\operatorname{Exh}\Omega}$ everywhere and replace $D$ with $\Omega$.*

**Theorem 5.** *Let $\Omega$ be a domain of type* **I**. *Let* (see (1))

$$f = \frac{p}{q} \in \operatorname{Mer}(\Omega),\ p, q \in \operatorname{Hol}(\Omega),\ p(0) = q(0) = 1,$$

*and* (see (4), (2), (3) resp.)

$$T_D(f) := \int_{\partial D} u_f \, \mathrm{d}\omega_D(0,\cdot),\ D \in \operatorname{Exh}(\Omega),$$

*where*

$$u_f := \max\{\log|p|, \log|q|\} \in \operatorname{sbh}(\Omega) \quad \text{or } u_f := \log\sqrt{|p|^2 + |q|^2} \in \operatorname{sbh}(\Omega).$$

*Assume that one of the following two conditions holds.*

1) $\sup\limits_{D\in \operatorname{Exh}\Omega} T_D(f) < +\infty$;

2) $p, q \in \operatorname{Hol}^\infty(\Omega)$.

*Then there are* $p_0, q_0 \in \operatorname{Hol}^\infty(\Omega)$ *without common zeros such that* $f = p_0/q_0$.

**Remark 1.** If the domain $\Omega$ is regular for the Dirichlet problem, then we can remove $\sup\limits_{D\in \operatorname{Exh}\Omega}$ in 1) and replace $D$ with $G$.

**Theorem 6.** *Assume* $\Lambda \subset \Omega$, *and let* $\Omega$ *be a domain of type* **II**. *If* $\Lambda$ *is a zero sequence for* $\operatorname{Hol}^\infty(\Omega)$, *then*

2) $\Lambda$ *is a zero subsequence for* $\operatorname{Hol}^\infty(\Omega)$;

3) *for each (for some) function* $f_\Lambda \in \operatorname{Hol}(\Omega)$ *with* $\operatorname{Zero}_{f_\Lambda} = \Lambda$

$$\sup_{D\in \operatorname{Exh}\Omega} \bigl(H_D \log|f_\Lambda|\bigr)(0) := \sup_{D\in \operatorname{Exh}\Omega} \int_D \log\bigl|f_\Lambda(z)\bigr| d\omega_D(0,z) < +\infty\,;$$

4) $\sup\limits_{D\in \operatorname{Exh}\Omega} \sum\limits_k g_D(\lambda_k, 0) < +\infty$.

*Conversely, assume that one of the conditions 2)–4) holds. Then there is a constant* $b < m$ *such that* $\Lambda$ *is a zero sequence for every space* $\operatorname{Hol}(\Omega, M)$ *with*

$$M\colon z \mapsto c_0^+ \log^+|z| + \sum_{k=1}^m c_k^+ \log^+ \frac{1}{|z-a_k|}\,, \tag{5}$$

*where* $\sum_{k=0}^m c_k = b$, $a_k \in \mathbb{C} \setminus \Omega$.

**Theorem 7.** *Let* $\Omega$ *be a domain of type* **II**. *Suppose* (see (1))

$$f = \frac{p}{q} \in \operatorname{Mer}(\Omega),\; p, q \in \operatorname{Hol}(\Omega),\; p(0) = q(0) = 1.$$

*Assume that one of the following two conditions holds.*

1) $\sup\limits_{D\in \operatorname{Exh}\Omega} T_D(f) < +\infty$;

2) $p, q \in \operatorname{Hol}^\infty(\Omega)$.

*Then there are constant* $b < m$ *such that, for every function* $M$ *from* (5), *there exist functions* $p_0, q_0 \in \operatorname{Hol}(\Omega; M)$ *without common zeros representing* $f = p_0/q_0$.

**Theorem 8 (on zeros).** *Let* $\Omega \subset \mathbb{C}$ *be a domain, and let* $\Lambda \subset \Omega$ *be a sequence. If* $\Lambda$ *is a zero (sub)sequence for* $\operatorname{Hol}^\infty(\Omega)$, *then*

3) *for each or some function* $f_\Lambda \in \operatorname{Hol}(\Omega)$ *with* $\operatorname{Zero}_{f_\Lambda} = \Lambda$

$$\sup_{D\in \operatorname{Exh}\Omega} \bigl(H_D \log|f_\Lambda|\bigr)(0) := \sup_{D\in \operatorname{Exh}\Omega} \int_D \log\bigl|f_\Lambda(z)\bigr| d\omega_D(0,z) < +\infty\,;$$

4) $\sup\limits_{D\in \operatorname{Exh}\Omega} \sum\limits_k g_D(\lambda_k, 0) < +\infty$.

*Conversely, let one of the conditions* 3), 4) *be fulfilled. Then* $\Lambda$ *is a zero* <u>*sub*</u>*sequence for the space* $\operatorname{Hol}(\Omega, M)$ *with*

$$M\colon z \mapsto \log \frac{1}{\operatorname{dist}(z, \partial\Omega)}\,, \quad z \in \Omega,$$

*where* $\operatorname{dist}(z, \partial\Omega)$ *is the Euclidean distance from* $z$ *up to* $\partial\Omega$, *if* $\Omega$ *is* <u>*bounded*</u>*, and with any*

$$M\colon z \mapsto \log \frac{1}{\operatorname{dist}(z, \partial\Omega)} + c_0 \log^+ |z| + c_1 \log^+ \frac{1}{|z-a|}\,, \quad z \in \Omega,$$

*where* $c_0 + c_1 = 9$, $a \in \mathbb{C} \setminus \Omega$, *if* $\Omega$ *is* <u>*unbounded*</u>.

**Theorem 9.** *Let* $\Omega$ *be a subdomain of* $\mathbb{C}$. *Let* (see (1))

$$f = \frac{p}{q} \in \operatorname{Mer}(\Omega),\ p, q \in \operatorname{Hol}(\Omega),\ p(0) = q(0) = 1.$$

*Let us choose a function* $M$ *as in Theorem* 3 (on zeros). *Suppose that*

$$\sup_{D \in \operatorname{Exh}\Omega} T_{D}(f) < +\infty.$$

*Then there exist functions* $p_0, q_0 \in \operatorname{Hol}(\Omega; M)$ *representing* $f = p_0/q_0$.

**Remark 2.** If $\Omega$ possesses the Green function then we can remove $\sup\limits_{D \in \operatorname{Exh}\Omega}$ everywhere in Theorems 3 and replace $D$ with $\Omega$.

**7. General results.** If $M \in \operatorname{sbh}(\Omega)$ with the Riesz measure $\nu_M := \frac{1}{2\pi}\Delta M \geq 0$, then there is a global Riesz representation (decomposition)

$$M(z) = \int_{\Omega} k(\zeta, z)\, \mathrm{d}\nu_M(\zeta) + H(z),\ z \in \Omega, \tag{6}$$

where $H \in \operatorname{harm}(\Omega)$,

$$k(\zeta, z) = \log|\zeta - z| + h_M(\zeta, z) \tag{7}$$

is a special subharmonic kernel with harmonic component $h(\zeta, z)$ of $z \in \Omega$ for each $\zeta \in \Omega$. Let $Q\colon \Omega \to \mathbb{R}$ be an upper semicontinuous function such that

$$\int_{\Omega} \bigl(k(\zeta, 0) - k(\zeta, z)\bigr)^+ \mathrm{d}\nu_M(\zeta) \leq Q(z) \tag{8}$$

for almost all $z \in \Omega$ with respect to the Lebesgue measure on $\Omega$.

Denote by $\mathcal{U}_0^d(\Omega)$ the class of all connected unions $D \ni 0$ of finitely many open disks from $\Omega$ whose complement has no one-point connected components.

**Theorem 10.** *Let* $M \in \operatorname{sbh}(\Omega) \cap C(\Omega)$ *with the Riesz measure* $\nu_M$.

**[Z]** *If $\Lambda=\{\lambda_k\}\subset\Omega$ is a zero (sub)set for* $\operatorname{Hol}(\Omega;M)$, *then*

$$\sup_{0\in D\Subset\Omega}\Bigl(\sum_k g_D(\lambda_k,0)-\int g_D(\zeta,0)\,\mathrm{d}\nu_M(\zeta)\Bigr)<+\infty. \tag{9}$$

*Conversely, if we have (9) where domains $D$ run through the class $\mathcal{U}_0^d(\Omega)$ only, then $\Lambda$ is a zero <u>sub</u>sequence (non-uniqueness sequence) for $\operatorname{Hol}(\Omega;\widehat{M})$ where $\widehat{M}(z):=$*

$$\inf_{0<t<\operatorname{dist}(z,\partial\Omega)}\Bigl(\frac{1}{2\pi}\int_0^{2\pi}M\bigl(z+te^{i\theta}\bigr)\,\mathrm{d}\theta+\log\bigl(1+1/t\bigr)\Bigr)+9\log^+|z|,$$

*and a zero sequence for $\operatorname{Hol}(\Omega;M+Q)$.*

*Thus, every zero <u>sub</u>sequence for $\operatorname{Hol}(\Omega;M)$ is a zero sequence for $\operatorname{Hol}(\Omega;M+Q)$.*

**[M]** *Let $f=g/q$ be a meromorpic function and $g,q\in\operatorname{Hol}(\Omega;M)$. Then*

$$\sup_{0\in D\Subset\Omega}\Bigl(\int_\Omega\log\max\{|g|,|q|\}(z)\,\mathrm{d}\omega_D(0,z)-\int M(z)\,\mathrm{d}\omega_D(0,z)\Bigr)<+\infty. \tag{10}$$

*Conversely, if, under the assumptions of (6) and (8), we have (10) where domains $D$ run through the class $\mathcal{U}_0^d(\Omega)$ only, then there are functions $g,q\in\operatorname{Hol}(\Omega;\widehat{M})$ and $g_0,q_0\in\operatorname{Hol}(\Omega;M+Q)$ such that $f=g/q=g_0/q_0$ and $g_0,q_0$ have no common zeros.*

*Thus, if $f=g/q$ with $g,q\in\operatorname{Hol}(\Omega;M)$, then there exist functions $g_0,q_0\in\operatorname{Hol}(\Omega;M+Q)$ without common zeros such that $f=g_0/q_0$.*

**8. Nevanlinna's theorems wit nonradial and nonpositive weight.** In [4, Theorem 1], we investigate also an slowly counterpart Nevanlinna theorems.

Let $M\colon\mathbb{D}\to\mathbb{R}$, and let $M\in\operatorname{sbh}(\mathbb{D})$ with the Riesz measure $\nu_M$. Given $z=re^{i\theta}$, $0\le r<1$, $\theta\in\mathbb{R}$, and $a>0$ we consider a polar rectangle

$$\boxminus(z;a):=\{\,\zeta=te^{i\psi}\colon\bigl(r-a\sqrt{1-r^2}\bigr)^+\le t<1,\ |\sin(\psi-\theta)|<a\sqrt{1-r^2}\,\} \tag{11}$$

of relative size $a$ and the function

$$q_M^{[a]}(z):=\frac{1}{1-|z|}\int_{\boxminus(z;a)}(1-|\zeta|)\,\mathrm{d}\nu_M(\zeta). \tag{12}$$

We set

$$\mathrm{A}_M^{[\varepsilon]}(z):=\frac{1}{2\pi}\int_0^{2\pi}M\bigl(z+\varepsilon(1-|z|)e^{i\theta}\bigr)\,\mathrm{d}\theta,\quad 0<\varepsilon<1. \tag{13}$$

**Theorem 11.** *Let $M$ be a subharmonic function $\mathbb{D}$, $M(0)>-\infty$ and*

$$\sup_{r<1}\int_0^{2\pi}M(re^{i\theta})\,\mathrm{d}\theta<+\infty, \tag{14}$$

*that is equivalent to the Blaschke condition*

$$\int_0^1(1-t)\,\mathrm{d}\nu_M(t)<+\infty. \tag{15}$$

*For a function $f_\Lambda$ with $\operatorname{Zero}_{f_\Lambda}=\Lambda$,*

(Z) *if it is fulfilled, at least, one condition*

$$\sup_{D\in\mathcal{U}_0^d(\mathbb{D})}\Bigl(\int_{\mathbb{D}}\log\bigl|f_\Lambda(z)\bigr|\,\mathrm{d}\omega_D(0,z)-\int_{\mathbb{D}}M(z)\,\mathrm{d}\omega_D(0,z)\Bigr)<+\infty, \tag{16}$$

$$\sup_{D\in\mathcal{U}_0^d(\mathbb{D})}\Bigl(\sum_k g_D(\lambda_k,0)-\int_{\mathbb{D}\setminus\{0\}}g_D(\zeta,0)\,\mathrm{d}\nu_M(\zeta)\Bigr)<+\infty, \tag{17}$$

$$\sup_{D\in\mathcal{U}_0^d(\mathbb{D})}\Bigl(\sum_k g_D(\lambda_k,0)-\int_{\mathbb{D}}M(z)\,\mathrm{d}\omega_D(0,z)\Bigr)<+\infty, \tag{18}$$

*then for any* $\varepsilon\in(0,1)$ *and* $1<a<2$ *the sequence* $\Lambda\subset\mathbb{D}$ *is a zero sequence for the space*

$$\operatorname{Hol}\Bigl(\mathbb{D};\mathrm{A}_M^{[\varepsilon]}+\frac{C_\varepsilon}{2-a}\,q_M^{[a]}\Bigr), \tag{19}$$

*where a constant* $C_\varepsilon$ *dependent only on* $\varepsilon$*;*

(U) *if* $\Lambda$ *is a zero subsequence for* $\operatorname{Hol}(\mathbb{D};M)$*, then* $\Lambda$ *is a zero sequence for* (19)*;*

(M) *if a meromorphic function* $f=g/q$ *in* $\mathbb{D}$ *is represented as a ratio of functions* $g,q\in\operatorname{Hol}(\mathbb{D})$*,* $\max\bigl\{|g(0)|,|q(0)|\bigr\}\neq 0$*, and, at least one of the following conditions holds*

$$\sup_{D\in\mathcal{U}_0^d(\mathbb{D})}\Bigl(\int_{\mathbb{D}}\log\max\bigl\{|g(z)|,|q(z)|\bigr\}\,\mathrm{d}\omega_D(0,z)-\int_{\mathbb{D}}M(z)\,\mathrm{d}\omega_D(0,z)\Bigr)<+\infty, \tag{20}$$

$$g,q\in\operatorname{Hol}(\mathbb{D};M), \tag{21}$$

*then there are functions* $g_0$ *and* $q_0$ *from class (19) without a common zero, such that* $f=g_0/q_0$ *in* $\mathbb{D}$.

A function $M\colon z=re^{i\theta}\to[-\infty,+\infty]$ is called *radial in a sector*

$$\measuredangle(\alpha,\beta):=\{z=re^{i\theta}\colon 0\le r<1,\ \alpha<\theta<\beta\} \tag{22}$$

*from* $\mathbb{D}$*,* if for each $0\le r<1$ the function $M(re^{i\theta})$ is independent of $\theta\in\measuredangle(\alpha,\beta)$.

**Corollary 1.** *If there is a sector* $\measuredangle(\alpha',\beta')\Supset\measuredangle(\alpha,\beta)$ *such that* $\alpha'<\alpha<\beta<\beta'$*, and the weight* $M$ *from Theorem 11 is radial and differentiable in* $r$*, then, for some number* $a$ *from a small neighborhood of 1, for all assertions* (Z), (U) and (M) *of this theorem at points* $z\in\measuredangle(\alpha,\beta)$ *the summand* $\frac{C_\varepsilon}{2-a}\,q_M^{[a]}(z)$ *in (19) can be changed to a summand*

$$\frac{aC_\varepsilon}{2(2-a)}\,\frac{1}{\sqrt{1-|z|}}\int_{\left(|z|-a\sqrt{1-|z|^2}\right)^+}^{1}(1-t)\,\mathrm{d}(tM'(t)). \tag{23}$$

*for small* $a>1$.

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Bashkir State University,
Russia, Ufa,
khabib-bulat@mail.ru


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